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Trace of the Adjacency Matrix of the Star Graph and Complete Bipartite Graph Raised to a Positive Integer Power

Corry Corazon Marzuki*, Fitri Aryani, Sri Basriati, and Yuslenita Muda

Department of Mathematics, Faculty of Sciences and Technology, Universitas Islam Negeri Sultan Syarif Kasim Riau, Jl. HR. Soebrantas No. 155 Simpang Baru, Panam, Pekanbaru, 28293

Abstract

This research aims to derive the general form of the trace matrix of adjacency from star graphs and complete bipartite graphs with size $n \times n$ and raised to a positive integer power. To obtain the general form of the trace matrix of adjacency for these graphs, we first derive the general form of the adjacency matrix raised to a positive integer power for each given graph. The general form of matrix exponentiation is proven using mathematical induction. The trace matrix of adjacency for each graph raised to a positive integer power is obtained through a direct proof based on the definition of the trace matrix. Additionally, applications of the trace matrix of adjacency from star graphs and complete bipartite graphs with size $n \times n$ and raised to a positive integer power are provided in the form of examples.

Keywords: adjacency matrix; bipartite graphs; mathematical induction; star graphs; trace matrix.

1 Introduction

The trace of a matrix is the sum of the main diagonal elements of a square matrix. This means that calculating the trace of a matrix is not particularly difficult, but when the matrix is raised to the power of n, computing its trace requires performing matrix multiplication n times. Only then can the trace of the matrix raised to the power of n be determined. Therefore, calculating the trace of a matrix raised to a power is quite complex. In other words, it is interesting to investigate how to find the exact formula for computing the trace of a matrix raised to a power without explicitly performing the matrix exponentiation or multiplication. By substituting the matrix entries into the formula, the value of the trace of the matrix raised to a power can be obtained without going through the lengthy process of matrix exponentiation or multiplication.

Determining the trace of a matrix raised to a power has been a subject of considerable attention. In 1976, Datta et al. [1] developed an algorithm for calculating the trace of a matrix raised to the power of $Tr(A^k)$, where k is an integer, and A is a Hessenberg matrix with unit codiagonal. Furthermore, in 1985, Chu. Mt [2] discussed symbolic calculations for the trace of a tridiagonal matrix raised to a power. The study of trace also finds applications in various areas of matrix theory and numerical linear algebra. In 1990, Pan.V [3] in his paper was able to

*Corresponding author. E-mail: corry@uin-suska.ac.id

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determine the eigenvalues of a symmetric matrix, providing basic procedures for estimating the trace of A^n and A^{-n}, where n is an integer.

According to Zarelua in 2008 [4], in the theory of number theory and combinatorics, the trace of a matrix raised to an integer power is related to Euler's congruence:

$$Tr(Ap^r) = Tr(Ap^{r-1}) \mod (p^r)$$

for all integer matrices A, where p is a prime number and r is an integer, the paper also discusses invariants in dynamic systems described in the form of the trace of a matrix raised to an integer power. An example provided in the paper is the Lefschetz number. In the field of network analysis, specifically in triangle counting in a graph, according to Avron in 2010 [5], when analyzing a complex network, the most crucial issue is counting the total number of triangles in a connected simple graph. This count is equivalent to $Tr(A^3)/6$, where A is the adjacency matrix of the graph. According to Brezinski et al. in 2012 [6], the trace of a matrix raised to a power is frequently discussed in various mathematical fields such as Network Analysis, Number Theory, Dynamical Systems, Matrix Theory, and Differential Equations.

The discussion on the formula for the trace of a matrix raised to a power has also been addressed by Pahade and Jha in 2015 [7]. The article derives the formula for the trace of a 2×2 matrix raised to a positive integer power. In the article, two forms of the formula for the trace of a matrix raised to a power are presented. First, the formula for the trace of a matrix raised to a power for even n is as follows:

$$tr(A^n) = \sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots(\text{up to terms})(\det(A))^r(\text{tr}(A))^{n-2r} \tag{1}$$

Second, the formula for the trace of a matrix raised to a power for odd n is as follows:

$$tr(A^n) = \sum_{r=0}^{n-\frac{1}{2}} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots(\text{up to terms})(\det(A))^r(\text{tr}(A))^{n-2r} \tag{2}$$

Aryani and Solihin in 2017 [8] discussed the trace of a 2×2 matrix raised to a negative integer power. In the article, two forms of the formula for the trace of a matrix raised to a power are presented. First, the formula for the trace of a matrix raised to a power for even n is as follows:

$$\frac{tr(A^n) = \sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots(\text{up to terms})(\det(A))^r(\text{tr}(A))^{n-2r}}{(\det(A))^n} \tag{3}$$

Second, the formula for the trace of a matrix raised to a power for odd n is as follows:

$$\frac{tr(A^n) = \sum_{r=0}^{n-\frac{1}{2}} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]\cdots(\text{up to terms})(\det(A))^r(\text{tr}(A))^{n-2r}}{(\det(A))^n} \tag{4}$$

In 2017, research on the trace of a matrix raised to a power was revisited by Pahade and Jha [9], who investigated the adjacency matrix or the adjacency matrix of a complete graph raised to a positive integer power. In their paper, given A as the symmetric adjacency matrix of a simple complete graph with n vertices, for even k, the following is obtained:

$$Tr(A^k) = \sum_{r=1}^{n/2} s(k,r)n(n-1)^r(n-2)^{k-2r} \tag{5}$$

and for k odd number is obtained:

$$Tr(A^k) = \sum_{r=1}^{\frac{n-1}{2}} s(k,r)n(n-1)^r(n-2)^{k-2r} \tag{6}$$

where $s(k,r)$ is a number that depends on k and r, and is defined as:

$$s(k,r) = 1, s(k, \frac{k}{r}) = 1, s(k, \frac{k-1}{2}), and, s(k,r) = s(k-1,r) + s(k-2,r-1) \tag{7}$$

In 2019, [10] extended the research by Pahade and Jha from 2017 regarding the trace of the adjacency matrix in a complete graph raised to negative integer powers: negative two, three, and four. The results obtained from this research are:

$$Tr(A^{-2}) = \frac{n((n-1)(n-2))^2}{(n-1)^2}, n \geq 2. \tag{8}$$

The general form of the trace of the adjacency matrix of an $n \times n$ complete graph raised to the negative third power is:

$$Tr(A^{-3}) = \frac{n - 2((n-1)(n-2) - (n-2)^3)}{(n-1)^3}, n \geq 2. \tag{9}$$

and the general form of the trace of the $n \times n$ adjacency matrix raised to the negative fourth power of a complete graph is:

$$Tr(A^{-4}) = \frac{n((n-1)^2 + 3(n-1)(n-2)^2 + (n-2)^4}{(n-1)^4}, n \geq 2. \tag{10}$$

In addition to complete graphs, there are also cycle graphs that can be represented by an adjacency matrix. A cycle graph is a graph where each vertex has a degree of two, and there exists a path that starts and ends at the same vertex [11]. Therefore, for each cycle graph C_n, it has n vertices and can be represented by an adjacency matrix C of size $n \times n$. Research on the trace of the adjacency matrix of cycle graphs raised to positive integer powers two to five, conducted by [12], obtained the following results:

$$\begin{aligned} Tr(C_n^2) &= 2n, \quad n \geq 6 \\ Tr(C_n^3) &= 0, \quad n \geq 8 \\ Tr(C_n^4) &= 6n, \quad n \geq 10 \\ Tr(C_n^5) &= 0, \quad n \geq 12 \end{aligned} \tag{11}$$

The adjacency matrix can also be formed from star graphs and complete bipartite graphs. The star graph S_n is a graph constructed from one central node by adding a certain number of n leaf nodes to that central node [13], with the general form of its adjacency matrix as follows:

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{12}$$

While the complete bipartite graph is a graph whose set of vertices can be partitioned into two sets, where each vertex in one set is adjacent to all vertices in the other set. However, vertices within the same set are not adjacent [13]. The general form of its adjacency matrix is:

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{13}$$

This article will elaborate on the trace of the adjacency matrix for $n \times n$ star graphs and complete bipartite graphs raised to positive integer powers. The proof of the general form of matrix exponentiation using mathematical induction rules is detailed in **karmakar2015some** and **das2015some**. Meanwhile, for the proof of the trace of the matrix, a direct proof is employed, utilizing the definition of the matrix trace found in **kolman2000introductory**, **strang2009introduction**, and [9]. Elaboration on the definitions of matrix multiplication, matrix exponentiation, and matrix trace can be found in **lay2012linear** and **poole2014linear**.

2 Methods

The research methodology refers to the steps taken by the author to address the problem in this study, namely to obtain the trace formula of the $n \times n$ adjacency matrix of star graphs and complete bipartite graphs raised to an integer power. The research methodology used in this study is a literature review. This research begins by studying previously established findings, particularly regarding the trace of the adjacency matrix in complete graphs. It is then further developed for the adjacency matrices of different types of graphs, namely star graphs and complete bipartite graphs raised to an integer power. The steps taken are as follows:

1. Steps to Determine the Trace Formula of the Adjacency Matrix of an $n \times n$ Star Graph B_n Raised to a Positive Integer Power.
   (a) A star graph S_{n-1} is given.
   (b) Obtain the adjacency matrix of the star graph S_{n-1}, denoted as matrix B_n, with dimensions $n \times n$ as shown in Equation 12.
   (c) Determine the powers of the adjacency matrix of the star graph $(B_n)^m$ with $n = 1, 2, 3, 4, 5$ and $m = 1, 2, 3, 4, 5, 6, 7$ so that the matrix pattern is visible.
   (d) Conjecture the general formula for the powers of the adjacency matrix of the star graph $(B_n)^m$ with m as a positive integer.
   (e) Prove the general formula for the powers of the adjacency matrix of the star graph $(B_n)^m$ with m as a positive integer, using mathematical induction.
   (f) Derive the general trace formula for the adjacency matrix of the star graph $(B_n)^m$ with m as a positive integer, denoted as $tr(B_n^m)$, using the definition of trace of a matrix.
   (g) Apply the formula for the powers of the adjacency matrix of the star graph $(B_n)^m$ with m as a positive integer in an example.
   (h) Apply the trace formula for the adjacency matrix of the star graph $(B_n)^m$ with m as a positive integer in an example, calculating $tr(B_n^m)$.
2. Steps to Determine the Trace Formula of the Adjacency Matrix of an $2n \times 2n$ Complete Bipartite Graph A_{2n} Raised to a Positive Integer Power.
   (a) A Complete Bipartite Graph $K_{m,m}$ is given.
   (b) Obtain the adjacency matrix of the Complete Bipartite Graph $K_{m,m}$, denoted by A_{2m}, which has dimensions $2m \times 2m$ as shown in Equation 13.

(c) Determine the powers of the adjacency matrix of the Complete Bipartite Graph A_{2m}, with $m = 2, 3, 4, 5$ and $n = 1, 2, 3, 4, 5, 6, 7$.

(d) Conjecture the general formula of the adjacency matrix of the complete bipartite graph $(A_{2m})^n$ for positive integer n.

(e) Prove the general formula of the adjacency matrix of the complete bipartite graph $(A_{2m})^n$ for positive integer n, using mathematical induction.

(f) Obtain the trace of the adjacency matrix of the complete bipartite graph $tr(B_n^m)$, for positive integer n and prove it using the definition of the trace of a matrix.

(g) Apply the general formula for the power of the adjacency matrix of an $m \times m$ graph on a cycle graph from $(A_{2m\times 2m})^n$ with a positive integer n, using an example.

(h) Apply the trace formula of the adjacency matrix of an $m \times m$ graph on a cycle graph from $(A_{2m\times 2m})^n$ with a positive integer n, using an example.

3 Results and Discussion

3.1 Matrix Exponentiation of the Adjacency Matrix of a Star Graph with a Positive Integer Power

The star graph S_m is a graph constructed from one central node by adding a certain number of m leaf nodes to that central node. The adjacency matrix of the star graph S_m is as follows:

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The following theorem stating the general form of matrix exponentiation for the adjacency matrix of a star graph raised to a positive integer power.

Theorem 1. *Suppose B is the adjacency matrix of the star graph S_m, that is*

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

then the exponentiation of B can be expressed as

$$B^n = \begin{pmatrix} (m-1)^{\frac{n}{2}} & 0 & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \end{pmatrix}$$

if n is even, and

$$B^n = \begin{pmatrix} 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ (m-1)^{\frac{n}{2}} & 0 & 0 & \cdots & 0 \\ (m-1)^{\frac{n}{2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-1)^{\frac{n}{2}} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

if n is odd.

Proof.

The proof is conducted using mathematical induction.

For n is even. Let $p(n)$: if

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is the adjacency matrix of a star graph of order m, then

$$B^n = \begin{pmatrix} (m-1)^{\frac{n}{2}} & 0 & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \end{pmatrix}$$

The base step. Would be proved $p(2)$ true.

$$B^2 = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} m-1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Since it satisfies Equation (1), it is concluded that $p(2)$ is true.

The induction step. Assume $p(k)$ is true, i.e. holds

$$B^k = \begin{pmatrix} (m-1)^{\frac{k}{2}} & 0 & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{k-2}{2}} & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \\ 0 & (m-1)^{\frac{k-2}{2}} & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{k-2}{2}} & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \end{pmatrix}$$

It will be proved $p(k+2)$ is also true, so that

$$B^{k+2} = \begin{pmatrix} (m-1)^{\frac{k+2}{2}} & 0 & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{k-2}{2}} & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \\ 0 & (m-1)^{\frac{k-2}{2}} & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{k-2}{2}} & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \end{pmatrix}$$

Note that

$$
\begin{aligned}
B^{k+2} &= B^k \cdot B^2 \\
&= \begin{pmatrix} (m-1)^{\frac{k}{2}} & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{k-2}{2}} & \cdots & (m-1)^{\frac{k-2}{2}} \end{pmatrix} \cdot \begin{pmatrix} m-1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{pmatrix} \\
&= \begin{pmatrix} (m-1)^{\frac{k}{2}}(m-1) & 0 & \cdots & 0 \\ 0 & (m-1)(m-1)^{\frac{k-2}{2}} & \cdots & (m-1)(m-1)^{\frac{k-2}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)(m-1)^{\frac{k-2}{2}} & \cdots & (m-1)(m-1)^{\frac{k-2}{2}} \end{pmatrix} \\
&= \begin{pmatrix} (m-1)^{\frac{k+2}{2}} & 0 & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{k}{2}} & (m-1)^{\frac{k}{2}} & \cdots & (m-1)^{\frac{k}{2}} \\ 0 & (m-1)^{\frac{k}{2}} & (m-1)^{\frac{k}{2}} & \cdots & (m-1)^{\frac{k}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{k}{2}} & (m-1)^{\frac{k}{2}} & \cdots & (m-1)^{\frac{k}{2}} \end{pmatrix}
\end{aligned}
$$

Then, $p(k+2)$ is true. Since the base step and the induction step have been proven, then $p(n)$ is proven to be true.

For n is odd number. Let $p(n)$: if

$$
B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}
$$

is the adjacency matrix of a star graph of order m, then

$$
B^n = \begin{pmatrix} 0 & (m-1)^{\frac{n-1}{2}} & (m-1)^{\frac{n-1}{2}} & \cdots & (m-1)^{\frac{n-1}{2}} \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \end{pmatrix}
$$

The base step. Would be proved $p(1)$ true.

$$
B^1 = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} = B
$$

Since it satisfies Equation (2) it is concluded that $p(1)$ is true.

Process of induction. Assume $p(k)$ is true, such that

$$B^k = \begin{pmatrix} 0 & (m-1)^{\frac{k-1}{2}} & (m-1)^{\frac{k-1}{2}} & \cdots & (m-1)^{\frac{k-1}{2}} \\ (m-1)^{\frac{k-1}{2}} & 0 & 0 & \cdots & 0 \\ (m-1)^{\frac{k-1}{2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-1)^{\frac{k-1}{2}} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

would be proved $p(k+2)$ is true as well. It means, will be proved that

$$B^{k+2} = \begin{pmatrix} 0 & (m-1)^{\frac{k+1}{2}} & (m-1)^{\frac{k+1}{2}} & \cdots & (m-1)^{\frac{k+1}{2}} \\ (m-1)^{\frac{k+1}{2}} & 0 & 0 & \cdots & 0 \\ (m-1)^{\frac{k+1}{2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-1)^{\frac{k+1}{2}} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Note that $B^{k+2} = B^k \cdot B^2$.

$$B^n = \begin{pmatrix} 0 & (m-1)^{\frac{n-1}{2}} & \cdots & (m-1)^{\frac{n-1}{2}} \\ (m-1)^{\frac{n-1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (m-1)^{\frac{n-1}{2}} & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} m-1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{pmatrix}$$

□

3.2 Trace of the Adjacency Matrix of Star Graphs with Positive Integer Ranks

To obtain the formula for the trace of the power of the adjacency matrix of a star graph, we use the formula derived in Theorem 1. The following is the theorem regarding the trace of the power of the adjacency matrix of a star graph along with its direct proof.

Theorem 2. *Let A be the adjacency matrix of a star graph of order m, that is:*

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$tr(B^n) = \begin{cases} 2(m-1)^n; & \text{if } n \text{ is even} \\ 0; & \text{if } n \text{ is odd} \end{cases}$$

Proof.

Because of

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is the adjacency matrix of a star graph of order m. Based on the Theorem 2.1., for even n we obtain:

$$B^n = \begin{pmatrix} (m-1)^{\frac{n}{2}} & 0 & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \end{pmatrix}$$

$tr(B^n) = (m-1)^{\frac{n}{2}} + (m-1)(m-1)^{\frac{n-2}{2}} = 2(m-1)^{\frac{n}{2}}$ Meanwhile, for n being an odd number, based on Theorem 2.1, we obtain:

$$A^n = \begin{pmatrix} 0 & (m-1)^{\frac{n-1}{2}} & (m-1)^{\frac{n-1}{2}} & \cdots & (m-1)^{\frac{n-1}{2}} \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

so that, yields $tr(B^n) = 0$. □

3.3 Trace Example of the Use of Exponentiation Formula and the Trace of the Power of the Adjacency Matrix of a Star Graph with Positive Integer Exponents

The following provides an example of the application of the formula obtained in Theorems 1 and 2 to the adjacency matrix of a star graph with positive integer exponents.

Example 1. *Let B is an adjacent matrix of star graph S_8. Determine B^{10} and B^{11}!*

Solution:

Let the following star graph S_8:

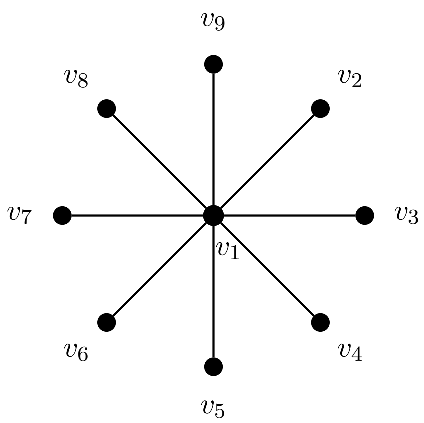

Figure 1: Star graph S_8

Based on Figure 1, the adjacency matrix of the graph S_8 is obtained as follows.

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Based on Theorem 2.1, yields:

$$B^{10} = \begin{pmatrix} (8-1)^5 & 0 & 0 & \cdots & 0 \\ 0 & (8-1)^4 & (8-1)^4 & \cdots & (8-1)^4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (8-1)^4 & (8-1)^4 & \cdots & (8-1)^4 \end{pmatrix}$$

$$B^{10} = \begin{pmatrix} 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 \\ 0 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 \\ 0 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 \\ 0 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 \\ 0 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 \\ 0 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 \\ 0 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 & 2401 \end{pmatrix}$$

and

$$B^{11} = \begin{pmatrix} 0 & (8-1)^5 & (8-1)^5 & \cdots & (8-1)^5 \\ (8-1)^5 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (8-1)^5 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B^{11} = \begin{pmatrix} 0 & 16807 & 16807 & 16807 & 16807 & 16807 & 16807 & 16807 \\ 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By Theorem 2.2, we obtain: $tr(B^{10}) = 2(8-1)^5 = 33614$ and $tr(B^{11}) = 0$.

3.4 Exponentiation of the Adjacency Matrix of the Complete Bipartite Graph $K_{(m,m)}$ with Positive Integer Exponents.

Let $K_{(m,m)}$ be the Complete Bipartite Graph with the set of vertices $V_1 = \{v_1, v_2, v_3, \ldots, v_m\}$ and $V_2 = \{v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{2m}\}$ as in Figure 2.

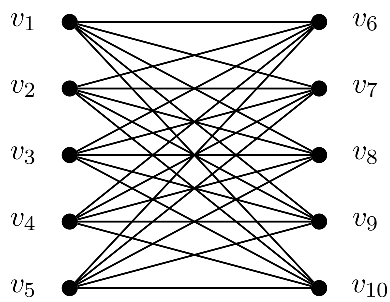

Figure 2: Complete Bipartite Graph $K_{5,5}$

From Figure 2, the adjacency matrix of size $2m \times 2m$ is obtained as follows

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The next step is to determine the exponentiation of the adjacency matrix of the complete bipartite graph A of size $2 \times 2, 4 \times 4, 6 \times 6$, and 8×8 for exponents $n = 1, 2, 3, 4, 5, 6, 7$. This exponentiation process was computed using Maple, and the results are presented in Table 1. By observing the exponentiation pattern of the adjacency matrix obtained from Maple, a formula for the exponentiation of the adjacency matrix for arbitrary m dan n was estimated. The estimated results are also presented in Table 1.

Table 1: Results of the Exponentiation of the Adjacency Matrix of the Complete Bipartite Graph K(m,m) with Positive Integer Exponents Using Maple.

n	Exponentiation Result
1	A
2	$mI_{2m} + (mJ_{2m})$
3	$m^2 A$
4	$m^2(mI_{2m} + mJ_{2m})$
5	$m^4 A$

Then, the conjecture obtained in Table 1 will be proven in the following Theorem 3:

Theorem 3. *Let matrix $A_{2m\times 2m}$ be an adjacency matrix of the complete bipartite graph $K_{m,m}$ with the set of vertices $V_1 = \{v_1, v_2, v_3, \ldots, v_m\}$ and $V_2 = \{v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{2m}\}$, then, the exponentiation of matrix A can be expressed as $A^n = (a_{i,j})$ with $a_{i,j} = m^{n-1}$; for $i = 1, 2, \ldots, m$ or $i = m+1, m+2, \ldots, 2m$ and $j = m+1, m+2, \ldots, 2m$, $a_{i,j} = 0$; elsewhere if n is even and $a_{i,j} = 0$; for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$ or $i = m+1, m+2, \ldots, 2m$ and $j = m+1, m+2, \ldots, 2m$, $a_{i,j} = m^{n-1}$; elsewhere if n is odd.*

Proof.

Let $K_{(m,m)}$ be an adjacency matrix of the complete bipartite graph $V_1 = \{v_1, v_2, v_3, \ldots, v_m\}$ and $V_2 = \{v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{2m}\}$.

For n is even. The proof will be done using mathematical induction. Let $p(n)$: If A is an adjacency matrix of the complete bipartite graph $K_{(m,m)}$, then $A^n = (a_{i,j})$ where $a_{i,j} = m^{\frac{n}{2}-1}$ for $i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2m$.

The base step. We will prove that $p(2)$ is true.

$$\begin{aligned}
A^2 &= A \cdot A \\
&= \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} \\
&= \begin{pmatrix} m & \cdots & m & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ m & \cdots & m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & m & \cdots & m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & m & \cdots & m \end{pmatrix} \\
&= m \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}
\end{aligned}$$

The result of A^2 satisfies the formula. Thus, $p(2)$ is true.

The induction step. Assume $p(k)$ is true, that is, for k is even,

$$A^k = m^{\frac{k-2}{2}} \begin{pmatrix} mI_m & 0 \\ 0 & mI_m \end{pmatrix} = m^{\frac{k}{2}} I_{2m}$$

and we will prove that $p(k+2)$ is also true.

$$\begin{aligned}
A^{k+2} &= A^k \cdot A^2 \\
&= m^{\frac{k}{2}} I_{2m} \cdot m \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} \\
&= m^{\frac{k}{2}} m \begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} \\
&= m^{\frac{k+2}{2}} I_{2m}
\end{aligned}$$

This completes the proof by induction for n even.

For n is odd. The proof will be done by mathematical induction. Let $p(n)$: If A is an adjacency matrix of the complete bipartite graph $K_{(m,m)}$, then $A^n = m^{\frac{n-1}{2}} A$.

The base step. We will prove that $p(1)$ is true. $A^1 = m^{\frac{1-1}{2}}A = m^0 A = A$. This is true. Thus, $p(1)$ is true.

The induction step. Assume $p(k)$ is true, that is, for k is odd, $A^k = m^{\frac{k-1}{2}}A$. We will prove that $p(k+2)$ is also true.

$$\begin{aligned} A^{k+2} &= A^k \cdot A^2 \\ &= (m^{\frac{k-1}{2}}A) \cdot (mI_{2m}) \\ &= m^{\frac{k-1}{2}} mAI_{2m} \\ &= m^{\frac{k+1}{2}}A \end{aligned}$$

This completes the proof by induction for n odd. □

3.5 Trace of the Adjacency Matrix of Complete Bipartite Graphs with Positive Integer Exponents

Theorem 4. *Let A be the adjacency matrix of a complete bipartite graph $K_{m,m}$, that is,*

$$A = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

Then, the trace of A^n is given by

$$tr(A^n) = \begin{cases} 2m^{\frac{n}{2}+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Proof.

Based on Theorem 3, for n even, we have $A^n = m^{\frac{n}{2}} I_{2m}$. The trace is the sum of the diagonal elements, so $tr(A^n) = \sum_{i=1}^{2m} (A^n)_{ii} = \sum_{i=1}^{2m} m^{\frac{n}{2}} = 2m \cdot m^{\frac{n}{2}} = 2m^{\frac{n}{2}+1}$. For n odd, based on Theorem 3, we have $A^n = m^{\frac{n-1}{2}}A$. The diagonal elements of A are all zero, so

$$tr(A^n) = \sum_{i=1}^{2m} (A^n)_{ii} = \sum_{i=1}^{2m} 0 = 0.$$

□

3.6 Example of the Application of the Exponentiation and Trace Formulas for the Adjacency Matrix of a Complete Bipartite Graph

Example 2. *Let A be the adjacency matrix of a complete bipartite graph $K_{3,3}$. Determine A^2 and $tr(A^2)$.*

Solution:

The adjacency matrix of the complete bipartite graph $K_{3,3}$ is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Based on Theorem 3, for $n = 2$ (even), we get

$$A^2 = m^{\frac{n}{2}} I_{2m} = 3^{\frac{2}{2}} I_{2(3)} = 3I_6 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Based on Theorem 4, we get $tr(A^2) = 2m^{\frac{n}{2}+1} = 2(3)^{\frac{2}{2}+1} = 2(3)^2 = 18$. Alternatively, based on the definition of trace, $tr(A^2) = 3 + 3 + 3 + 3 + 3 + 3 = 18$.

4 Conclusion

Based on the results and discussion, the following conclusions can be drawn.

1. The general form of the adjacency matrix of a star graph S_m raised to a positive integer power, for n even, is

$$B^n = \begin{pmatrix} (m-1)^{\frac{n}{2}} & 0 & 0 & \cdots & 0 \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (m-1)^{\frac{n-2}{2}} & (m-1)^{\frac{n-2}{2}} & \cdots & (m-1)^{\frac{n-2}{2}} \end{pmatrix}$$

   and for n odd, is

$$B^n = \begin{pmatrix} 0 & (m-1)^{\frac{n-1}{2}} & (m-1)^{\frac{n-1}{2}} & \cdots & (m-1)^{\frac{n-1}{2}} \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-1)^{\frac{n-1}{2}} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

   The trace of the adjacency matrix of a star graph S_m raised to a positive integer power is

$$tr(B^n) = \begin{cases} 2(m-1)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

2. The general form of the adjacency matrix of a complete bipartite graph $K_{m,m}$ raised to a positive integer power, for n even, is $A^n = m^{\frac{n}{2}} I_{2m}$, and for n odd is $A^n = m^{\frac{n-1}{2}} A$. The trace of the adjacency matrix of a complete bipartite graph $K_{m,m}$ raised to a positive integer power is

$$tr(A^n) = \begin{cases} 2m^{\frac{n}{2}+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

CRediT Authorship Contribution Statement

Corry Corazon Marzuki: Conceptualization, Methodology, Writing: the Original Draft. **Fitri Aryani:** Conceptualization, Methodology, Formal Analysis. **Sri Basriati:** Project Administration, Funding Acquisition, Validation. **Yuslenita Muda:** Writing: Review and Editing.

Declaration of Generative AI and AI-assisted technologies

No generative AI or AI-assisted technologies were used during the preparation of this manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

This study did not involve any datasets or source code that can be publicly shared. Therefore, no data or code are available for this research.

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