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Explicit Determinant and Inverse Formulas of Skew Circulant Matrices with Alternating Fibonacci Numbers

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Abstract

Skew circulant matrices have various applications such as cryptography, signal processing, and many more. Their structure can potentially simplify their determinant and inverse computations. This study presents explicit formulas for the determinant and inverse of skew circulant matrices with entries from the alternating Fibonacci sequence. Elementary row and column operations are used to derive simple explicit formulas for the determinant and inverse. Computational tests using Wolfram Mathematica show that the algorithm built from these explicit formulas performs with much faster execution time than the built-in functions, especially for large matrix size. The proposed approach offers a practical method for the numerical computation of the determinant and inverse of these matrices.

Keywords: alternating Fibonacci matrix; explicit matrix determinant; explicit matrix inverse; efficient algorithm

1 Introduction

Circulant and skew circulant matrices have many applications in several fields, such as cryptography, differential equations, network engineering, signal processing, and many more. One of the applications is a new and fast algorithm for optimal design of block digital filters [1]. In another field, circulant and skew circulant matrices are used to solve the continuous Sylvester equations [2] and the fractional diffusion-advection equations [3]. In the field of cryptography, modified public key cryptosystem is built based on circulant matrices [4]. In the field of signal processing, compressive signal processing is built using circulant sensing matrices [5]. The determinant and inverse of these matrices can be utilized in their applications. However, the calculation of their determinant and inverse is inefficient due to the very slow computation time for large matrices. The characteristics of circulant matrices are square matrices whose entries in each row are identical to the entries in the previous row shifted one position to the right in the next row, while skew circulant matrices are circulant matrices whose entries below the main diagonal are negative of the entries below the main diagonal of the circulant matrices. Based on its characteristics, the determinant and inverse can be formulated explicitly and efficiently if the entries also have a simple pattern.

Let $(c_0, c_1, \ldots, c_{n-1})$ be a sequence of real numbers. A skew circulant matrix of size n×n is defined as [6]

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$$\mathrm{SCirc}\,(c_0, c_1, \dots, c_{n-1}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & c_{n-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

If $C = \mathrm{SCirc}\,(c_0, c_1, \dots, c_{n-1})$, λ_k are the eigenvalues and $\mathbf{v}_k$ are the corresponding eigenvectors of C for $k = 0, 1, \dots, n-1$, then λ_k and $\mathbf{v}_k$ are well-known formulated as [7]

$$\lambda_k = \sum_{j=0}^{n-1} c_j (\psi\omega^k)^j \text{ and } \mathbf{v}_k = \left(1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k}\right), \tag{1}$$

where $\omega = e^{\frac{2\pi}{n}i} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ and $i = \sqrt{-1} \in \mathbb{C}$. From Eq. 1, it is clear that

$$\det(C) = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} c_j (\psi\omega^k)^j \text{ and } C^{-1} = \mathrm{SCirc}\,(b_0, b_1, b_2, \dots, b_{n-1}), \tag{2}$$

where $b_j = \frac{1}{n}\sum_{k=0}^{n-1} \mu_k (\psi\omega^k)^{-j}$ for $j = 0, 1, 2, \dots, n-1$, and $\mu_k = \begin{cases} 0, & \lambda_k = 0 \\ \frac{1}{\lambda_k}, & \lambda_k \neq 0 \end{cases}$.

Based on all the formulas above, it can be seen that the calculation of the formula is not efficient to apply if the matrix size or n is very large. However, if the sequence $(c_0, c_1, \dots, c_{n-1})$ has a simple pattern, then the formulas can be simplified so that better explicit formulas are obtained. This is expected to make the computation time for calculating the determinant and inverse of the matrices much faster.

Some researchers have created explicit formulas for determining the determinant and inverse of circulant and skew circulant matrices with entries of various numbers, such as the determinant and inverse of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers [8], complex Fibonacci numbers [9], Fermat and Mersenne numbers [10], VanderLaan numbers [11], ratio of the elements of Fibonacci and geometric numbers [12], generalized Tribonacci numbers [13], Gaussian Pell numbers [14], Gaussian nickel Fibonacci numbers [15], arithmetic numbers [16], alternating Fibonacci numbers [17], geometric numbers [18], Lucas numbers [19], and the determinant of circulant matrices with generalized Tetranacci numbers [20].

The determinant and inverse of skew circulant matrices have also been formulated for any continuous Fibonacci numbers [21], generalized Lucas numbers [22], Tribonacci numbers [23], geometric numbers [24], sum of Fibonacci and Lucas numbers [25], product of Fibonacci and Lucas numbers [26], Pell-Lucas numbers [27], product of Pell and Pell-Lucas numbers [28], and Fibonacci numbers [29].

In this article, we study the explicit formulations of the determinant and inverse of skew circulant matrices with alternating Fibonacci sequence entries through a series of elementary row and column operations. This sequence has never been used as the entries of skew circulant matrices in previous studies to formulate the determinant and inverse explicitly. The explicit formulas of the determinant and inverse of the skew circulant matrices obtained later are much simpler than those for the general case without calculating the eigenvalues and can be shown by the computation time of the algorithm built based on these formulas. An alternating Fibonacci sequence $\mathcal{F} = (a_j)_{j=0}^{\infty}$ is defined recursively by the second-order linear homogeneous recurrence relation with constant coefficients as [17]

$$a_j = -a_{j-1} + a_{j-2} \text{ for all } j \geq 2, \text{ where } a_0 = 0, a_1 = 1.$$

This sequence has a solution that can be written in the following explicit formula.

$$a_j = \frac{\alpha^j - \beta^j}{\alpha - \beta},$$

where $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$. If $(f_n)_{n=0}^{\infty}$ is a Fibonacci sequence and $(a_n)_{n=0}^{\infty}$ is an alternating Fibonacci sequence. A skew circulant matrix $\mathcal{C}_n(\mathcal{F})$ of size $n \times n$ with alternating Fibonacci entries $\mathcal{F}$ is defined as follows.

Definition 1. *For all integers $n \geq 2$, a skew circulant matrix of size $n \times n$ with entries in the first row are $\mathcal{F} = (a_j)_{j=1}^{n}$, is the matrix*

$$\mathcal{C}_n(\mathcal{F}) = \mathrm{SCirc}\,(a_1, a_2, a_3, \ldots, a_n)$$

2 Methods

In Subsection 3.1, we give a theorem that the matrix $\mathcal{C}_n(\mathcal{F})$ is invertible and its proof is mainly based on the formula of eigenvalues for a general circulant matrix and the well-known theorem that a matrix is invertible if and only if all its eigenvalues are nonzero. Then, we derive explicit formulas of the determinant and inverse of the matrix $\mathcal{C}_n(\mathcal{F})$ in Theorem 4. The proof of this theorem is based on elementary row and column operations on a matrix and their relation to the determinant of a matrix. Elementary row operations on a matrix are described as [30]

- the ith row is interchanged with the jth row, denoted as $E_{ij},\ i \neq j$;
- the ith row is multiplied by a constant $k \neq 0$, denoted as $E_{i(k)}$; and
- the ith row is added to k times the jth row, denoted as $E_{ij(k)},\ i \neq j$.

Elementary column operations are described the same as elementary row operations, but the word “row” is replaced with “column” and denoted as K_{ij}, $K_{i(l)}$, and $K_{ij(l)}$ [30]. If elementary row and column operations are applied to a matrix X to obtain a matrix Y, then Y can be expressed as in the following theorem.

Theorem 1 ([30])**.** *Let X be a matrix, and I be the identity matrix of the same size. If Y is a matrix obtained by applying the series of elementary row and column operations $R_1, R_2, \ldots, R_n$ and $C_1, C_2, \ldots, C_n$ on X, then there exist non-singular matrix P and Q such that $Y = PXQ$, where $P = R_n R_{n-1} \ldots R_1(I)$ and $P = C_n C_{n-1} \ldots C_1(I)$.*

The relationship between the determinant of a matrix and the elementary row and column operations can be stated in the following theorem.

Theorem 2 ([30])**.** *Let X be an $n \times n$ matrix and k be a constant. Then, the following statements hold:*

- $\det\left(E_{ij}(X)\right) = -\det(X),\ i \neq j$;
- $\det\left(E_{i(k)}(X)\right) = k\det(X),\ k \neq 0$; *and*
- $\det\left(E_{ij(k)}(X)\right) = \det(X),\ i \neq j$.

These statements are analogous to elementary column operations.

In Subsection 3.2, we show how to apply the formulas in Theorem 4 and construct an algorithm for the formula. We also compare the computation time of the algorithm and the built-in functions from Wolfram Mathematica on the same computer specifications for various values of n. If the overall computing time of one tends to be faster than the other, then it is more efficient.

3 Results and Discussion

In this section, we present the main objectives of this study, including the explicit determinant and inverse formulas for skew circulant matrices with alternating Fibonacci entries. The results are derived using elementary row and column operations, and illustrative example of their use is presented. The computational performance of the formulas is also analyzed.

3.1 The Determinant and Inverse Formulations

This section begins with the invertibility theorem of matrices defined in Definition 1. The proof of the theorem follows using the eigenvalues formula for general skew circulant matrices.

Theorem 3. *Let $\mathcal{C}_n(\mathcal{F}) = \mathrm{SCirc}\,(a_1, a_2, a_3, \dots, a_n)$ be the matrix defined in Definition 1 for all integers $n \geq 2$. Then, $\mathcal{C}_n(\mathcal{F})$ is invertible.*

Proof. According to Eq. 1, the eigenvalues of $\mathcal{C}_n(\mathcal{F})$ are

$$\lambda_k = \sum_{j=1}^{n} a_j(\psi\omega^k)^{j-1},$$

where $\omega = e^{\frac{2\pi i}{n}}$, $\psi = e^{\frac{\pi i}{n}}$, and $i = \sqrt{-1} \in \mathbb{C}$ for $k = 0, 1, 2, \dots, n-1$. Note that $a_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}$, where $\alpha = \frac{-1+\sqrt{5}}{2}$, $\beta = \frac{-1-\sqrt{5}}{2}$, and $\alpha + \beta = -1 = \alpha\beta$. Then, for $k = 0, 1, 2, \dots, n-1$,

$$\lambda_k = \frac{1}{\alpha - \beta} \sum_{j=1}^{n} \left(\alpha(\alpha\psi\omega^k)^{j-1} - \beta(\beta\psi\omega^k)^{j-1} \right).$$

Since $\alpha\psi\omega^k \neq 1$, $\beta\psi\omega^k \neq 1$, $\alpha\psi\omega^k \neq 0$, and $\beta\psi\omega^k \neq 0$, then for $k = 0, 1, 2, \dots, n-1$,

$$\lambda_k = \frac{1}{\alpha - \beta} \left(\frac{\alpha(1 - (\alpha\psi\omega^k)^n)}{1 - \alpha\psi\omega^k} - \frac{\beta(1 - (\beta\psi\omega^k)^n)}{1 - \beta\psi\omega^k} \right) = \frac{1 + a_{n+1} + \psi\omega^k a_n}{1 + \psi\omega^k - \psi^2\omega^{2k}}.$$

Assume to the contrary that $\lambda_0 = 0$. Then, $1 + a_{n+1} + \psi a_n = 0$, and hence it follows that $\psi = -\frac{1+a_{n+1}}{a_n}$ is a real number where $a_n \neq 0$ for $n \geq 2$. It yields that the imaginary part of ψ is $\sin\frac{\pi}{n} = 0$. Therefore, $1/n$ must be an integer. This contradicts the fact that $1/n$ is not an integer for all integers $n \geq 2$. Thus, $\lambda_0 \neq 0$. Furthermore, assume to the contrary that $\lambda_m = 0$ where $m = 1, 2, \dots, n-1$. Then, $1 + a_{n+1} + \psi\omega^m a_n = 0$, and hence it follows that $\psi\omega^m = -\frac{1+a_{n+1}}{a_n}$ is a real number where $a_n \neq 0$ for $n \geq 2$. It yields that the imaginary part of $\psi\omega^m$ is $\sin\frac{(2m+1)\pi}{n} = 0$. Therefore, $\psi\omega^m = \cos\frac{(2m+1)\pi}{n} = -1$ for $0 < \frac{(2m+1)\pi}{n} < 2\pi$. Then, $1 + a_{n+1} + \psi\omega^m a_n \neq 0$, and this contradicts with $1 + a_{n+1} + \psi\omega^m a_n = 0$ for all integers $n \geq 2$. Thus, $\lambda_k \neq 0$ for all $k = 0, 1, 2, \dots, n-1$. So, A is invertible for $n \geq 2$. □

We have proven that the matrix defined in Definition 1 is invertible for $n \geq 2$. Next, we formulate the determinant and inverse of the matrix in the following theorem.

Theorem 4. *Let $\mathcal{C}_n(\mathcal{F}) = \mathrm{SCirc}\,(a_1, a_2, a_3, \dots, a_n)$ be the matrix defined in Definition 1 for all integers $n \geq 2$. Then,*

$$\det\left(\mathcal{C}_n(\mathcal{F})\right) = x_n^{n-1} - \sum_{j=0}^{n-2} (-1)^j a_{n-1-j} a_n^j x_n^{n-2-j}, \tag{3}$$

and

$$\mathcal{C}_n^{-1}(\mathcal{F}) = \frac{1}{\det\left(\mathcal{C}_n(\mathcal{F})\right)} \mathrm{SCirc}\,(s_1, s_2, \dots, s_n), \tag{4}$$

where $x_n = -a_n + a_{n-1} + 1$, $s_1 = \frac{\det(\mathcal{C}_n(\mathcal{F})) + (-1)^{n-2} a_n^{n-2}}{x_n}$, $s_2 = \frac{x_n^{n-2} - \det(\mathcal{C}_n(\mathcal{F}))}{a_n}$, and $s_j = (-1)^{j-2} a_n^{j-3} x_n^{n-j}$ for $j = 3, 4, \dots, n$.

Proof. For simplicity of writing the proof, let

$$\mathcal{C}_n^{-1}(\mathcal{F}) = A_n = \begin{pmatrix} 1 & -1 & 2 & \cdots & a_{n-2} & a_{n-1} & a_n \\ -a_n & 1 & -1 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ -a_{n-1} & -a_n & 1 & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 3 & -5 & 8 & \cdots & 1 & -1 & 2 \\ -2 & 3 & -5 & \cdots & -a_n & 1 & -1 \\ 1 & -2 & 3 & \cdots & -a_{n-1} & -a_n & 1 \end{pmatrix}.$$

Based on Theorem 3, A_n is invertible for $n \geq 2$.

- For the case $n = 2$, it is clear that

$$\det(A_2) = 2 \text{ and } A_2^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

  and each of them satisfies Eq. 3 and Eq. 4, respectively.
- For the case $n = 3$, it is clear that

$$\det(A_3) = 4 \text{ and } A_3^{-1} = \frac{1}{4}\begin{pmatrix} -1 & -3 & -1 \\ 1 & -1 & -3 \\ 3 & 1 & -1 \end{pmatrix},$$

  and each of them satisfies Eq. 3 and Eq. 4, respectively.
- For the case $n \geq 4$, let B_j be the matrix resulting from the elementary row and column operations in the jth step. The proof is described by applying elementary row and column operations on A_n in step by step as follows.
  (1) The $(i+1)$th row is added to the $(i+2)$th row and subtracted by the $(i+3)$th row, for $i = 1, 2, \ldots, n-3$:

$$\begin{aligned} B_1 &= E_{(n-2)n(-1)}E_{(n-2)(n-1)(1)} \cdots E_{35(-1)}E_{34(1)}E_{24(-1)}E_{23(1)}(A_n) \\ &= \begin{pmatrix} 1 & -1 & 2 & \cdots & a_{n-1} & a_n \\ 0 & x_n & a_n & \cdots & 0 & 0 \\ 0 & 0 & x_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n & 0 \\ -2 & 3 & -5 & \cdots & 1 & -1 \\ 1 & -2 & 3 & \cdots & -a_n & 1 \end{pmatrix}, \end{aligned}$$

  where

$$x_n = 1 - a_n + a_{n-1}. \tag{5}$$

  (2) The $(n-1)$th row is added to the nth row and the first row:

$$B_2 = E_{(n-1)1(1)}E_{(n-1)n(1)}(B_1) = \begin{pmatrix} 1 & -1 & 2 & \cdots & a_{n-1} & a_n \\ 0 & x_n & a_n & \cdots & 0 & 0 \\ 0 & 0 & x_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n & 0 \\ 0 & 0 & 0 & \cdots & x_n & a_n \\ 1 & -2 & 3 & \cdots & -a_n & 1 \end{pmatrix}.$$

(3) The nth row is subtracted by the first row:

$$B_3 = E_{n1(-1)}\left(B_2\right) = \begin{pmatrix} 1 & -1 & 2 & -3 & \cdots & a_{n-1} & a_n \\ 0 & x_n & a_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_n & a_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_n & a_n \\ 0 & -1 & 1 & -2 & \cdots & -a_{n-2} & x_n - a_{n-1} \end{pmatrix}.$$

Based on Theorem 1, there exists a non-singular matrix

$$\begin{aligned} P_1 &= E_{n1(-1)}E_{(n-1)1(1)}E_{(n-1)n(1)}E_{(n-2)n(-1)}E_{(n-2)(n-1)(1)}\dots E_{24(-1)}E_{23(1)}\left(I_n\right) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \end{aligned}$$

such that $B_3 = P_1 A_n$.

(4) The $(j+1)$th column is subtracted by a_{j+1} times the first column, for $j = 1, 2, \dots, n-1$:

$$B_4 = K_{n1(-a_n)}\dots K_{31(-a_3)}K_{21(-a_2)}\left(B_3\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_n & a_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_n & a_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_n & a_n \\ 0 & -1 & 1 & -2 & \cdots & -a_{n-2} & x_n - a_{n-1} \end{pmatrix}.$$

Based on Theorem 1, there exists a non-singular matrix

$$Q_1 = K_{n1(-a_n)}\dots K_{31(-a_3)}K_{21(-a_2)}\left(I_n\right) = \begin{pmatrix} 1 & -a_2 & -a_3 & -a_4 & \cdots & -a_{n-1} & -a_n \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

such that $B_4 = P_1 A_n Q_1$.

(5) The $(i+1)$th row is multiplied by $\frac{1}{x_n}$, for $i = 1, 2, \dots, n-2$:

$$B_5 = E_{(n-1)\left(\frac{1}{x_n}\right)}\dots E_{3\left(\frac{1}{x_n}\right)}E_{2\left(\frac{1}{x_n}\right)}\left(B_4\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & y_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & y_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & y_n \\ 0 & -1 & 1 & -2 & \cdots & -a_{n-2} & x_n - a_{n-1} \end{pmatrix},$$

where

$$y_n = \frac{a_n}{x_n}. \tag{6}$$

Based on Theorem 1, there exists a non-singular matrix

$$P_2 = E_{(n-1)\left(\frac{1}{x_n}\right)} \cdots E_{3\left(\frac{1}{x_n}\right)} E_{2\left(\frac{1}{x_n}\right)} (P_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{x_n} & \frac{1}{x_n} & -\frac{1}{x_n} & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{x_n} & \frac{1}{x_n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{x_n} & 0 & 0 & 0 & \cdots & \frac{1}{x_n} & \frac{1}{x_n} \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

such that $B_5 = P_2 A_n Q_1$.

(6) The $(j+2)$th column is subtracted by y_n times the $(j+1)$th column, for $j = 1, 2, \ldots, n-2$:

$$B_6 = K_{n(n-1)(-y_n)} \ldots K_{34(-y_n)} K_{32(-y_n)} (B_5) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & g_1 & g_2 & g_3 & \cdots & g_{n-2} & d_n \end{pmatrix},$$

where

$$g_1 = -a_1 = 1 \text{ and } g_j = -a_j - y_n g_{j-1} \text{ for } j = 2, 3, \ldots, n-2, \tag{7}$$

$$d_n = x_n - a_{n-1} - y_n g_{n-2} = 1 - a_n - y_n g_{n-2}. \tag{8}$$

Therefore, based on Theorem 2, $\det(A_n) = x_n^{n-2} d_n$. Based on Eq. 7, it is obtained that

$$\begin{aligned} g_1 &= -a_1, \\ g_2 &= -a_2 - y_n g_1 = -a_2 - a_1(-y_n), \\ g_3 &= -a_3 - y_n g_2 = -a_1(-y_n)^2 - a_2(-y_n) - a_3, \\ g_4 &= -a_4 - y_n g_3 = -a_1(-y_n)^3 - a_2(-y_n)^2 - a_3(-y_n) - a_4, \end{aligned}$$

and so on, so that it is obtained that

$$g_{n-2} = \sum_{j=1}^{n-2} \left(-a_j (-y_n)^{n-2-j} \right).$$

Thus,

$$\det(A_n) = x_n^{n-2} \left(x_n - a_{n-1} + \sum_{j=1}^{n-2} \left(-a_j (-y_n)^{n-1-j} \right) \right).$$

Then, by using Eq. 6 and changing the counter variable, it is clear that

$$\det(A_n) = x_n^{n-1} - \sum_{j=0}^{n-2} (-1)^j a_{n-1-j} a_n^j x_n^{n-2-j}.$$

Moreover, based on Theorem 1, there exists a non-singular matrix

$$\begin{aligned} Q_2 &= K_{n(n-1)(-y_n)} \ldots K_{34(-y_n)} K_{32(-y_n)} (Q_1) \\ &= \begin{pmatrix} 1 & h_2 & h_3 & h_4 & \cdots & h_{n-1} & h_n \\ 0 & 1 & -y_n & (-y_n)^2 & \cdots & (-y_n)^{n-3} & (-y_n)^{n-2} \\ 0 & 0 & 1 & -y_n & \cdots & (-y_n)^{n-4} & (-y_n)^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -y_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \end{aligned}$$

such that $B_6 = P_2 A_n Q_2$, where $h_2 = -a_2 = 1$, and for $j = 3, 4, \ldots, n$, $h_j = -a_j - y_n h_{j-1}$.

(7) The nth row is subtracted by g_i times the $(i+1)$th row, for $i = 1, 2, \ldots, n-2$:

$$B_7 = E_{n(n-1)(-g_{n-2})} \ldots E_{n3(-g_2)} E_{n2(-g_1)} (B_6) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & d_n \end{pmatrix}.$$

Based on Theorem 1, there exists a non-singular matrix

$$P_3 = E_{n(n-1)(-g_{n-2})} \ldots E_{n3(-g_2)} E_{n2(-g_1)} (P_2) = \frac{1}{x_n} \begin{pmatrix} x_n & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 & \cdots & z_{n-1} & z_n \end{pmatrix},$$

such that $B_7 = P_3 A_n Q_2$. Based on Eq. 5, Eq. 6, Eq. 7, and Eq. 8, notice that

$$z_1 = -x_n - g_{n-2} = -\frac{a_n}{y_n} - g_{n-2} = \frac{d_n - 1}{y_n} = \frac{x_n}{a_n}(d_n - 1), \tag{9}$$

$$z_2 = 1, \tag{10}$$

$$z_3 = -g_1 - g_2 = 1 - (1 + y_n) = -y_n = (-y_n) z_2, \tag{11}$$

$$z_4 = g_1 - g_2 - g_3 = y_n^2 = (-y_n) z_3, \tag{12}$$

$$z_5 = g_2 - g_3 - g_4 = -y_n^3 = (-y_n) z_4, \tag{13}$$

and so on, for $j = 4, 5, \ldots, n-1$, it is obtained that

$$z_j = g_{j-3} - g_{j-2} - g_{j-1} = (-1)^{j-2} y_n^{j-2} = (-y_n) z_{j-1}, \tag{14}$$

where $z_2 = 1$ and $z_3 = -y_n$. Then, it is obtained that

$$z_n = g_{n-3} - g_{n-2} + x_n = d_n - y_n z_{n-1} = (-1)^{n-2} y_n^{n-2} + d_n. \tag{15}$$

(8) Since $B_7 = P_3 A_n Q_2$, then $B_7^{-1} = (P_3 A_n Q_2)^{-1} = Q_2^{-1} A_n^{-1} P_3^{-1}$, so that

$$A_n^{-1} = Q_2 B_7^{-1} P_3 = \begin{pmatrix} 1 & h_2 & h_3 & \cdots & h_{n-1} & \frac{h_n}{d_n} \\ 0 & 1 & -y_n & \cdots & (-y_n)^{n-3} & \frac{(-y_n)^{n-2}}{d_n} \\ 0 & 0 & 1 & \cdots & (-y_n)^{n-4} & \frac{(-y_n)^{n-3}}{d_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{y_n}{d_n} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{d_n} \end{pmatrix} P_3. \tag{16}$$

Note that A_n^{-1} is also skew circulant based on Eq. 2. Based on Eq. 16, it is obtained that

$$A_n^{-1} = \frac{1}{d_n x_n} \mathrm{SCirc}(z_n, -z_1, -z_2, \ldots, -z_{n-2}, -z_{n-1})$$

$$\Leftrightarrow A_n^{-1} = \frac{x_n^{n-3}}{\det(A_n)} \mathrm{SCirc}(z_n, -z_1, -z_2, \ldots, -z_{n-2}, -z_{n-1}). \tag{17}$$

For simplification, Eq. 17 can be rewritten as

$$A_n^{-1} = \frac{1}{\det(A_n)} \mathrm{SCirc}(s_1, s_2, \ldots, s_{n-1}, s_n),$$

where

$$s_1 = x_n^{n-3} z_n = \frac{\det(A_n) + (-1)^{n-2} a_n^{n-2}}{x_n},$$

$$s_2 = -x_n^{n-3} z_1 = -x_n^{n-3} \left(\frac{x_n(d_n - 1)}{a_n} \right) = \frac{x_n^{n-2} - \det(A_n)}{a_n},$$

$$s_3 = -x_n^{n-3} z_2 = -x_n^{n-3},$$

and for $j = 4, 5, \ldots, n$, it applies that

$$s_j = -x_n^{n-3} z_{j-1} = -x_n^{n-3} \left((-1)^{j-3} y_n^{j-3} \right) = (-1)^{j-2} a_n^{j-3} x_n^{n-j},$$

based on Eq. 6 and Eq. 9 – Eq. 15.

□

To clarify the method used in the proof of Theorem Theorem 4, we provide an illustrative example using elementary row and column operations with a small matrix size. The example below shows the steps for obtaining the determinant and inverse of the skew circulant matrix with alternating Fibonacci entries using this method.

Example 1. *Let*

$$A_4 = \mathrm{SCirc}(1, -1, 2, -3) = \begin{pmatrix} 1 & -1 & 2 & -3 \\ 3 & 1 & -1 & 2 \\ -2 & 3 & 1 & -1 \\ 1 & -2 & 3 & 1 \end{pmatrix}.$$

(1) The second row is added to the third row and subtracted by the fourth row:

$$B_1 = E_{24(-1)} E_{23(1)}(A_4) = \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 6 & -3 & 0 \\ -2 & 3 & 1 & -1 \\ 1 & -2 & 3 & 1 \end{pmatrix}.$$

(2) The third row is added to the fourth row and the first row:

$$B_2 = E_{31(1)} E_{34(1)}(B_1) = \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & 6 & -3 \\ 1 & -2 & 3 & 1 \end{pmatrix}.$$

(3) The fourth row is subtracted by the first row:

$$B_3 = E_{41(-1)}(B_2) = \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & -1 & 1 & 4 \end{pmatrix}.$$

Based on 1, there exists a non-singular matrix

$$P_1 = E_{41(-1)} E_{31(1)} E_{34(1)} E_{24(-1)} E_{23(1)}(I_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

such that $B_3 = P_1 A_4$.

(4) The $(j+1)$th column is subtracted by a_{j+1} times the first column, for $j = 1,2,3$:

$$B_4 = K_{41(-a_4)}K_{31(-a_3)}K_{21(-a_2)}(B_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & -1 & 1 & 4 \end{pmatrix}.$$

Based on 1, there exists a non-singular matrix

$$Q_1 = K_{41(-a_4)}K_{31(-a_3)}K_{21(-a_2)}(I_4) = \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that $B_4 = P_1A_4Q_1$.

(5) The $(i+1)$th row is multiplied by $\frac{1}{6}$, for $i = 1,2$:

$$B_5 = E_{3(\frac{1}{6})}E_{2(\frac{1}{6})}(B_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & -1 & 1 & 4 \end{pmatrix}.$$

Based on 1, there exists a non-singular matrix

$$P_2 = E_{3(\frac{1}{6})}E_{2(\frac{1}{6})}(P_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

such that $B_5 = P_2A_4Q_1$.

(6) The third column is subtracted by $-\frac{1}{2}$ times the second column, then the fourth column is subtracted by $-\frac{1}{2}$ times the new third column:

$$B_6 = K_{43(\frac{1}{2})}K_{32(\frac{1}{2})}(B_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{17}{4} \end{pmatrix}.$$

Based on Theorem 2, we obtain that

$$\det(A_4) = 6^2 \det(B_4) = 36 \cdot \frac{17}{4} = 153.$$

Moreover, based on Theorem 1, there exists a non-singular matrix

$$Q_2 = K_{43(\frac{1}{2})}K_{32(\frac{1}{2})}(Q_1) = \begin{pmatrix} 1 & 1 & -\frac{3}{2} & \frac{9}{4} \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that $B_6 = P_2A_4Q_2$.

(7) The fourth row is subtracted by -1 times the second row and subtracted by $\frac{1}{2}$ times the third row:

$$B_7 = E_{43(-\frac{1}{2})}E_{42(1)}(B_6) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{17}{4} \end{pmatrix}.$$

Based on Theorem 1, there exists a non-singular matrix

$$P_3 = E_{43(-\frac{1}{2})}E_{42(1)}(P_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ -\frac{13}{12} & \frac{1}{6} & \frac{1}{12} & \frac{9}{12} \end{pmatrix},$$

such that $B_7 = P_3A_4Q_2$.

(8) *Since* $B_7 = P_3A_4Q_2$, *then* $B_7^{-1} = (P_3A_4Q_2)^{-1} = Q_2^{-1}A_4^{-1}P_3^{-1}$, *so that*

$$A_4^{-1} = Q_2B_7^{-1}P_3 = \begin{pmatrix} 1 & 1 & -\frac{3}{2} & \frac{9}{4} \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{4}{17} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ -\frac{13}{12} & \frac{1}{6} & \frac{1}{12} & \frac{9}{12} \end{pmatrix}$$

$$\Leftrightarrow A_4^{-1} = \frac{1}{153}\begin{pmatrix} 27 & 39 & -6 & -3 \\ 3 & 27 & 39 & -6 \\ 6 & 3 & 27 & 39 \\ -39 & 6 & 3 & 27 \end{pmatrix}.$$

The explicit determinant and inverse formulas derived above allow for efficient computation. To demonstrate this, we next evaluate their computational time and compare them with built-in functions from Wolfram Mathematica.

3.2 Computational Performance

We present a simple illustration first to show how to apply the formula in Theorem 4. Then, by considering the illustration, we construct an algorithm.

Example 2. *Let*

$$A_5 = \text{SCirc}(1, -1, 2, -3, 5) = \begin{pmatrix} 1 & -1 & 2 & -3 & 5 \\ -5 & 1 & -1 & 2 & -3 \\ 3 & -5 & 1 & -1 & 2 \\ -2 & 3 & -5 & 1 & -1 \\ 1 & -2 & 3 & -5 & 1 \end{pmatrix}.$$

Then, by using Theorem 4, we obtain that $x_5 = 1 - a_5 + a_4 = -7$, *and the determinant and inverse are*

$$\det(A_5) = (-7)^4 - (-3\cdot(-7)^3 - 2\cdot 5\cdot(-7)^2 - 5^2\cdot(-7) - 5^3) = 1812,$$

$$A_5^{-1} = \frac{1}{1812}\text{SCirc}\left(\frac{1812 + (-1)^3\cdot 5^3}{-7}, \frac{(-7)^3 - 1812}{5}, -(-7)^2, 5\cdot(-7), -5^2\right)$$

$$= \frac{1}{1812}\text{SCirc}(-241, -431, -49, -35, -25)$$

$$\Leftrightarrow A_5^{-1} = \frac{1}{1812}\begin{pmatrix} -241 & -431 & -49 & -35 & -25 \\ 25 & -241 & -431 & -49 & -35 \\ 35 & 25 & -241 & -431 & -49 \\ 49 & 35 & 25 & -241 & -431 \\ 431 & 49 & 35 & 25 & -241 \end{pmatrix}.$$

From the above illustration, the iteration process in computing the determinant stores information that can be reused in computing the inverse. Thus, an algorithm can be constructed to compute both efficiently.

Algorithm 1 Determinant and inverse of skew circulant matrices with alternating Fibonacci

Require: $A_n = \text{SCirc}(a_1, a_2, \ldots, a_n)$ with alternating Fibonacci entries $\mathcal{F} = (a_j)_{j=1}^{n}$ for $n \geq 2$

$x_n \leftarrow 1 - a_n + a_{n-1}$; $w \leftarrow x_n^{n-2}$; $v \leftarrow 1$; $\Delta \leftarrow w(x_n - a_{n-1})$; $s_1 \leftarrow 1$; $s_2 \leftarrow w$; {Store variables for the iteration process}

for $j = 1$ to $n - 2$ **do**

  $w \leftarrow \frac{w}{x_n}$;

  $s_1 \leftarrow -s_1$; $s_{j+2} \leftarrow s_1 v w$; {Compute s_j for the inverse using x_n and a_n}

  $v \leftarrow v a_n$; $\delta \leftarrow s_1 a_{n-1-j} v w$; $\Delta \leftarrow \Delta - \delta$; {Compute determinant using x_n and a_n}

end for

$s_1 \leftarrow \frac{\Delta + s_1 v}{x_n}$; $s_2 \leftarrow \frac{s_2 - \Delta}{a_n}$; {Finalize s_1 and s_2 for the inverse using Δ, x_n, and a_n}

return (Δ, A_n^{-1})

Algorithm 1 is implemented using Wolfram Mathematica. Both are run for various values of n once each and the running time is compared with the built-in functions from Wolfram Mathematica on the same computer specifications. The running time is shown in Table 1 below.

Table 1: Running time of Algorithm 1 and built-in functions from Wolfram Mathematica

n	**Running time (second)**	
	Algorithm 1	**Built-in Functions for Determinant and Inverse**
100	0.0124702	6.59187
110	0.0164727	11.57160
120	0.0245279	19.74890
130	0.0362658	33.33000
140	0.0429187	53.04420
150	0.0619016	79.21960

According to Table 1, the running time of Algorithm 1 and the built-in functions from Wolfram Mathematica to calculate the determinant and inverse of the matrices tends to increase as the matrix size increases. However, Algorithm 1 is able to calculate the determinant and inverse much faster than the built-in functions. In fact, Algorithm 1 can work 1,280 times faster than the built-in functions when $n = 150$. This can indicate that the explicit formulas of the determinant and inverse in Theorem 4 are computationally efficient formulas and are robust to the size of the matrix used.

Numerical instability may arise due to the accumulation of floating-point rounding errors for large matrix size or large n, particularly in computing powers of x_n and repeated multiplications in the iteration process. Further work may involve error analysis to formally assess the stability.

4 Conclusion

The determinant and inverse of the skew circulant matrices with alternating Fibonacci entries are formulated explicitly through a series of elementary row and column operations. The formulas have a simpler expression than the general formulas known previously. Then, the algorithm built to calculate the determinant and inverse can perform much faster than the built-in functions from Wolfram Mathematica. However, the method assumes exact arithmetic and rounding errors may occur when the matrix size becomes very large. The methods used in this article are limited to matrices whose entries have certain simple patterns. Future work may include applying this method to other recursive sequence such as complex Fibonacci, harmonic, or Pell-Lucas, and to other matrices such as Toeplitz or Foeplitz matrices. Furthermore, numerical stability analysis and implementation in other software such as Python, Julia, or Maple, could be future research directions.

CRediT Authorship Contribution Statement

Author One: Conceptualization, Methodology, Writing-Original Draft, Software, Validation, Visualization. **Author Two:** Resources, Formal Analysis, Supervision, Validation. **Author Three:** Formal Analysis, Supervision, Validation. **Author Four:** Writing-Review & Editing, Supervision, Validation.

Declaration of Generative AI and AI-assisted technologies

ChatGPT version 5 was used for writing assistance in improving the clarity and readability of the manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

The data and code supporting the findings of this study are available from the corresponding author upon reasonable request and subject to confidentiality agreements.

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