MODIFIED CHEBYSHEV (VIETA–LUCAS POLYNOMIAL) COLLOCATION METHOD FOR SOLVING DIFFERENTIAL EQUATIONS

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Abstract

This paper presents derivation of alternative numerical scheme for solving differential equations, which is modified Chebyshev (Vieta-Lucas Polynomial) collocation differentiation matrices. The Scheme of modified Chebyshev (Vieta-Lucas Polynomial) collocation method is applied to both Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) cases. Finally, the performance of the proposed method is compared with finite difference method and the exact solution of the example. It is shown that modified Chebyshev collocation method more effective and accurate than FDM for some example given.

Keywords: Vieta-Lucas Polynomial, collocation method, Ordinary Differential Equations, Partial Differential Equations, Finite difference methods

INTRODUCTION

Chebyshev polynomials have become alternatively numerical solution for differential problems, which has developed after digital computerizing growth rapidly. It was discovered by Chebyshev and developed by some mathematicians; Clenshaw [1], Nidekker, Fox and Parker [2], etc. after computer was discovered.

It is known that there are four kinds of Chebyshev polynomial as developed that commonly applied in numerical analysis [3]. For the first through four kinds, it is symbolized by $T_N(x), U_N(x), V_N(x),\ and\ W_N(x)$ respectively. First and second kind of Chebyshev polynomial had practiced to solving linear differential equation [4, 5]. Furthermore, the other kinds were used to find solution of some problems numerically [6-9]. However, for some reasons of the advantages, modified Chebyshev and its statistical properties was discovered by Witula and Damian and symbolized by $\Omega_N(x)$ [10].

The main objective of the present article is to develop algorithms based on modified Chebyshev polynomial (Vieta-Lucas Polynomial) [10] for solving differential equations. And In this paper, it is practiced for some DEs example and compared with finite difference method.

The contents of this article are organized as follows. In Section 2, we discuss about a literature review of methods, FDM and modified Chebyshev collocations method. In addition, we discuss discovering of Differentiation matrices of Modified Chebyshev (Vieta-Lucas Polynomial) collocation and the result. Finally, some concluding remarks are presented in Section 3.

A LITERATURE REVIEW

Finite Difference Method

Finite difference method is a superior method to approach the derivative of a function based on the Taylor series. The accuracy of finite difference method can be adapted to taking into first part account of the Taylor series. In this article, the first and second derivatives are approximated using central difference (centered difference approximation) second order as follows [11];

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h} + O(h^2) \tag{1}$$

$$f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + O(h^2) \tag{2}$$

$$f^{(3)}(x) \approx \frac{f(x+2h)-2f(x+h)+2f(x-h)-f(x-2h)}{2h^3} + O(h^2) \tag{3}$$

By this scheme, some of differential equations examples shall be solve and compared with the exact solution.

Chebyshev Polynomial Collocation Method

Chebyshev collocation is one of spectral methods based on trial functions Chebyshev polynomial. Many practical problems are performed by mathematician using Chebyshev collocation [1, 4, 5, 7-9]. This is because some advantages such as accurate and memory minimizing. There are several kinds of Chebyshev polynomial describe below,

First kind,

$$T_N(x) = \cos(N\theta). \tag{4}$$

Second kind,

$$U_N(x) = \frac{sin((N+1)\theta)}{\sin(\theta)}. \tag{5}$$

Third kind,

$$V_N(x) = \frac{\cos((N+\frac{1}{2})\theta)}{\cos(\frac{1}{2}\theta)}. \tag{6}$$

Fourth kinds,

$$W_N(x) = \frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}. \tag{7}$$

When $\theta = arccos(x)$ [3] and x is collocation points.

Let we have boundary value problem on range [-1,1]:

$$\frac{d^2}{dx^2}u(x) = f(x), \quad u(-1) = a,\ u(+1) = b, \tag{8}$$

where $f(x)$ is a function and boundary value is given above.

Suppose that we approximate $u(x)$ by,

$$u_N(x) = \sum_{k=0}^{N} c_k\, \varphi_k(x) \tag{9}$$

It is involving $n+1$ unknown coefficients $\{c_k\}$, and $\varphi_k(x)$ is a basis function; in this case, we perform Chebyshev polynomial as a basis function. Then we may select $n-1$ points $\{x_1, \dots, x_{n-1}\}$ in the range of integration called collocation points and require $u_N(x)$ satisfying the differential equation (8). This requires us to solve just the system of $n+1$ linear equations by differentiation matrix of the method. It is well-known as Chebyshev collocation method.

Modified Chebyshev Collocation Method

Furthermore, in present paper, we perform first kind of modified Chebyshev polynomial as a basis function,

$$\Omega_N(x) = 2T_N\left(\frac{x}{2}\right) \tag{10}$$

where $T_N(x)$ is the first Chebyshev Polynomial. Some properties of Modified Chebyshev polynomial presented by Witula, et all [3]. It is also called *n-th* Vieta-Lucas Polynomial.

RESULTS AND DISCUSSION

Finite Difference Scheme

To approximate first, second and third derivative, we employ central difference second order as seen in equation (1)-(3). In this part, we can perform its differentiation matrices scheme of FDM equations (1)-(3) respectively as follows,

$$D_{N,1} = \frac{1}{2h}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ 0 & -1 & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \dots & & -1 & 0 \end{pmatrix} \tag{11.a}$$

$$D_{N,2} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \dots & & 1 & -2 \end{pmatrix} \tag{11.b}$$

$$D_{N,3} = \frac{1}{2h^3}\begin{pmatrix} 0 & -2 & -1 & \dots & 0 \\ 2 & 0 & -2 & \dots & \vdots \\ 1 & 2 & \ddots & \ddots & -1 \\ \vdots & & \ddots & \ddots & -2 \\ 0 & \dots & 1 & 2 & 0 \end{pmatrix} \tag{11.c}$$

However, for solving PDEs, the system of ODE, which is generated by FDM, can be advanced in time by forward Euler scheme or a third order Runge-Kutta scheme.

Modified Chebyshev (Vieta-Lucas Polynomial) Differentiation Matrices

Basically, solving DEs cannot be denied involving system of linear equations. Hence, we need differentiation matrices derived by collocation method. In this section, we shall explain how to derive differentiation matrices of modified Chebyshev (Vieta-Lucas Polynomial) collocation method.

Firstly, we define Vieta-Lucas Polynomial as a basis function of collocation approximations. We have first kind of modified Chebyshev (Vieta-Lucas Polynomial) Collocation Method defined by Witula [3] as equation (10), consequently,

$$\Omega_N(x) = 2.cos\left(N.arccos\left(\frac{x}{2}\right)\right) \quad [-2,2] \quad (12)$$

Furthermore, discover zero and extrema points of first kind of modified Chebyshev (Vieta-Lucas Polynomial) used to construct differentiation matrices. We show that zero points of modified Chebyshev (Vieta-Lucas Polynomial) is

$$x_k = 2.cos\left(\frac{2k-1}{2N}\right)\pi \qquad k \in \mathbb{Z} \qquad (13)$$

By first derivative, $\frac{d\Omega_N(x)}{dx} = 0$, we discover the extrema points of modified Chebyshev as follows,

$$x_j = 2.cos\left(\frac{\pi j}{N}\right), \quad 0 \le j \le N \qquad (14)$$

It is well-known as first kind modified Chebyshev (Vieta-Lucas Polynomial) Collocation points. Considering the Chebyshev points above, we construct differentiation matrices of Modified Chebyshev Collocation method as follows. First, we need the following results:

$$\Omega_N(x_j) = 2.cos\left(N.arccos\left(\frac{x_j}{2}\right)\right) \quad [-2,2]$$

$$\Omega'_N(x_j) = \frac{N.\sin(N\theta)}{\sin(\theta)} = 0, \quad 0 \le j \le N-1 \quad (15.a)$$

$$\Omega''_N(x_j) = \frac{(-1)^{j+1}.2.N^2}{(4-x_j{}^2)}, \quad 0 \le j \le N-1 \quad (15.b)$$

$$\Omega'''_N(x_j) = \frac{(-1)^{j+1}.6.N^2 x_j}{(4-x_j{}^2)^2}, \quad 0 \le j \le N-1 \quad (15.c)$$

$$\Omega'_N(\pm 2) = (\pm 1)^{N+1} N^2 \quad \text{and}$$

$$\Omega''_N(\pm 2) = \frac{(\pm 1)^N N^2(N^2-1)}{6} \quad (15.d)$$

Since modified Chebyshev $\Omega'_N(x)$ is polynomial and we know all zeros points of it (called as Vieta-Lucas Collocation points) we may write,

$$\Omega'_N(x) = \alpha_{\Omega N} \prod_{j=1}^{N-1}(x - x_j) \qquad (16)$$

For $x_{0=-2}$ and $x_{N=2}$, we can write,

$$\gamma_N(x) = \prod_{j=0}^{N}(x - x_j) = \beta_N(x^2 - 4)\Omega'_N(x) \quad (17)$$

Where β_N is a positive a constant such that coefficient x^{N+1} equals to 1.

From equation (15.a) and (15.d) and if $x = x_j$ and $0 \le j \le N$, it follows that

$$\gamma'_N(x) = \beta_N 2x\Omega'_N(x) + \beta_N(x^2 - 4)\Omega''_N(x) \quad (18)$$

However, for $0 \le j \le N-1$, $\Omega'_N(x_j) = 0$ and when $j = 0$ and $j = N$ as seen in equation (15.d), In consequence, we have

$$\gamma'_N(x_j) = (-1)^j \tilde{c}_j N^2 \beta_N \qquad (19)$$

where $\tilde{c}_0 = \tilde{c}_N = 4$ and $\tilde{c}_j = 2$ for $1 \le j \le N-1$. For an arbitrary function $f(x)$ and nodes $x = x_j$, we can estimate the function using Lagrange's interpolation polynomial below,

$$p(x) = \sum_{j=0}^{N} F_j(x).f(x) \qquad (20)$$

where

$$F_j(x) = \frac{\gamma_N(x)}{\gamma'_N(x_j)(x-x_j)} \ , 0 \le j \le N \qquad (21)$$

since $\gamma_N(x)$ and $\gamma'_N(x_j)$ are described by (17) and (19) respectively. By definition of differentiation matrix, hence we obtain differentiation matrix below,

$$D^1_{kj} = F'_j(x_k)$$

The element of the matrices can be evolved from first derivative of $F(x_j)$ with some point's defined equation (14). Scheme below is determining the element of differentiation matrices;

1. For $1 \le j \ne k \le N-1$ for equation (21), together with (15.a) and (15.d), leads to

$$F'_j(x_k) = \frac{(-1)^{j+k}}{\tilde{c}_j(x_k - x_j)} \qquad (22.a)$$

2. For $j \neq k$ and $k = 0,\ 1 \leq j \leq N-1$

$$F'_j(x_0) = \frac{(-1)^j.4}{\tilde{c}_j(x_0 - x_j)} \quad (22.b)$$

3. For $j \neq k$ and $j = N,\ 1 \leq k \leq N-1$

$$F'_N(x_k) = \frac{(-1)^{N+k}.2}{\tilde{c}_N(x_k - x_N)} \quad (22.c)$$

4. For $j \neq k$ and $k = N,\ 1 \leq j \leq N-1$

$$F'_j(x_N) = \frac{(-1)^{j+N}.4}{\tilde{c}_j(x_N - x_j)} \quad (22.d)$$

5. For $j \neq k$ and $j = 0,\ 1 \leq k \leq N-1$

$$F'_0(x_k) = \frac{(-1)^k.2}{\tilde{c}_0(x_k - x_0)} \quad (22.e)$$

6. For $j \neq k$, and $k = 0, j = N$,

$$F'_N(x_0) = \frac{1}{\tilde{c}_N} \quad (22.f)$$

7. For $j \neq k$, and $j = 0, k = N$,

$$F'_0(x_N) = -\frac{(-1)^N}{\tilde{c}_0} \quad (22.g)$$

From equation (22.a) to (22.g), in general, for $0 \leq j \neq k \leq N$, we have

$$F'_j(x_k) = \frac{(-1)^{j+k}\tilde{c}_k}{\tilde{c}_j(x_k - x_j)} \quad (23)$$

where $\tilde{c}_0 = \tilde{c}_N = 4$ and $\tilde{c}_j = 2$.

Furthermore, by applying L'Hospital for $j = k = 0$ we have,

$$F'_0(x_0) = \left(\frac{1+2N^2}{6}\right) \quad (24)$$

and for $j = k = N$

$$F'_N(x_N) = -\left(\frac{1+2N^2}{6}\right) \quad (25)$$

Again using L'Hospital's rule for $1 \leq j = k \leq N-1$ shows that

$$F'_j(x_j) = \frac{-x_j}{\tilde{c}_j(4 - x_j^2)} \quad (26)$$

Vieta-Lucas Differentiation Matrices

For each $N \geq 1$, modified Chebyshev (Vieta-Lucas Polynomial) collocation differentiation matrix is defined below

$$(D_N^1)_{00} = \left(\frac{1+2N^2}{6}\right) \quad (27.a)$$

$$(D_N^1)_{NN} = -\left(\frac{1+2N^2}{6}\right) \quad (27.b)$$

$$(D_N^1)_{kk} = \frac{-x_k}{2(4 - x_k^2)},\ 1 \leq k \leq N-1 \quad (27.c)$$

$$(D_N^1)_{kj} = \frac{(-1)^{j+k}\tilde{c}_k}{\tilde{c}_j(x_k - x_j)},\ 0 \leq j \neq k \leq N \quad (27.d)$$

where $\tilde{c}_0 = \tilde{c}_N = 4$ and $\tilde{c}_j = 2$.

Furthermore, for high order differential problem, modified Chebyshev (Vieta-Lucas Polynomial) differentiation matrices can be applied by define high order modified Chebyshev (Vieta-Lucas Polynomial) differentiation as power of matrix below,

$$D_N^m = (D_N^1)^m \quad (28)$$

Examples and Solutions

For comparison the performance of Vieta-Lucas Collocation scheme and FDM, some of example are given below.

Table 1. Example of Boundary and initial value of differential equations problems.

No	Example
1	$y^{(2)} + y = x^2 + x$ $y(-2) = y(2) = 0, y'(-2) = 2.285$
2	$y^{(2)} - y^{(-1)} = \sin(x)$ $y(-2) = y(2) = 0, y'(-2) = 0.67$
3	$u_t + u_x = 0, \quad x \in [-1,1]$ $u(x,0) = \sin(\pi x)$ $u(-1,t) = \sin(\pi(-1-t))$

NUMERICAL EXPERIMENT

Numerical result below is comparison belong to the Vieta-Lucas collocation, FDM and its exact solution.

Example of ODE no 1 can be solved by Vieta-Lucas Collocation scheme as equation $D_N^2 u_N + u_N = F$. Where $F = (X_0, X_1, \ldots, X_N)$, X is right hand side of the example of ODE no 1 and N=64. While for FDM, we perform the second order of FDM and N=400 to solve the example. The result of both scheme is compared with the exact solution as table and the picture below,

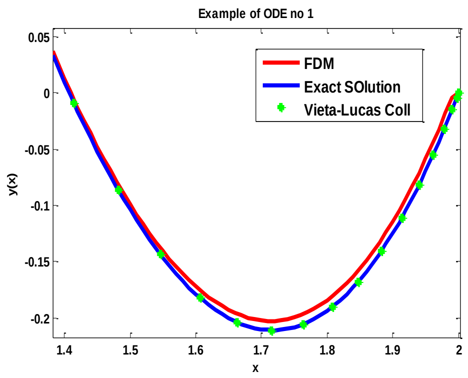

Figure 1. Numerical Solution of example 1

Table 2. Error of the example 1

Scheme	Error
Vieta-Lucas	2.3×10^{-6}
FDM	1.5×10^{-2}

Furthermore, for ODE example no 2, Vieta-Lucas Scheme can be write as $D_N^2 u_N + D_N u_N = F$ and N=64. Whereas FDM is performed with N=400.

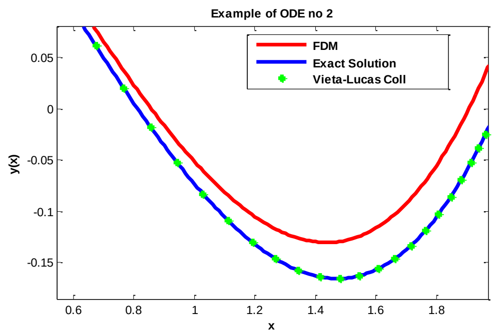

Figure 2. Numerical Solution of example 2

Table 3. Error of the example 2

Scheme	Error
Vieta-Lucas	3.9×10^{-5}
FDM	3.5×10^{-2}

In addition, the boundary and initial problem of linear PDE is solved by Vieta-Lucas Collocation for spatial part with N=64, then it is evaluate by forward Euler scheme for time differencing as equation

$$\mathbf{u}_N(t^{i+1}) = \mathbf{u}_N(t^i) + \Delta t\, D_N \mathbf{u}_N(t^i), \quad \mathbf{u}(t^0) = sin(\pi X_N).$$

Moreover, second order of FDM with N=400 is applied to solve it with time differencing as Vieta-Lucas Collocation.

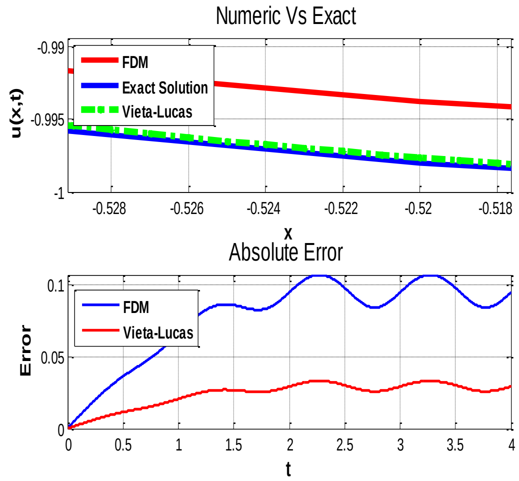

Figure 3. Numerical Solution of example 3

CONCLUSION

The problem of solving differential equations occurs frequently in mathematical work. In this paper, we have introduced modified Chebyshev collocation scheme called Vieta-Lucas collocation to for solving differential equation. Numerical Experiment shows that Vieta-Lucas Collocation scheme s more effective and accurate than FDM even with small number of N. Meanwhile, the scheme introduced in this paper can be used to more class of nonlinear differential equations

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