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Confidence Intervals for the Mean Function of a Compound Cyclic Poisson Process in the Presence of Power Function Trend

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ABSTRACT

Asymtotic normality of an estimator for the mean function of a compound cyclic Poisson process in the present of power function trend which introduced by Safitri in 2002. To provided information on parameters guarantees (mean function) covered in an interval, it is necessary to find a convidence interval for the mean function of a compound cyclic Poisson process in the presence of power function trend. The objectives of this paper are: (i) to construct confidence interval for the mean function of a compound cyclic Poisson process with significance level $0 < \alpha < 1$, (ii) to prove that the probability that the mean function contained in the confidence interval converges to $1 - \alpha$, and (iii) to observe, using simulation study, that the probabilities of the mean function contained in the confidence intervals for bounded length of observation interval. This paper showed that a confidence interval for the mean function and a theorem about convergence of the probability that the mean function contained in confidence interval. The simulation study shows that the probability that the mean function contained in the confidence interval is in accordance with the theorem. The contribution of this study is to provide information for users regarding confidence interval for the mean function of a compound cyclic Poisson process in the presence of power function trend.

Keywords: compound cyclic poisson process; power function tren; mean function; confidence interval; poisson process.

INTRODUCTION

There are many events in everyday life that are uncertain, such as the birth and death process [1] the queue process [2] and the estimation of total insurance claims [3], which can be modeled using a stochastic process. A stochastic process is process that describes series of random events at certain time intervals [4]. A special form of stochastic process is the compound Poisson process. A compound Poisson process is a process of adding sequencess random variables of independent and identically distributed (i.i.d) with certain distribution as many as Poisson random variables, and independent of the Poisson process. Based on the time aspect, stochastic process can be classified in two categories, namely discrete time stochastic process and continuous time stochastic process. A special form of continuous time stochastic process is the Poisson process. The Poisson process is a counting a process in which the number of events in a Poisson distribution time interval.

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Based on the intensity aspect, the Poisson process can be classified in two categories, that is homogeneous Poisson process with constant intensity function (not dependent on time) and the non-homogeneous Poisson Process with intensity function depends on time. One type of non-homogeneous Poisson process is the cyclic or periodic Poisson process [5]. The period can be daily, weekly, yearly or in other forms [6]. This non-homogeneous Poisson process is widely applied to real phenomena, such as th phenomenon of earthquakes [7], healthcare [8], radio burst rates [9], and traffic accidents [10].

The study of the compound periodic Poisson process is widely. This research begins with the estimation of the expected value function on the compound periodic Poisson process [11] [12], then it was developed with a power trend [13] [14]. The compound cyclic Poisson does not follow the usual distribution patern. One aspect which can be estimated is the mean value. Since this value depends on the time of observation, it is called the mean function. In [15], an estimator for the mean function of a compound cyclic Poisson process has been constructed and studied. The asymptotic normality of this estimator also has been proven. Furthemore, to give assurance information that the mean function is included in an interval, it is necessary to construct a confidence interval for mean function of the compound cyclic Poisson process in the presence of power function trend.

As an application of the asymptotic normality, it can be determined the confidence interval of the estimator for the periodic component. In [16] studied the confidence intervals for the mean and variance functions of compound Poisson process with power function intensity have been studied, while in [17] confidence intervals for the mean and variance functions of compound Poisson process with exponential of linear function intensity have been studied. Specifically, this research was conducted to (i) to construct confidence interval for the mean function of a compound cyclic Poisson process in the presence of power function trend, (ii) to prove convergence to $1-\alpha$ of probability that the mean function included in the confidence interval, and (iii) to check using simulation study that the probabilities of the mean function contained in the confidence intervals for bounded length of observation interval. The contribution of this study is to provide information for users regarding confidence interval for the mean function of a compound cyclic Poisson process in the presence of power function trend.

METHODS

The Estimator for the Mean Function

Suppose that $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with (unknown) intensity function λ which is assumed to be locally integrable. Suppose that λ has two components, namely a cyclic component (λ_c) with (known) period $\tau > 0$ and a power function trend component (as^b). In other words, for all $a \geq 0$ and $s \geq 0$, the intensity function $\lambda(s)$ can be expressed as

$$\lambda(s) = \lambda_c(s) + as^b. \tag{1}$$

The value of b is assumed to be known real number and $0 < b < \frac{1}{2}$. We do not assumed any parametric form for the cyclic component λ_c, except that it is cyclic or periodic, which satisfies

$$\lambda_c(s) = \lambda_c(s + k\tau) \tag{2}$$

for all $s \geq 0$ and all $k \in \mathbb{N}$.

Suppose that $\{Y(t), t \geq 0\}$ is a process where

$$Y(t) = \sum_{i=1}^{N(t)} X_i \tag{3}$$

with $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables which having mean $\mu < \infty$ and variance $\sigma^2 < \infty$, and also independent of $\{N(t), t \geq 0\}$. The process $\{Y(t), t \geq 0\}$ is called a compound cyclic Poisson process with power function trend [7].

Suppose that $\psi(t)$ is notation of the mean function of $Y(t)$, that is

$$\psi(t) = E(Y(t)) = E[N(t)]E[X_1] = \Lambda(t)\mu \tag{4}$$

with $\mu = E(X_i)$ and

$$\Lambda(t) = \int_0^t \lambda(s)\, ds. \tag{5}$$

Let $t_r = t - \left\lfloor \frac{t}{\tau} \right\rfloor \tau$, where $\left\lfloor \frac{t}{\tau} \right\rfloor$ represents the largest integer less than or equal to $\frac{t}{\tau}$, $\frac{t}{\tau} \in \mathbb{R}$ and $k_{t,\tau} = \left\lfloor \frac{t}{\tau} \right\rfloor$. Then, for any real number $t \geq 0$, t can be expressed as $t = k_{t,\tau}\tau + t_r$ with $0 \leq t_r \leq \tau$.

Let $\theta = \frac{1}{\tau}\int_0^\tau \lambda_c(s)\, ds$ denotes the global intensity of the periodic component in the process $\{N(t), t \geq 0\}$ and it is assumed that $\theta > 0$. This θ can be written as $\Lambda_c(t_r) + {\Lambda_c}^c(t_r)$ with

$$\Lambda_c(t_r) = \int_0^{t_r} \lambda_c(s)\, ds \tag{6}$$

and

$${\Lambda_c}^c(t_r) = \int_{t_r}^{\tau} \lambda_c(s)\, ds. \tag{7}$$

By using (6) and (7) and substituting (1) into (5), then for any $t \geq 0$, $\Lambda(t)$ can be written as

$$\Lambda(t) = (k_{t,\tau} + 1)\Lambda_c(t_r) + k_{t,\tau}{\Lambda_c}^c(t_r) + \frac{a}{b+1}t^{b+1}. \tag{8}$$

By substituting (8) into (4), we have

$$\psi(t) = \left((k_{t,\tau} + 1)\Lambda_c(t_r) + k_{t,\tau}{\Lambda_c}^c(t_r) + \frac{a}{b+1}t^{b+1}\right)\mu. \tag{9}$$

Estimation of Mean Function

In [11] an estimator of the mean function $\psi(t)$ has been formulated as follows

$$\hat{\psi}_{n,b}(t) = \left((k_{t,\tau} + 1)\hat{\Lambda}_{c,n,b}(t_r) + k_{t,\tau}{\hat{\Lambda}_{c,n,b}}^c(t_r) + \frac{\hat{a}_{m,b}}{b+1}t^{b+1} \right)\hat{\mu}_n \tag{10}$$

where

$$\hat{\Lambda}_{c,n,b}(t_r) = \frac{(1-b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{1}{k^b} N([k\tau, k\tau + t_r]) - \hat{a}_{m,b}(1-b)n^b t_r, \tag{11}$$

$${\hat{\Lambda}_{c,n,b}}^c(t_r) = \frac{(1-b)\tau^{1-b}}{n^{1-b}} \sum_{k=1}^{k_{n,\tau}} \frac{1}{k^b} N([k\tau + t_r, k\tau + \tau]) + \hat{a}_{m,b}(1-b)n^b(t_r - \tau), \tag{12}$$

$$\hat{a}_{m,b} = \frac{(1+b)N([0,m])}{m^{(1+b)}} - \frac{(1+b)}{m^b}\tilde{\theta}_n, \tag{13}$$

$$\tilde{\theta}_n = \frac{(1-b)}{n^{1-b}\tau^b b^2}\sum_{k=1}^{k_{n,\tau}} \frac{1}{k^b} N([k\tau, k\tau+\tau]) - \frac{(1+b)(1-b)n^b N([0,n])}{n^{(1+b)}b^2}, \tag{14}$$

$$\hat{\mu}_n = \frac{1}{N[0,n]}\sum_{i=1}^{N([0,n])} X_i. \tag{15}$$

with $\hat{\mu}_n = 0$ when $N([0,n]) = 0$.

Asymptotic Normally of the Estimator for the Mean Function

Theorem 1 (The Asymptotic Normally of the Estimator for the Mean Function)

Suppose that the intensity λ statisfies (1) and locally integrable. If $Y(t)$ statisfies (2), then

$$\sqrt{n^{1-b}}\left(\hat{\psi}_{n,b}(t) - \psi(t)\right) \xrightarrow{d} \text{Normal}\left(0, \left(k_{t,\tau}+1\right)^2 a(1-b)\tau t_r \mu^2 + {k_{t,\tau}}^2 a(1-b)\tau(\tau - t_r)\mu^2\right) \tag{16}$$

as $n \to \infty$.

The proofs of Theorem 1 can be proved through a rough analysis [11].

RESULTS AND DISCUSSION

Our main results are a confidence interval for the mean function $\boldsymbol{\psi(t)}$ and a theorem about convergence of the probability that $\boldsymbol{\psi(t)}$ contained in the confidence interval.

Corollary 1 (The confidence interval for $\boldsymbol{\psi(t)}$)

For given a significant level α, where $0 < \alpha < 1$, the confidence interval for $\psi(t)$ in the case $0 < b < \frac{1}{2}$ is given by

$$I_{\psi,n} = \left[\hat{\psi}_{n,b}(t) - \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{V_{n,b}}\ , \hat{\psi}_{n,b}(t) + \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{V_{n,b}}\right]$$

with

$$V_{n,b} = \frac{\left(k_{t,\tau}+1\right)^2 \hat{a}_{m,b}(1-b)\tau t_r {\hat{\mu}_n}^2 + {k_{t,\tau}}^2 \hat{a}_{m,b}(1-b)\tau(\tau - t_r){\hat{\mu}_n}^2}{n^{1-b}},$$

where Φ denotes the standard normal distribution and $V_{n,b}$ denotes the studenize version of (16).

Theorem 2 (Convergence of Probability that $\boldsymbol{\psi(t)} \in \boldsymbol{I_{\psi,n}}$)

For confidence interval $I_{\psi,n}$ of ψ(t) given in Corollary 1, we have that

$$P\left(\psi(t) \in I_{\psi,n}\right) \to 1 - \alpha$$

as $n \to \infty$.

Proof of Theorem 2:

The probability that $\psi(t)$ contained in the confidence interval $I_{\psi,n}$ can be computed as follows.

$$P\left(\hat{\psi}_{n,b}(t) - \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}} \le \psi(\mathrm{t}) \le \hat{\psi}_{n,b}(t) + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}}\right)$$

$$= P\left(-\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}} \le -\hat{\psi}_{n,b}(t) + \psi(\mathrm{t}) \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}}\right)$$

$$= P\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}} \ge \hat{\psi}_{n,b}(t) - \psi(\mathrm{t}) \ge -\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}}\right)$$

$$= P\left(-\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}} \le \hat{\psi}_{n,b}(t) - \psi(\mathrm{t}) \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{V_{n,b}}\right)$$

$$= P\left(-\Phi^{-1}\left(1-\frac{\alpha}{2}\right) \le \frac{\hat{\psi}_{n,b}(t)-\psi(\mathrm{t})}{\sqrt{V_{n,b}}} \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right).$$

By the Studentize version of (16), we have that

$$\frac{\hat{\psi}_{n,b}(t)-\psi(\mathrm{t})}{\sqrt{V_{n,b}}} \xrightarrow{d} \text{Normal}(0,1),$$

as $n \to \infty$. Therefore $P(\psi(t) \in I_{\psi,n})$ converges to

$$P\left(-\Phi^{-1}\left(1-\frac{\alpha}{2}\right) \le Z \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

as $n \to \infty$, where Z is the standard normal random variable.

Further we can simplify the above probability as follows.

$$\begin{aligned}
&P\left(-\Phi^{-1}\left(1-\frac{\alpha}{2}\right) \le Z \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\\
&\quad = P\left(Z \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - P\left(Z \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\\
&\quad = P\left(Z \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - P\left(Z \ge \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\\
&\quad = P\left(Z \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - \left(1 - P\left(Z \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right)\\
&\quad = \Phi\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - \left(1 - \Phi\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)\right)\\
&\quad = \left(1-\frac{\alpha}{2}\right) - \left(1 - \left(1-\frac{\alpha}{2}\right)\right)\\
&\quad = 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 2.

Simulation of the Confidence Interval for the Mean Function

The purpose of this simulation is to check the probability that the mean function $\psi(t)$ is contained in the confidence intervals for some different significant levels, period, and length of observation interval, using generated data. This simulation was carried out with the help of R software and Scilab software for illustration the results.

The programing stage is carried out by generating the realization of compound periodic Poisson process with a power function trend with the formulation of the intensity function:

$$\lambda(s) = \sin\frac{2\pi s}{\tau} + 1 + as^{b}.$$

In this simulation, we choose significant levels $\alpha = 1\%$, 5% and 10%, $\tau = 1, s = 2.5$, $a = 0.1, b = 0.4, n = 20, 50$ and 100 with 1000 repetitions.

Table 1. Simulation results of confidence interval for the mean function $\psi(t)$

	n	A	B	C	D	E
	20	985	15	98.5%	1.5%	0.5%
1%	50	988	12	98.8%	1.2%	0.2%
	100	990	10	99.0%	1.0%	0.0%
	20	941	59	94.1%	5.9%%	0.9%
5%	50	950	50	95.0%	5.0%	0.0%
	100	952	48	95.2%	4.8%	0.2%
	20	891	109	89,1%	10.9%	0.9%
10%	50	907	93	90.7%	9.3%	0.7%
	100	917	83	91.7%	8.3%	1.7%

(A= the number confidence interval containing the parameter, B= the number confidence interval that do not contain the parameter, C= percentage of confidence interval containing the parameter, D= percentage of confidence interval that does not contain the parameter, E= absolute error between α and percentage of confidence interval that does not contain the parameters)

Based on simulation results, percentage of confidence interval that does not contain parameter at $s = 2.5$ and $\tau = 1$ with $\alpha = 1\%$, 5% and 10% fir observation interval $[0, n]$ with $n = 20, 50$ and 100 respectively from $0.0\% - 0.5\%, 0.0\% - 0.9\%$ and from $0.7\% - 1.7\%$. The error that obtained between α and percentage of confidence interval that does not contain parameters also tend to be small, between 0% and 1.7%. This shows that the result of the simulation of the confidence interval for the mean function $\psi(t)$ for the compound Poisson process with different significant levels is in accordance with the theory obtained. The simulation results based on the first 200 estimators can be seen in Figure 1.

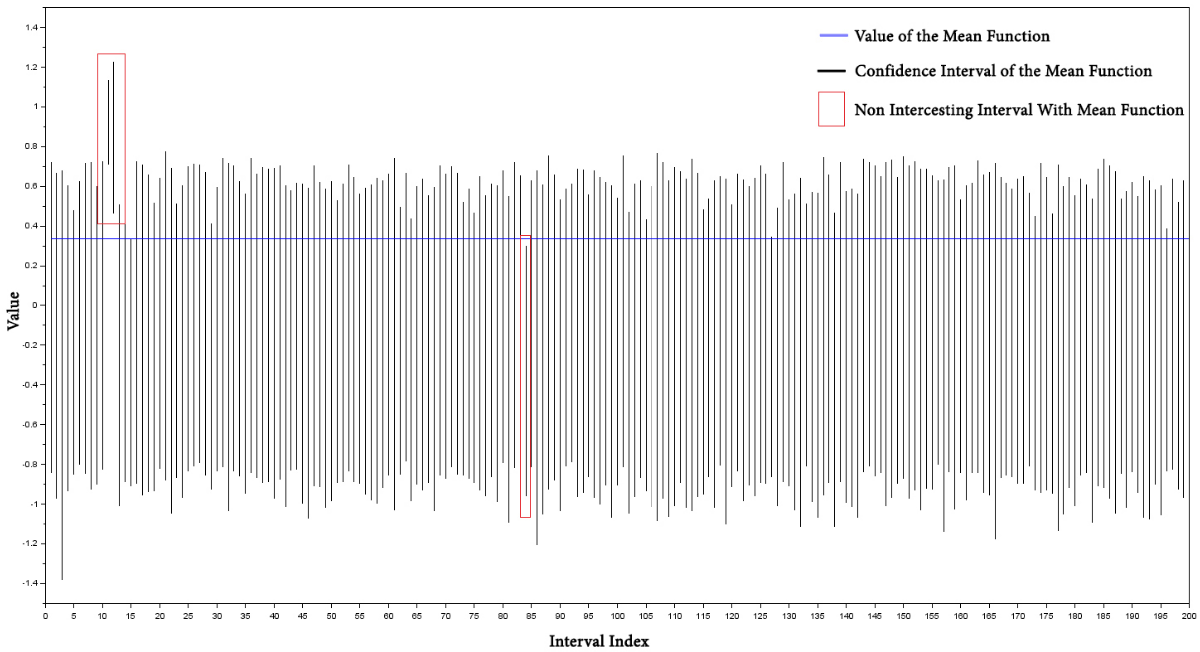

Figure 1. Confidence interval for the mean function $\psi(t)$ based on the first 200 estimators with $s = 2.5, \tau = 1, \alpha = 1\%$ and $n = 100$

The illustration in Figure 1 shows some of the results of the confidence interval simulation for the mean function $\psi(t)$ at $s = 2.5$ and $\tau = 1$ based on the first 200 estimators with significance level of $\alpha = 1\%$ and $n = 100$. It can be seen in the figure that the horizontal line is the true value of the mean function $\psi(t)$ and vertical lines are the confidence intervals of the mean function $\psi(t)$. If the horizontal and vertical lines do not intersect each other, this indicates that the value of the mean function $\psi(t)$ is not in that interval. In Figure 1, there are three non intersecting lines which indicates there are three confidence intervals based on the first 200 estimators do not contain the value of the mean function $\psi(t)$. Since in Table 1 there are 10 confidence intervals that do not contain the mean function $\psi(t)$, this shows that there are seven confidence intervals based on the 201-st to 1000-th estimators do not contain the value of the mean function $\psi(t)$.

The illustration results in Figure 1 show that the probability of the mean function $\psi(t)$ is contained in the confidence interval already close to $1 - \alpha$ for $\tau = 1$ and 5, $\alpha = 1\%, 5\%$ and 10% for bounded time interval observation.

CONCLUSIONS

According to the main results, it can be concluded that confidence interval for the mean function $\psi(t)$ of compound cyclic Poisson process in the presence of power function trend is

$$I_{\psi,n} = \left[\hat{\psi}_{n,b}(t) - \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{V_{n,b}}\ ,\ \hat{\psi}_{n,b}(t) + \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{V_{n,b}}\right],$$

where Φ denotes the standard normal distribution and

$$V_{n,b} = \frac{\left(k_{t,\tau} + 1\right)^2 \hat{a}_{m,b}(1 - b)\tau t_r \hat{\mu}_n{}^2 + k_{t,\tau}{}^2 \hat{a}_{m,b}(1 - b)\tau(\tau - t_r)\hat{\mu}_n{}^2}{n^{1-b}}.$$

Convergence of the probability that the mean function $\psi(t)$ contained in the confidence interval is

$$P\big(\psi(t) \in I_{\psi,n}\big) \rightarrow 1-\alpha, \qquad \text{as } n \rightarrow \infty.$$

The simulation results show that the probability of the mean function $\psi(t)$ included in the confidence interval already close to $1-\alpha$ for a finite length observation interval.

A recommendation for futher research can be to use different intensity functiom and different observation function from this study at thr simulation stage, so that they can show more diverse simulation results.

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