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Exploring the (h, m)-Convexity for Operators in Hilbert Space

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ABSTRACT

This study examines the concept of operator (h, m)-convexity within the context of Hilbert spaces, aiming to advance the understanding of operator convex functions. Operator convex functions play a pivotal role in various mathematical disciplines, particularly in optimization and the study of inequalities. The paper introduces the notion of an operator (h, m)-convex function, which generalizes existing classes of operator convexity, and explores its fundamental properties. The methodological framework relies on a theoretical analysis of bounded operators and their relationships with other forms of operator convex functions. Key findings demonstrate that, under certain conditions, the product of two operator convex functions retains operator convexity. Furthermore, the study establishes convergence results for matrix (h, m)-convex functions. These contributions enhance the theoretical foundation of operator convexity, offering a basis for future research and applications. The results not only deepen the understanding of operator (h, m)-convex functions but also support the development of sharper inequalities, thereby broadening the applicability of operator convexity within mathematical analysis.

Keywords: operator convexity; (h, m)-convexity; (h, m)-convex functions; matrix convexity; Hilbert space operators

INTRODUCTION

The concept of convexity is central to the study of applied mathematics [1]. Convexity was initially defined and used in the study of arc length by Archimedes. The definition of convexity proposed by Archimedes is similar to the current definition, which states that a set is called convex if the set contains all line segments formed by any two points in the set. It is written mathematically as in the Definition 1 below.

Definition 1 (see [2]). Let X be a non-empty set. $X \subseteq \mathbb{R}^n$ is called a convex set if for any $x, y \in X$ and $\alpha \in (0,1)$ holds

$$\alpha x + (1-\alpha)y \in X.$$

Similarly, a curve (function) is called as convex if the line segment connecting any two points on the function's graph lies above the curve[1]. We can write it mathematically as in the Definition 2 below.

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Definition 2 (see [2]). Let X be a convex set. A functions $\varphi: X \to \mathbb{R}$ is called a convex, if for any $x, y \in X$ and $\alpha \in (0,1)$ holds

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y). \quad (1)$$

If the inequality (1) is reversed, then φ is called a concave. In other words if φ is a convex, then $-\varphi$ is a concave, vice versa.

Various studies based on the definition and classical convexity theories have been developed, leading to various variations in the definitions and convexity theories that are more general than classical convexity. In 1984, Toader [3] introduced the definition of m-convex functions, in which it must be defined on a m-convex set. In this class, when $m = 0$, it is called starshaped functions [4]. In 2007, Varošanec [5] introduced a new class of convexity, which is h-convexity. He studied the h-convex function definition and their properties. Here, $h: J \to [0, \infty)$ is nonnegative and the interval $J \supseteq (0,1)$. This is also the generalization of some classes in convexity, such as Godunova-Levin functions (see [6]), s-convex functions (see [7], [8]), P-functions (see [9]), and s-Godunova-Levin functions (see [10]). Combining the concepts of h-convexity and m-convexity, Özdemir et al. [11] studied (h, m)-convex functions's definition and their properties. In this class of convex functions, the domain should be a m-convex set. The special class that generalized from this class is (s, m)-convex functions (see [12], [13]) and (s, m)-Godunova-Levin functions (see [14]).

Operator convex functions are important tools in functional analysis, matrix analysis, quantum information and so on (For instance, see [15], [16], [17], [18], [19]). These are real functions whose extensions to self-adjoint operator preserve order. The motivation for constructing the definition of operator convex function came from Theorem 1. First, we denote $\mathcal{H}$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, while $\mathcal{B}(\mathcal{H}) = \{B | B: \mathcal{H} \to \mathcal{H} \text{ is a bounded and linear operator}\}$ and $1_{\mathcal{H}}$ is the identity operator in Hilbert space $\mathcal{H}$.

Theorem 1 (see [20]). Let $\mathcal{B}(\mathcal{H}) = \{B | B: \mathcal{H} \to \mathcal{H} \text{ is a bounded and linear operator}\}$. Then, there exist a unique $*$-isometric isomorphism Gelfand map $\Phi: C(\sigma(B)) \to \mathcal{B}(\mathcal{H})$, such that is for any $\varphi_1, \varphi_2 \in C(\sigma(B))$ holds

(i) $\Phi(\varphi_1 + \varphi_2) = \Phi(\varphi_1) + \Phi(\varphi_2)$.
(ii) $\Phi(c\varphi_1) = c\Phi(\varphi_1)$.
(iii) $\Phi(\varphi_1\varphi_2) = \Phi(\varphi_1)\Phi(\varphi_2)$.
(iv) $\Phi(\overline{\varphi_1}) = \Phi(\varphi_1)^*$.
(v) If $\varphi_1(u) = u$ and $\varphi_1(u) = 1$, then $\Phi(\varphi_1) = B$ and $\Phi(\varphi_1) = 1_{\mathcal{H}}$ respectively.
(vi) $\varphi_1(x) \geq 0$, then $\Phi(\varphi_1)$ is positive operator.

For convenience, before we go further to the definition of operator convex functions, we give the notation of operator orders

$$A \preccurlyeq B \quad (2)$$

means

$$\langle Ax, x\rangle \leq \langle Bx, x\rangle, \quad (3)$$

for any $x \in \mathcal{H}$. Thus, changing the sign in (2) with "$\prec$", "$\succcurlyeq$", "$\succ$", and "$=$" means changing the sign in (3) with "$<$", "$\geq$", "$>$", and "$=$" respectively. We also denote $\mathcal{B}^+(\mathcal{H})$ is a family of bounded linear positive operator in Hilbert space and $\wp$ be a subspace from the family $\mathcal{B}^+(\mathcal{H})$.

By using that theorem Bhatia [15] gives the definition of the operator convex function as follows.

Definition 3 (see [15]). A function $\varphi: I \to \mathbb{R}$ is called to be operator convex function, if for any self-adjoint operator $A, B \in \mathcal{B}(\mathcal{H})$ with $\sigma(A), \sigma(B) \subseteq I$ and $\alpha \in (0,1)$ holds

$$\varphi(\alpha A + (1-\alpha)B) \preccurlyeq \alpha\varphi(A) + (1-\alpha)\varphi(B).$$

By applying the same approach used to construct the definition of operator convex functions, Salaş et al. [21] studied about the operator (h, m)-convex function's definition in Hilbert space using the concept of (h,m)-convex functions studied by Özdemir et al. [11]. By using Theorem 1 Salaş et al. [21] constructed the definition of the operator (h, m)-convex functions for operators in $\wp$ as written in the following Definition 4.

Definition 4 (see [21]). Let $m \in [0,1]$ and $h: J \supseteq (0,1) \to [0, \infty)$. A function $\varphi: [0, b] \to \mathbb{R}$ is called a operator (h, m)-convex function, if for any self-adjoint operator $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0, b]$ and $\alpha \in (0,1)$ holds

$$\varphi(\alpha A + m(1-\alpha)B) \preccurlyeq h(\alpha)\varphi(A) + mh(1-\alpha)\varphi(B).$$

From Definition 4 above, we can get some other classes in operator convexity in Hilbert space, if we take $h(\alpha) = \alpha$, then we get the definition of operator m-convex functions (see [18], [22], [23]). Meanwhile, if we take the value of m equals one, then we obtain the definition of operator h-convex functions (see [24]).

They also constructed the Hermite-Hadamard type inequalities for operator (h, m)-convex functions. However, the investigation of the fundamental properties of (h, m)-convex operator functions is essential for establishing a solid theoretical framework upon which further analysis and applications can be built. For instance, if we want to construct an extension of the Hermite-Hadamard type inequality—namely, the Hermite-Hadamard inequality for the product of two operator convex functions. When we study this topic, an important issue arises regarding whether the product of two operator convex functions remains operator convex, and under what conditions the resulting function can maintain operator convexity. This consideration motivates us to conduct research on the fundamental properties of (h, m)-operator convex functions.

Based on developments that have been presented, in this paper, we provide some fundamental properties for operator (h, m)-convex functions in Hilbert space. Using the fact that matrix is a bounded linear operator, we also provide the definition and the properties of the matrix (h, m)-convex functions as a special case, when the operator is a self-adjoint matrix. Then, we study about the relation between operator (h, m)-convex functions and matrix (h, m)-convex functions.

METHODS

Our methodology for composing this research is literature review. This research is composed by using a theoretical approach, that is reviewing, analyzing, and extending existing concepts of (h, m)-convex functions to operators in Hilbert spaces. Thus, our methods for composing this article can be written as follows:

(i) Conduct a comprehensive literature review related to (h, m)-convexity and operator convexity.
(ii) Using the definition of operator (h, m)-convex functions that have been studied by Salaş et al. [21] and the characteristic of the h function in Varošanec [5] to derive new fundamental properties.
(iii) Formulate the definition of matrix (h, m)-convex functions.
(iv) Investigate the relationship between operator (h, m)-convex functions and matrix (h, m)-convex functions.

All theoretical proofs are constructed based on operator theory, functional analysis, and properties of convex functions without the use of empirical or computational experiments.

RESULTS AND DISCUSSION

In this section, we discuss some basic properties in the operator version of (h,m)-convexity class, where the operator is bounded linear operator in Hilbert space. We also give the matrix (h,m)-convex function definition as a special case of this class of convexity.

Firstly, we present an example of the operator (h,m)-convex functions in the following Example 1.

Example 1. Let $h(\alpha) = 3\alpha$, $m \in (0,1]$, $\mathcal{g} = \{A|A \in \mathcal{B}^+(\mathcal{H})$ with $A = A^*$ and $\sigma(A) = \{\lambda \in \mathbb{R}^+|\lambda \leq 2\}\}$. A function φ, defined by

$$\varphi(x) = \begin{cases} \frac{1}{2}x, & 0 \leq x < 1, \\ \frac{3}{2}x, & 1 \leq x \leq 2, \end{cases}$$

is an operator (h,m)-convex functions for operator in $\mathcal{g}$.

We can prove Example 1 by using contradiction. So, if we take two operators A, B at least one exists $\alpha \in [0,1]$ such that the condition in the Definition 4 is not valid. Without loss generalization we choose $\sigma(A) \subseteq [0,1)$ and $\sigma(B) \subseteq [1,2]$. Yet, by using spectral theorem for every eigenvalue contained in $\{\lambda \in \mathbb{R}^+|\lambda \leq 1\}$, we obtain that the eigenvalues of whether $h(\alpha)\frac{1}{2}(A) + mh(1-\alpha)\frac{3}{2}(B) - \frac{3}{2}(\alpha A + m(1-\alpha)B) = 3m(1-\alpha)B$ or $h(\alpha)\frac{1}{2}(A) + mh(1-\alpha)\frac{3}{2}(B) - \frac{1}{2}(\alpha A + m(1-\alpha)B) = 3\alpha A$ are all positive. Hence, for any $\alpha \in [0,1]$ we have

$$h(\alpha)\varphi(A) + mh(1-\alpha)\varphi(B) - \varphi(\alpha A + m(1-\alpha)B) \succcurlyeq 0.$$

Now, we construct the linearity property of operator (h,m)-convex functions. This property is essential to develop some inequalities that are generated from operator (h,m)-convex functions. We present it as in the following Proposition 1 and Proposition 2.

Proposition 1. Let $[0,b]$ be an interval, $m \in [0,1]$, and $h: J \supseteq (0,1) \to [0,\infty)$. Suppose $\varphi_1, \varphi_2: [0,b] \to \mathbb{R}$ are operator (h,m)-convex functions. Then, $\varphi_1 + \varphi_2$ is also operator (h,m)-convex function on $[0,b]$.

Proof: Let $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0,b]$. Since φ_1, φ_2 are two operator (h,m)-convex functions, then for any $\alpha \in (0,1)$ we have

$$\begin{aligned}(\varphi_1 + \varphi_2)(\alpha A + m(1-\alpha)B) &= \varphi_1(\alpha A + m(1-\alpha)B) + \varphi_1(\alpha A + m(1-\alpha)B) \\ &\preccurlyeq h(\alpha)\varphi_1(A) + mh(1-\alpha)\varphi_1(B) + h(t)\varphi_2(A) \\ &\quad + mh(1-\alpha)\varphi_2(B) \\ &\preccurlyeq h(\alpha)(\varphi_1 + \varphi_2)(A) + mh(1-\alpha)(\varphi_1 + \varphi_2)(B).\end{aligned}$$

That means $\varphi_1 + \varphi_2$ is operator (h,m)-convex function on the interval [0,b].

■

Proposition 2. Let $m \in [0,1]$, $h: J \supseteq (0,1) \to [0,\infty)$, and $k \in (0,\infty)$. Suppose $\varphi: [0,b] \to \mathbb{R}$ be an operator (h,m)-convex function. Then, $k\varphi$ is also an operator (h,m)-convex function on $[0,b]$.

Proof: Let $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0,b]$. Since φ is an operator (h,m)-convex function, then for any $\alpha \in (0,1)$ we have

$$\begin{aligned}(k\varphi)(\alpha A + m(1-\alpha)B) &\preccurlyeq k[h(\alpha)\varphi(A) + mh(1-\alpha)\varphi(B)] \\ &\preccurlyeq h(\alpha)(k\varphi)(A) + mh(1-\alpha)(k\varphi)(B).\end{aligned}$$

That means $k\varphi$ is an operator (h,m)-convex function.

■

The following proposition discusses the properties of operator (h,m)-convex functions through the composition of two functions. However, a certain condition must be satisfied, which is related to the monotonicity of one of the functions.

Proposition 3. Let $[0,b], [0,c]$ be two intervals, $m \in [0,1]$, and $h: J \supseteq (0,1) \to [0,\infty)$. Suppose $\varphi_1: [0,b] \to \mathbb{R}$ be an operator (h,m)-convex function for operators in $\wp$ and monotone operator function and $\varphi_2: [0,c] \to \mathbb{R}$ be an operator m-convex function for operators in $\wp$, with $\varphi_2([0,c]) \subseteq [0,b]$. Then, the composition function $\varphi_1 \circ \varphi_2$ is also operator (h,m)-convex on $[0,c]$ for operators in $\wp$.

Proof: Let $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0,c]$. Since φ_2 is an operator m-convex function and φ_1 is an operator monotone function, then for any $\alpha \in (0,1)$ we have

$$(\varphi_1 \circ \varphi_2)(\alpha A + m(1-\alpha)B) \preccurlyeq \varphi_1\big(\alpha\varphi_2(A) + m(1-\alpha)\varphi_2(B)\big).$$

Indeed, since $\varphi_2([0,c]) \subseteq [0,b]$, also φ_1 is operator (h,m)-convex and monotone, we have

$$\begin{aligned}(\varphi_1 \circ \varphi_2)(\alpha A + m(1-\alpha)B) &\preccurlyeq h(\alpha)\varphi_1\big(\varphi_2(A)\big) + mh(1-\alpha)\varphi_1\big(\varphi_2(B)\big) \\ &= h(\alpha)\,(\varphi_1 \circ \varphi_1)(A) + m\,h(1-\alpha)(\varphi_1 \circ \varphi_2)(B).\end{aligned}$$

That means, $\varphi_1 \circ \varphi_2$ is an operator (h,m)-convex function on the interval $[0,c]$ for operators in $\wp$.

■

We consider two functions that map from an interval $J \supseteq (0,1)$ to the set of real numbers $\mathbb{R}$, namely h_1 and h_2. The next property is obtained by comparing those two functions.

Proposition 4. Let $m \in [0,1]$ and $h_1, h_2: J \supseteq (0,1) \to [0,\infty)$ with $h_1 \leq h_2$. If $\varphi: [0,b] \to \mathbb{R}$ is an operator (h_1,m)-convex function for operators in $\wp$, then φ is also an operator (h_2,m)-convex function for operators in $\wp$.

Proof: Let $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0,b]$. Since φ is an operator (h_1,m)-convex function and $h_1 \leq h_2$, then for any $\alpha \in (0,1)$ we have

$$\begin{aligned}\varphi(\alpha A + m(1-\alpha)B) &\preccurlyeq h_1(\alpha)\varphi(A) + mh_1(1-\alpha)\varphi(B) \\ &\preccurlyeq h_2(\alpha)\varphi(A) + mh_2(1-\alpha)\varphi(B).\end{aligned}$$

Hence, φ is also an operator (h_2,m)-convex function.

■

Similar to the previous property, by considering two scalars $m_1, m_2 \in [0,1]$, we can obtain the following property.

Proposition 5. Let $h: J \supseteq (0,1) \to [0, \infty)$ and $m_1, m_2 \in [0,1]$ with $m_1 \leq m_2$. If $\varphi: [0, b] \to \mathbb{R}$ is an operator (h, m_1)-convex function for operators in $\wp$ and monotone decreasing, then φ is also an operator (h, m_2)-convex function for operators in $\wp$.

Proof: Let $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0, b]$. Since φ is an operator (h, m_1)-convex function and $m_1 \leq m_2$, then for any $\alpha \in (0,1)$ we have

$$\begin{aligned} \varphi(\alpha A + m_2(1-\alpha)B) &\preccurlyeq h(\alpha)\varphi(A) + m_1 h(1-\alpha)\varphi(B) \\ &\preccurlyeq h(\alpha)\varphi(A) + m_2 h(1-\alpha)\varphi(B). \end{aligned}$$

Hence, φ is also an operator (h, m_2)-convex function.

■

Now, we move to the multiplicative property of two (h, m)-convex operator functions that are presented, which support the development of the Hermite-Hadamard type inequalities for the product of two (h, m)-convex operator functions. In the following proposition, we need to assume that $\wp$ is also a family of commutative and comparable operators.

Proposition 6. Let $h: J \supseteq (0,1) \to [0, \infty)$, and $m \in [0,1]$ such that $h(\alpha) + mh(1-\alpha) \leq 1$. Also suppose $A, B \in \wp$ be commutative and comparable (i.e. $AB = BA$ and $A \preccurlyeq B$ or $A \succcurlyeq B$) with $\sigma(A), \sigma(B) \subseteq [0, b]$. If both $\varphi_1, \varphi_2: [0, b] \to [0, \infty)$ are operator (h, m)-convex and operator monotone functions, then $\varphi_1\varphi_2$ is also operator (h, m)-convex functions on the interval $[0, b]$.

Proof: By the assumptions $A, B \in \wp$ be commutative and comparable with $\sigma(A), \sigma(B) \subseteq [0, b]$. Without loss the generalization, we take $A \preccurlyeq B$. According to [10], it holds that $\varphi_1(A)\varphi_2(B) = \varphi_2(B)\varphi_1(A)$. Since φ_1, φ_2 are operator monotone functions, then we have $\varphi_1(B) - \varphi_1(A) \succcurlyeq 0$ and $\varphi_2(B) - \varphi_2(A) \succcurlyeq 0$. So, by multiplication we have

$$\big(\varphi_1(B) - \varphi_1(A)\big)\big(\varphi_2(B) - \varphi_2(A)\big) \succcurlyeq 0$$

or

$$\varphi_1(B)\varphi_2(B) + \varphi_1(A)\varphi_2(A) \succcurlyeq \varphi_1(A)\varphi_2(B) + \varphi_1(B)\varphi_2(A)\,. \tag{5}$$

Next, because φ_1, φ_2 are operator (h, m)-convex functions, then we get

$$\begin{aligned} \varphi_1\varphi_2(\alpha A + m(1-\alpha)B) &= \varphi_1(\alpha A + m(1-\alpha)B) \cdot \varphi_2(\alpha A + m(1-\alpha)B) \\ &\preccurlyeq [h(\alpha)\varphi_1(A) + mh(1-\alpha)\varphi_1(B)] \\ &\quad \times [h(\alpha)\varphi_2(A) + mh(1-\alpha)\varphi_2(B)] \\ &= h^2(\alpha)\varphi_1(A)\varphi_2(A) + m^2h^2(1-\alpha)\varphi_1(B)\varphi_2(B) \\ &\quad + mh(\alpha)h(1-\alpha)[\varphi_1(A)\varphi_2(B) + \varphi_1(B)\varphi_2(A)]. \end{aligned}$$

Based on (5), we thus have

$$\begin{aligned} \varphi_1\varphi_2(\alpha A + m(1-\alpha)B) &\preccurlyeq h^2(\alpha)\varphi_1(A)\varphi_2(A) + m^2h^2(1-\alpha)\varphi_1(B)\varphi_2(B) \\ &\quad + mh(\alpha)h(1-\alpha)[\varphi_1(A)\varphi_2(A) + \varphi_1(B)\varphi_2(B)]. \\ &= h(\alpha)[h(\alpha) + mh(1-\alpha)]\varphi_1(A)\varphi_2(A) \\ &\quad + mh(1-\alpha)[h(\alpha) + mh(1-\alpha)]\varphi_1(B)\varphi_2(B) \end{aligned}$$

By the assumption that $h(\alpha) + mh(1-\alpha) \leq 1$. Hence, we obtain

$$\varphi_1\varphi_2(\alpha A + m(1-\alpha)B) \preccurlyeq h(\alpha)\varphi_1(A)\varphi_2(A) + mh(1-\alpha)\varphi_1(B)\varphi_2(B).$$

That means $\varphi_1\varphi_2$ is an operator (h, m)-convex functions on the interval $[0, b]$ for operators in $\wp$.

■

The characteristic of h in reference [5] has several consequences to the properties of the operator (h, m)-convex functions. In the following Theorem 2 and Theorem 3, we obtain the properties of the operator (h,m)-convex functions based on the characteristic of the h functions.

Theorem 2. Let $m \in [0,1]$ and $h: J \supseteq (0,1) \to [0,\infty)$. The following theorems are held:

(a) Let $\varphi: [0,b] \to \mathbb{R}$ be operator (h,m)-convex function and $\varphi(0) = 0$. If h is a super-multiplicative function, then holds
$$\varphi(\alpha A + m\beta B) \preccurlyeq h(\alpha)\varphi(A) + mh(\beta)\varphi(B),$$
where $A, B \in \wp$ such that $\sigma(A), \sigma(B) \subseteq [0,b]$ and $\alpha, \beta \in [0,\infty)$ with $\alpha + \beta \leq 1$.

(b) Let $\varphi: [0,b] \to \mathbb{R}$ such that $-\varphi$ be operator (h,m)-convex function and $\varphi(0) = 0$. If h is a sub-multiplicative function, then holds
$$\varphi(\alpha A + m\beta B) \succcurlyeq h(\alpha)\varphi(A) + mh(\beta)\varphi(B),$$
where $A, B \in \wp$ such that $\sigma(A), \sigma(B) \subseteq [0,b]$ and $\alpha, \beta \in [0,\infty)$ with $\alpha + \beta \leq 1$.

Proof: Let $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0,b]$. We separate our proof for each poin (a) and (b) as follows.

(a) Let α, β be positive numbers such that $\alpha + \beta = t \leq 1$ or we write $\frac{\alpha}{t} + \frac{\beta}{t} = 1$. Also let us define $\lambda_1 = \frac{\alpha}{t}$ and $\lambda_2 = \frac{\beta}{t}$. Since φ be operator (h,m)-convex function on I and $\varphi(0) = 0$, then have
$$\begin{aligned}\varphi(\alpha A + m\beta B) &= \varphi(\lambda_1 tA + m\lambda_2 tB) \preccurlyeq h(\lambda_1)\varphi(tA) + mh(\lambda_2)\varphi(tB)\\ &= h(\lambda_1)\varphi(tA + m(1-t)0) + mh(\lambda_2)\varphi(tB + m(1-t)0)\\ &\preccurlyeq h(\lambda_1)[h(t)\varphi(A) + mh(1-t)\varphi(0)]\\ &\quad + mh(\lambda_2)[h(t)\varphi(B) + mh(1-t)\varphi(0)]\\ &= h(\lambda_1)h(t)\varphi(A) + mh(\lambda_2)h(t)\varphi(B).\end{aligned}$$
Since h is a super-multiplicative function, then we obtain
$$\begin{aligned}\varphi(\alpha A + m\beta B) &\preccurlyeq h(\lambda_1 t)\varphi(A) + mh(\lambda_2 t)\varphi(B)\\ &= h(\alpha)\varphi(A) + mh(\beta)\varphi(B).\end{aligned}$$
The proof for case (b) in this theorem is similar to the proof for case (a).

■

Theorem 3. Let $m \in [0,1]$, $h_k: J_k \supseteq (0,1) \to [0,\infty)$ with $k = 1,2$, and $h_2(J_2) \subseteq J_1$ such that there exists $h_2(t) + h_2(1-t) \leq 1$ for every $t \in (0,1)$. Suppose $\varphi_1: [0,b] \to \mathbb{R}$ and $\varphi_2: [0,c] \to \mathbb{R}$ with $\varphi_2([0,c]) \subseteq [0,b]$ and $\varphi_1(0) = 0$.

(a) If the function h_1 is a super-multiplicative, φ_1 is an operator (h_1,m)-convex function and monotone on the interval $[0,b]$, and φ_1 is an operator (h_2,m)-convex function, then $\varphi_1 \circ \varphi_2$ is an operator (h_3,m)-convex function on the interval $[0,c]$.

(b) If the function h_1 is a sub-multiplicative, $-\varphi_1$ is an operator (h_1,m)-convex function and monotone, and $-\varphi_2$ is an operator (h_2,m)-convex function, then $-\varphi_1 \circ (-\varphi_2)$ is an operator (h_3,m)-convex function on the interval $[0,c]$.

Here $h_3 = h_1 \circ h_2$.

Proof: Let $A, B \in \wp$ with $\sigma(A), \sigma(B) \subseteq [0,c]$. We separate our proof for each case (a) and (b) as follows.

(a) Since φ_2 is an operator (h_2,m)-convex function on $[0,c]$ and φ_1 is a monotone operator, then for any $\alpha \in (0,1)$ we have
$$(\varphi_1 \circ \varphi_2)(\alpha A + m(1-\alpha)B) \preccurlyeq \varphi_1\big(h_2(\alpha)\varphi_2(A) + mh_2(1-\alpha)\varphi_2(B)\big).$$
Next, using Theorem 2 (a) and due to the assumption which says $\varphi_2([0,c]) \subseteq [0,b]$, φ_1 is operator (h_1,m)-convex function on $[0,b]$, and $h_2(J_2) \subseteq J_1$. If we choose $h_2(\alpha) + h_2(1-\alpha) \leq 1$, then for every $\alpha \in (0,1)$ we obtain

$$(\varphi_1 \circ \varphi_2)(\alpha A + m(1-\alpha)B) \preccurlyeq h_1\big(h_2(\alpha)\big)\varphi_1\big(\varphi_2(A)\big) + mh_1(h_2\alpha)\varphi_1\big(\varphi_2(B)\big)$$
$$= h_3(\alpha)\varphi_1\big(\varphi_2(A)\big) + mh_3(1-\alpha)\varphi_1\big(\varphi_2(B)\big).$$

The proof for case (b) in this theorem is similar to the proof for case (a).■

Now, we introduce the definition of matrix (h,m)-convex functions, which is a special case in our discussions. Let us denote $\mathcal{M}_n(\mathcal{H})$ as a family of matrices with n order. Thus, if $M \in \mathcal{M}_n(\mathcal{H})$, then $A:\mathcal{H}_n \to \mathcal{H}_n$, in which $\mathcal{H}_n \subseteq \mathcal{H}$ is a finite dimensional space. For more detailed information about matrix convex functions, it can be seen in [15], [22], and [25]. In [15, 22], it stated that if $M \in \mathcal{M}_n(\mathbb{C})$ is a self-adjoint matrix, then a unitary operator $U \in \mathcal{M}_n(\mathbb{C})$ and a diagonal matrix $D = \text{diag}[\lambda_1, \dots, \lambda_n] \in \mathcal{M}_n(\mathbb{C})$, with the eigenvalues $\lambda_i \in \sigma(A)(i = 1, \dots, n)$ exists such that $M = UDU^*$ and $\varphi(M) = U\varphi(D)U^*$. Here, $\varphi(D) \in \mathcal{M}_n(\mathbb{C})$. So, it can be written as $\varphi(D) = \text{diag}[\varphi(\lambda_1), \dots, \varphi(\lambda_n)]$. From the Definition 4, then we obtain the definition of matrix (h,m)-convex functions as in the following Definition 5.

Definition 5. Let $m \in [0,1], n \in \mathbb{N}$, and $h: J \supseteq (0,1) \to [0,\infty)$. A function $\varphi: [0,b] \to \mathbb{R}$ is called to be matrix (h,m)-convex function if for every self-adjoint matrices $M_1, M_2 \in \mathcal{M}_n(\mathcal{H})$ and $\alpha \in (0,1)$ holds

$$\varphi(\alpha M_1 + m(1-\alpha)M_2) \preccurlyeq h(\alpha)\varphi(M_1) + mh(1-\alpha)\varphi(M_2).$$

Note that, the matrix (h,m)-convex functions also have properties of operator (h,m)-convex functions, since they are a special case. In the following Proposition 7 and Proposition 8, we present two special properties of the matrix (h,m)-convex functions.

Proposition 7. Let $m \in [0,1]$ and $h: J \to [0,\infty)$. Suppose $\varphi: [0,b] \to \mathbb{R}$ be a continuous function on the interval $[0,b]$. φ is an operator (h,m)-convex function if only if φ is a matrix (h,m)-convex function.

Proof: $(\Rightarrow)$ Let $M_1, M_2 \in \mathcal{M}_n(\mathcal{H})$ be self-adjoint matrices and $\mathcal{H}_n \subseteq \mathcal{H}$ be a finite dimensional space. Also let $P_n \in \mathcal{M}_n(\mathcal{H})$ be the projection matrix on $\mathcal{H}_n$. Define $\widehat{M_1} \coloneqq M_1P_n: \mathcal{H} \to \mathcal{H}_n$ and $\widehat{M_2} \coloneqq M_2P_n: \mathcal{H} \to \mathcal{H}_n$. Since M_1, M_2 are two self-adjoint matrices, then for every $x, y \in \mathcal{H}$, we can obtain

$$\langle \widehat{M_1}x, y\rangle = \langle M_1P_nx, y\rangle = \langle M_1x, y\rangle = \langle x, M_1y\rangle = \langle x, M_1P_ny\rangle = \langle x, \widehat{M_1}y\rangle.$$

So, if M_1, M_2 are self-adjoint matrices, then $\widehat{M_1}, \widehat{M_2}$ also are self-adjoint matrices. It can be observed that $\varphi(M_1) = \varphi(\widehat{M_1})\big|_{\mathcal{H}_n}$. Thus, we have

$$\varphi(\alpha M_1 + m(1-\alpha)M_2) = \varphi\big(\alpha\widehat{M_1} + m(1-\alpha)\widehat{M_2}\big) \preccurlyeq h(\alpha)\varphi\big(\widehat{M_1}\big) + mh(1-\alpha)\varphi\big(\widehat{M_2}\big)$$
$$= h(\alpha)\varphi(M_1) + mh(1-\alpha)\varphi(M_2).$$

$(\Leftarrow)$ By the assumption φ is a continuous function on $[0,b]$, according to the Weierstrass approximation theorem (see [26, Theorem B]), then a sequence of polynomials exists, namely $\{p_n\}_{n\in\mathbb{N}}$, such that $p_n \to \varphi$ on $[0,b]$. Let us consider a subspace $\mathcal{H}_{k_n} \subseteq \mathcal{H}_n$, in which it is generated by the vectors $1_{\mathcal{H}}x, M_1x, {M_1}^2x, \dots, {M_1}^{k_n}x, M_2x, {M_2}^2x, \dots, {M_2}^{k_n}x$. Also let $P_{k_n} \in \mathcal{M}_{k_n}(\mathcal{H})$ be the projection matrix on $\mathcal{H}_{n_k}$. Thus, we can define $M_{1_n} \coloneqq M_1P_{k_n}: \mathcal{H}_n \to \mathcal{H}_{k_n}$ and $M_{2_n} \coloneqq M_2P_{k_n}: \mathcal{H}_n \to \mathcal{H}_{k_n}$. Since

$$\langle M_{1_n}x, y\rangle = \langle M_1P_{k_n}x, y\rangle = \langle M_1x, y\rangle = \langle x, M_1y\rangle = \langle x, M_1P_{k_n}y\rangle = \langle x, M_{1_n}y\rangle,$$

then M_{1_n} and M_{2_n} are self-adjoint matrices. This means $p_n\big(M_{1_n}\big)x = p_n(M_1)x$ or

$$p_n\big(M_{1_n}\big)x - p_n(M_1)x = 0.$$

Because φ is a continuous function on $[0,b]$, thus for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that for $n > N$ holds

$$\|[\varphi(M_1) - p_n(M_1)]x\| < \frac{\varepsilon}{2}.$$

Hence, we can calculate

$$\begin{aligned}\|[\varphi(M_{1_n}) - \varphi(M_1)]x\| &= \|[\varphi(M_{1_n}) - p_n(M_{1_n}) + p_n(M_{1_n}) - p_n(M_1) + p_n(M_1) - \varphi(M_1)]x\| \\ &\leq \|[\varphi(M_{1_n}) - p_n(M_{1_n})]x\| + \|[p_n(M_{1_n}) - p_n(M_1)]x\| \\ &\quad + \|[p_n(M_1) - \varphi(M_1)]x\| \\ &< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This means, $\varphi(M_{1_n})x$ converges to $\varphi(M_1)x$. Using the same way, we can obtain that $\varphi(M_{2_n})x$ converges to $\varphi(M_2)x$ and $\varphi(\alpha M_{1_n} + m(1-\alpha)M_{1_n})x$ converges to $\varphi(\alpha M_1 + m(1-\alpha)M_2)x$. Next, since φ is matrix (h,m)-convex functions, then we have

$$\varphi(\alpha M_{1_n} + m(1-\alpha)M_{2_n}) \preccurlyeq h(\alpha)\varphi(M_{1_n}) + mh(1-\alpha)\varphi(M_{2_n})$$

or

$$h(\alpha)\varphi(M_{1_n}) + mh(1-\alpha)\varphi(M_{2_n}) - \varphi(\alpha M_{1_n} + m(1-\alpha)M_{2_n}) \succcurlyeq 0.$$

It means

$$\langle [h(\alpha)\varphi(M_{1_n}) + mh(1-\alpha)\varphi(M_{2_n}) - \varphi(\alpha M_{1_n} + m(1-\alpha)M_{2_n})]x, x\rangle \geq 0.$$

Since n tends to infinity, then we have

$$\lim_{n\to\infty} \langle [h(\alpha)\varphi(M_{1_n}) + mh(1-\alpha)\varphi(M_{2_n}) - \varphi(\alpha M_{1_n} + m(1-\alpha)M_{2_n})]x, x\rangle \geq 0.$$

Hence, we get

$$\langle [h(\alpha)\varphi(M_1) + mh(1-\alpha)\varphi(M_1) - \varphi(\alpha M_1 + m(1-\alpha)M_2)]x, x\rangle \geq 0.$$

That can be expressed as

$$\varphi(\alpha M_1 + m(1-\alpha)M_2) \preccurlyeq h(\alpha)\varphi(M_1) + mh(1-\alpha)\varphi(M_2).$$

This is the end of our proof.

■

The contrapositive of the statement in Proposition 7 is: φ is not an operator (h,m)-convex function if and only if φ is not a matrix (h,m)-convex function. This is particularly interesting because we can prove that a function is not an operator (h,m)-convex function by taking two self-adjoint matrices and showing that they do not satisfy the condition in Definition 5. For instance, let us consider a function $g: [0,10] \to \mathbb{R}$ by the rule $g(u) = u^3$, for any $u \in [0,10]$. For $h(\alpha) = \alpha$ and $m = \frac{10}{11}$, g is not an operator (h,m)-convex function. The proof is quite simple, we just need take two arbitary self-adjoint matrices $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We can see the spectrum of each matrix $\sigma(M_1) = \{0, 1\}$ and $\sigma(M_2) = \{0.3820, 2.6180\}$ are contained in the interval $[0,10]$. However, if we choose $\alpha = \frac{5}{11}$, then we obtain

$$\alpha {M_1}^3 + \frac{10}{11}(1-\alpha){M_2}^3 - \left(\alpha M_1 + \frac{10}{11}(1-\alpha)M_2\right)^3 = \frac{1}{1771561}\begin{pmatrix} 5389860 & 4128180 \\ 4128180 & 3114300 \end{pmatrix} =: M_h.$$

Since M_h has one negative eigenvalue, then we can conclude that M_h is not positive definite matrix. Hence, according to Proposition 7, the function g is not an operator (h,m)-convex function for $h(\alpha) = \alpha$ and $m = \frac{10}{11}$.

The next proposition show the convergence of the sequence of matrix (h,m)-convex functions. However, by using Proposition 7, this property also holds for the operator (h,m)-convex functions as written in Corollary 1 below.

Proposition 8. Let $m \in [0,1]$, $n \in \mathbb{N}$, and $h: J \supseteq (0,1) \to [0,\infty)$. Also let the sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\varphi_n: [0,b] \to \mathbb{R}$ be a sequence of matrix (h,m)-convex functions. If φ_n converges to φ, then φ is also a matrix (h,m)-convex function on the interval $[0,b]$.

Proof: Let $M_1, M_2 \in \mathcal{M}_n(\mathcal{H})$ be self-adjoint matrices with $\sigma(M_1), \sigma(M_2) \subseteq [0,b]$. Thus, we can write $M_1 = UDU^*$, where U is a unitary matrix and diagonal matrix $D = \text{diag}[\lambda_1, \dots, \lambda_n]$ with the eigenvalues $\lambda_i (i = 1, \dots, n) \in \sigma(M_1)$. Since φ_n converges to φ, then we have

$$\varphi_n(M_1) = \varphi_n(UDU^*) = U\varphi_n(D)U^* = U\varphi(D)U^* = \varphi(UDU^*) = \varphi(M_1).$$

That means $\varphi_n(M_1) \to \varphi(M_1)$. Using the same way for matrix operator M_2, we obtain that $\varphi_n(M_2)$ converges to $\varphi(M_2)$.

Next, because $\{\varphi_n\}_{n\in\mathbb{N}}$ are sequence of (h,m)-convex function, then it holds

$$\varphi_n(\alpha M_1 + m(1-t)M_2) \preccurlyeq h(\alpha)\varphi_n(M_1) + mh(1-\alpha)\varphi_n(M_2)$$

or

$$h(\alpha)\varphi_n(M_1) + mh(1-\alpha)\varphi_n(M_2) - \varphi_n(\alpha M_1 + m(1-\alpha)M_2) \succcurlyeq 0$$

It means

$$\langle [h(\alpha)\varphi_n(M_1) + mh(1-\alpha)\varphi_n(M_2) - \varphi_n(\alpha M_1 + m(1-\alpha)M_2)]x, x\rangle \geq 0.$$

Since n tends to infinity, then we have

$$\lim_{n\to\infty} \langle [h(\alpha)\varphi_n(M_1) + mh(1-\alpha)\varphi_n(M_2) - \varphi_n(\alpha M_1 + m(1-\alpha)M_2)]x, x\rangle \geq 0.$$

Hence, we obtain

$$\langle [h(\alpha)\varphi(M_1) + mh(1-\alpha)\varphi(M_2) - \varphi(\alpha M_1 + m(1-\alpha)M_2)]x, x\rangle \geq 0$$

or it can be expressed as

$$\varphi(\alpha M_1 + m(1-\alpha)M_2) \preccurlyeq h(\alpha)\varphi(M_1) + mh(1-\alpha)\varphi(M_2).$$

This is the end of our proof.

■

Corollary 1. Let $m \in [0,1]$ and $h: J \to [0,\infty)$. If $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\varphi_n: [0,b] \to \mathbb{R}$ are a sequence of operator (h,m)-convex functions such that $\varphi_n \to \varphi$, then φ is an of operator (h,m)-convex functions on $[0,b]$.

The results obtained above provide a theoretical framework for further exploration in operator convexity. These properties have potential applications in areas such as functional analysis, matrix analysis, and quantum information theory. Moreover, by extending the operator convexity notions, one can derive sharper operator inequalities, which are valuable for optimization problems and mathematical physics.

CONCLUSIONS

In this study, we developed a foundational framework for operator (h,m)-convex functions, that is establishing several fundamental properties and extending our results for matrix case. We obtain the relation property between operator (h,m)-convex functions and matrix (h,m)-convex functions. This study has the potential to open new avenues for research in inequality theory and operator analysis.

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