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Numerical Solution of the Time-Fractional Black-Scholes Equation and Its Application to European Option Pricing

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Abstract

The classical Black-Scholes model is widely used in financial mathematics for option pricing but is constrained by assumptions such as constant volatility and memoryless market dynamics. To address these limitations, this study employs a time-fractional version of the model that incorporates memory effects through the Caputo fractional derivative. A finite difference method is developed to numerically solve the fractional model and applied to the pricing of European options. Simulations for various fractional orders demonstrate that option prices are sensitive to the memory parameter, with lower values resulting in higher prices. The results highlight the effectiveness of the proposed numerical approach and the enhanced flexibility of the fractional model in capturing complex market behaviors.

Keywords: Fractional Black-Scholes Model; European Option Pricing; Caputo Derivative; Finite Difference Method; Memory Effects in Finance

1 Introduction

In the field of financial mathematics, option pricing plays a central role in modern investment strategies and risk management. Derivative instruments such as options have become essential in both academic research and practical applications, as they provide investors with the flexibility to manage exposure to market fluctuations [1]. The development of mathematical models to accurately price these financial derivatives remains a critical area of study, especially given the dynamic and complex nature of financial markets [2].

The classical Black-Scholes model, introduced in 1973, has long served as a foundational framework for pricing European options. Its elegance and analytical tractability have made it a standard tool in both theoretical and applied finance [3]. However, this model assumes constant volatility and neglects memory effects in asset price movements, which often results in inaccuracies when applied to real-world markets [4]. In practice, asset prices exhibit features such as volatility clustering, jumps, and heavy tails—none of which are captured by the classical model. Empirical studies have shown that option prices predicted by the Black-Scholes model often deviate significantly from observed market data, especially during periods of high uncertainty or financial turbulence.

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To address these shortcomings, researchers have proposed fractional extensions of the classical model. Fractional calculus introduces memory and nonlocality through derivatives of non-integer order, making it possible to model long-term dependence in asset price dynamics [5]. By replacing the standard time derivative with a fractional derivative, particularly the Caputo derivative, the fractional Black-Scholes model accounts for historical effects and time-varying volatility, offering a more realistic representation of financial behavior [6], [7]. Several studies have investigated both analytical and numerical solutions for fractional models, using approaches such as Laplace transforms, series decompositions, and integral operators [8]–[14].

Despite these advancements, many existing solution methods are analytically intensive and difficult to implement for real financial datasets or high-dimensional problems. This motivates the development of stable and flexible numerical methods that can be used in practical contexts, such as option pricing [15]. Among the available techniques, the finite difference method (FDM) stands out due to its simplicity, adaptability, and suitability for solving partial differential equations involving fractional operators [16], [17].

The main objective of this paper is to numerically solve the time-fractional Black-Scholes equation using an implicit finite difference method and apply the results to the pricing of European call options. Unlike previous analytical studies, this work focuses on practical numerical implementation and demonstrates the sensitivity of option prices to the fractional order parameter α. The novelty of this study lies in integrating a classical numerical scheme with fractional calculus to capture memory effects in option pricing models. The results show that the fractional model offers greater flexibility in modeling real market behavior and that the FDM provides a reliable computational tool for fractional financial models. This study contributes to bridging the gap between theoretical developments in fractional calculus and their application in financial engineering.

Recent developments in mathematical finance have highlighted the limitations of classical models in capturing market anomalies, such as volatility clustering and heavy-tailed distributions in asset returns. These phenomena suggest the presence of memory and nonlocal behavior in financial time series, which are not addressed by integer-order models. Fractional derivatives, by incorporating non-local operators and historical dependence, provide a powerful framework for modeling these characteristics [5], [7]. Consequently, the time-fractional Black-Scholes model has emerged as a promising alternative that better reflects empirical observations in real markets [6], [11].

In response to the challenges of solving fractional differential equations, a variety of analytical and semianalytical methods have been developed, such as the Laplace decomposition method [8], Elzaki transform [11], and homotopy-based techniques [10], [12]–[14]. However, these methods often involve complex computations and are limited in scalability for high-dimensional problems or irregular boundary conditions. As such, numerical methods, particularly finite difference schemes, have become increasingly relevant for implementing fractional models in practical financial applications [16]. By discretizing both space and time domains, finite difference methods offer a flexible and structured approach that is well suited for option pricing under fractional dynamics [9], [15], [17].

2 Methods

This study employs a computational approach to solve the time-fractional Black-Scholes partial differential equation using finite difference methods (FDM). The goal is to analyze European option prices under fractional dynamics and to demonstrate the effectiveness of the Caputo derivative in capturing memory effects. The methodology is organized into three subsections: model formulation, numerical scheme (explicit and implicit), and implementation.

2.1 Model Formulation

The time-fractional Black-Scholes equation incorporates a Caputo fractional derivative of order $0 < \alpha \leq 1$ to model memory effects in asset price dynamics. The governing equation is formulated as:

$$\frac{\partial^\alpha V(S,t)}{\partial t^\alpha} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV, \tag{1}$$

where $V(S,t)$ is the option price, S is the stock price, t is time, σ is the volatility, and r is the risk-free interest rate. The fractional derivative is defined in the Caputo sense:

$$\frac{\partial^\alpha V(S,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial V(S,\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha}. \tag{2}$$

2.2 Numerical Scheme

The spatial domain $[0, S_{\max}]$ is divided into M intervals of size $h = \Delta S$, and the time domain $[0,T]$ is divided into N intervals of size $k = \Delta t$. The fractional time derivative is approximated using a Grünwald-Letnikov-type backward scheme, which is widely applied in solving fractional PDEs [18], [19].

2.2.1 Explicit Scheme

The explicit finite difference approximation of Eq. (1) is given by:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t^\alpha} = \frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{(\Delta S)^2} + rS_i \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta S} - rV_i^n, \tag{3}$$

where V_i^n is the option value at node i and time level n.

2.2.2 Implicit Scheme

The implicit scheme involves solving a tridiagonal system at each time step. The approximation becomes:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t^\alpha} = \frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{(\Delta S)^2} + rS_i \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2\Delta S} - rV_i^{n+1}. \tag{4}$$

This system is solved using the Thomas algorithm for tridiagonal matrices. In addition, irregular mesh discretizations can improve solution accuracy in regions of high gradient, as shown in [20].

2.3 Implementation

The implementation process consists of the following stages:

1. Initialization: Set parameters such as $S_{\max}$, K, r, σ, T, α, M, and N.
2. Grid Generation: Define spatial grid points $S_i = i\Delta S$ and time steps $t_n = n\Delta t$.
3. Initial Condition: At maturity ($t = T$), for a European call:

$$V_i^0 = \max(S_i - K, 0). \tag{5}$$

4. Boundary Conditions:

$$V_0^n = 0, \quad (\text{as } S \to 0) \tag{6}$$

$$V_M^n = S_{\max} - Ke^{-r(T-t_n)}, \quad (\text{as } S \to \infty) \tag{7}$$

5. Time Marching: For each time level n, solve the system using either Eq. (3) or Eq. (4).
6. Post-Processing: Plot option value surfaces and analyze results for different α.

3 Results and Discussion

This section presents the numerical results obtained from the implicit finite difference method applied to the fractional Black-Scholes equation for pricing European put options. The aim is to demonstrate the effectiveness of the proposed numerical scheme in solving time-fractional partial differential equations, particularly in capturing the dynamics of option pricing under memory effects. We begin by formulating the discretization framework and proceed to derive the complete finite difference scheme.

The explicit finite difference scheme can be employed to solve partial differential equations (PDEs) of both integer and fractional order. Prior to applying this scheme to the fractional Black-Scholes equation, we outline a general algorithm for solving fractional PDEs using an explicit finite difference approach. This algorithm is based on discretizing the time and/or space domains, with adjustments made for the fractional derivative operator, which is approximated using explicit schemes such as the Grünwald-Letnikov approximation or related formulations. In this method, the solution at a future time step is directly computed using values from previous steps, resulting in a computationally straightforward scheme, though stability limitations must be considered.

The fractional Black-Scholes equation is an extension of the classical Black-Scholes model that incorporates a fractional time derivative to capture memory effects and non-Markovian behavior in financial markets. The equation is subject to the terminal condition $V(S, 0) = \max(K - S, 0)$, representing the payoff of a European put option, along with boundary conditions $V(0, \tau) = Ke^{-r\tau}$ and $V(S_{\max}, \tau) = 0$, reflecting option value behavior as the asset price approaches zero and infinity.

To solve the equation numerically, both the time and stock price domains are discretized. The stock price domain $[0, S_{\max}]$ is divided into M intervals of size $h = S_{\max}/M$, while the time domain $[0, T]$ is divided into N intervals of size $k = T/N$. The discrete grid points in space and time are denoted as $S_m = mh$ and $\tau_n = nk$, respectively. The approximate numerical solution at these grid points is denoted by $V_n^m \approx V(S_m, \tau_n)$.

The fractional time derivative is approximated using the Caputo definition, expressed as:

$$\frac{\partial^\alpha V}{\partial \tau^\alpha} \approx \frac{1}{\Gamma(2-\alpha)} k^{-\alpha} \sum_{j=1}^{n} \omega_j^{(\alpha)} (V_m^{n-j+1} - V_m^{n-j}), \tag{8}$$

where $\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$. The first and second spatial derivatives are approximated using standard finite difference formulas:

$$\frac{\partial V}{\partial S} \approx \frac{V_{m+1}^n - V_{m-1}^n}{2h}, \tag{9}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{m+1}^n - 2V_m^n + V_{m-1}^n}{h^2}. \tag{10}$$

Substituting all components into the fractional Black-Scholes equation results in a tridiagonal linear system at each time level τ_n. The resulting equation at grid point (m, n) is:

$$\frac{1}{\Gamma(2-\alpha)} k^{-\alpha} \left[\omega_1^{(\alpha)} (V_m^n - V_m^{n-1}) + \sum_{j=2}^{n} \omega_j^{(\alpha)} (V_m^{n-j+1} - V_m^{n-j}) \right] + \frac{\sigma^2 S_m^2}{2h^2} (V_{m+1}^n - 2V_m^n + V_{m-1}^n) - \frac{rS_m}{2h} (V_{m+1}^n - V_{m-1}^n) + rV_m^n = 0. \tag{11}$$

This system is structured into tridiagonal coefficients A_m, B_m, and C_m for V_{m-1}^n, V_m^n, and V_{m+1}^n, respectively:

$$A_m = -\frac{\sigma^2 S_m^2}{2h^2} - \frac{rS_m}{2h}, \tag{12}$$

$$B_m = \frac{1}{\Gamma(2-\alpha)} k^{-\alpha} \omega_1^{(\alpha)} + r + \frac{\sigma^2 S_m^2}{h^2}, \tag{13}$$

$$C_m = -\left(\frac{\sigma^2 S_m^2}{2h^2} - \frac{rS_m}{2h}\right), \tag{14}$$

The right-hand side (RHS) is computed using previous time step values:

$$\text{RHS}_m = \frac{1}{\Gamma(2-\alpha)} k^{-\alpha} \left[\omega_1^{(\alpha)} V_m^{n-1} + \sum_{j=2}^{n} \omega_j^{(\alpha)} (V_m^{n-j+1} - V_m^{n-j})\right]. \tag{15}$$

The tridiagonal system is solved using the Thomas algorithm, which involves forward elimination followed by backward substitution. This time-stepping procedure is repeated from $n = 1$ to $n = N$, applying boundary conditions $V_0^n = Ke^{-r\tau_n}$ and $V_M^n = 0$ at each step. The initial condition at $n = 0$ is given by:

$$V_m^0 = \max(K - S_m, 0), \tag{16}$$

representing the payoff of the European put option at maturity.

Upon completion of the iterations, the full numerical solution matrix $V[n][m]$ is obtained, representing the option price at each time and asset price grid point. The vector V_m^N gives the option price at $t = 0$ for various stock prices S_m. This solution can be visualized to analyze the behavior of option prices and the impact of the fractional order α on the dynamics of the model. Through this approach, a complete numerical solution to the fractional Black-Scholes equation is achieved using the implicit finite difference method.

The resulting matrix can then be visualized to explore the influence of the fractional order α on option pricing behavior. Figure 1 presents the numerical solution surfaces of the European call option prices generated using the finite difference method for several values of α.

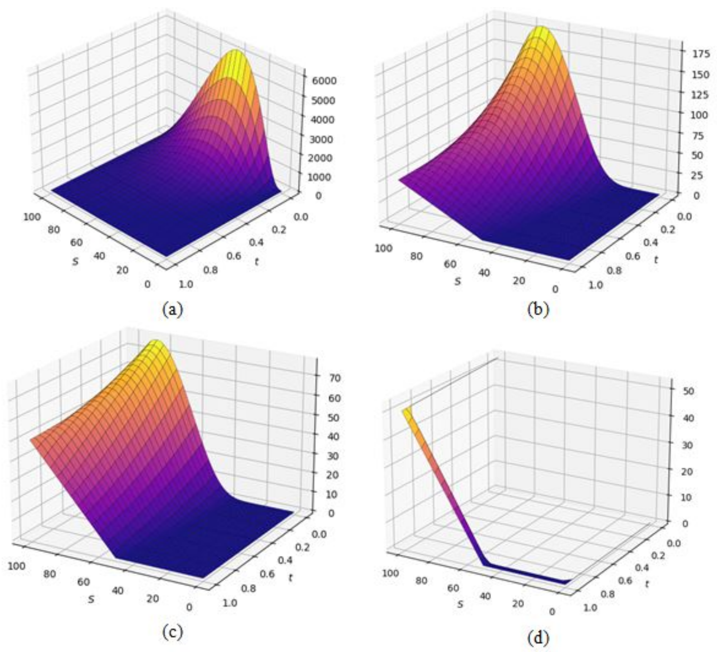

Figure 1: Graph of the solution with different α values: (a) $\alpha = 0.25$; (b) $\alpha = 0.75$; (c) $\alpha = 0.99$; (d) $\alpha = 1$.

Figure 1 presents the absolute error between the numerical and exact solutions. Errors remain minimal near the strike price, affirming grid-centered stability. The horizontal axes represent the asset price S and time t, while the vertical axis indicates the option value. As seen in the figure, panel (a) with $\alpha = 0.2$ exhibits a very steep surface with significantly high option prices, indicating a strong memory effect in the pricing dynamics. In panel (b), where $\alpha = 0.4$, the option value remains elevated but starts to decrease more smoothly. Panel (c), corresponding to $\alpha = 0.6$, shows a more regular and stable price surface, with a smoother gradient in time. Finally, panel (d) for $\alpha = 0.8$ illustrates a notably flatter surface, where the price evolution appears closer to the classical case, indicating reduced memory influence. These results reflect how increasing the fractional order α reduces the long-memory characteristics and brings the pricing behavior closer to the standard Black-Scholes model.

Example 1.

European put option pricing is based on the fractional Black-Scholes equation under the following parameters:

- Strike price: $K = 50$
- Risk-free interest rate: $r = 0.01$
- Volatility: $\sigma = 0.3$
- Time to maturity: $T = 1$
- Maximum stock price: $S_{\max} = 100$
- Number of spatial steps: $M = 20$
- Number of time steps: $N = 10$

The simulation is carried out for fractional orders $\alpha = 0.3, 0.5, 0.7, 0.9$. Option values are computed and plotted for time levels $t = 0.1, 0.5, 1.0$, as shown in Figure 2.

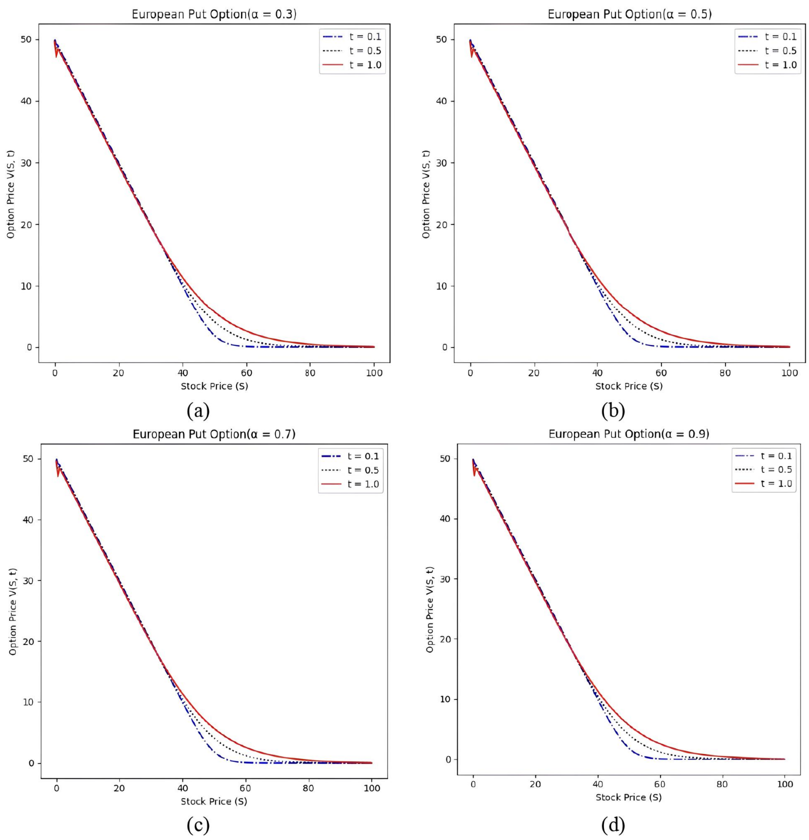

Figure 2: European put option prices for different fractional orders α at $t = 0.1, 0.5, 1.0$.

This visualization (Figure 2) demonstrates the impact of the fractional order α on the behavior of the European put option price over time. Smaller α (e.g., 0.3) leads to sharper changes in value, indicating more localized effects and less diffusion. In contrast, larger α (e.g., 0.9) results in smoother transitions, reflecting greater memory effects inherent in the fractional derivative model. These results highlight the flexibility of the fractional Black-Scholes model in capturing different levels of market memory and volatility behavior, offering a more realistic framework for option pricing in non-ideal financial environments.

Example 2.

Analyze the influence of the time-fractional order α on the pricing of a European call option. The option under consideration has a strike price of $K = 50$, a maturity of $T = 1$ year, a volatility of $\sigma = 0.4$, and a constant risk-free interest rate of $r = 5\%$. For the numerical scheme, the maximum stock price is set to $S_{\max} = 100$.

To investigate different market conditions, the call option price is evaluated under three scenarios: when the stock price equals the strike price ($S = K = 50$, at-the-money), when the stock price is less than the strike price ($S = 30 < K$, out-of-the-money), and when the stock price exceeds the strike price ($S = 70 > K$, in-the-money).

The option prices for each case are computed using the time-fractional Black-Scholes model, implemented through the implicit finite difference method. The analysis is performed for four different values of the fractional order α: 0.25, 0.5, 0.75, and 1.0. Here, $\alpha = 1.0$ corresponds to the classical Black-Scholes model, while $\alpha < 1$ captures the memory effects in asset price dynamics via the fractional extension.

Table 1: European call option prices computed using the FDM for various α values compared to the exact Black-Scholes solution

S	**FDM Method**				**Exact Solution**
	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1.0$	
$S < K$ (50)	0.1105	0.4918	0.9459	1.4067	1.4093
$S = K$ (60)	2.6949	3.8003	4.6997	5.4290	5.4896
$S > K$ (70)	10.4308	11.1226	11.6710	11.9301	12.5356

Table 1 compares numerical and analytical solutions. As α approaches 1, the model converges to the classical Black-Scholes pricing, validating the proposed numerical scheme. As the fractional order parameter α increases, the computed prices of European call options using the finite difference method become closer to the classical Black-Scholes solution. When the underlying stock price is below the strike price ($S < K$), the option is considered out-of-the-money, and its value is relatively low. In this case, smaller values of α yield significantly lower option prices compared to the exact solution. This underpricing indicates that the memory effect introduced by the fractional model with low α dampens the growth of the option value.

For at-the-money scenarios ($S = K$), the option price becomes more sensitive to changes in α. As α increases, the option value rises steadily and approaches the classical value when α reaches 1.0. This behavior demonstrates the transitional nature of the fractional model, where intermediate α values interpolate between a more constrained market dynamic (low α) and the classical Black-Scholes assumptions (when $\alpha = 1.0$).

In in-the-money cases ($S > K$), where the option has intrinsic value, the same trend persists. The finite difference method still underestimates the price for lower α, although the deviation is relatively smaller compared to the out-of-the-money region. When $\alpha = 1.0$, the numerical solution aligns closely with the exact analytical value, confirming the accuracy of the finite difference scheme and its consistency with the classical model in the limiting case.

The results highlight the significant influence of the fractional order α on option pricing. Lower α values introduce stronger memory effects and result in systematically lower option prices.

This demonstrates the potential of the time-fractional Black-Scholes model in capturing more complex market behaviors that deviate from the assumptions of the classical model.

The fractional Black-Scholes model demonstrates superior performance compared to the classical model under several non-ideal market conditions. Numerical results indicate that the fractional model provides more accurate and flexible pricing in environments where asset price dynamics exhibit memory effects, such as volatility clustering or persistent trends. In particular, lower values of the fractional order α (e.g., $\alpha = 0.25$ or 0.5) result in significantly higher option prices compared to the classical case ($\alpha = 1$), reflecting a stronger sensitivity to past market behavior.

This suggests that the fractional model is better suited for pricing options during periods of heightened uncertainty or market turbulence, where classical models tend to underestimate risk. Furthermore, the model implicitly accommodates heavy-tailed return distributions, which are common in real markets but not captured by the Gaussian assumptions of the classical framework. The increased sensitivity of the fractional model in short-term maturities also allows for more refined option price estimation near expiry.

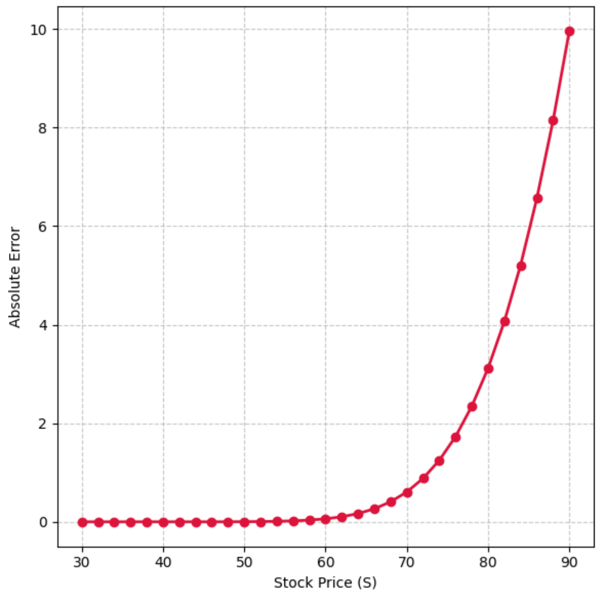

Figure 3: Absolute Error Between FDM ($\alpha = 1$) and Exact Black-Scholes Solution

Figure 3 presents the absolute error between the numerical solution obtained from the Finite Difference Method (FDM) with $\alpha = 1$ and the exact analytical solution derived from the classical Black-Scholes model for European call options. The error is computed for a range of asset prices from $S = 30$ to $S = 90$, with a fixed strike price of $K = 60$. The horizontal axis represents the underlying asset price, while the vertical axis shows the absolute difference between the two solutions.

The plot indicates that the FDM solution is highly accurate when the asset price is near the strike price ($S \approx K$), as the absolute error is very small in this region. This is expected, as the finite difference grid is typically centered around the strike price, allowing better approximation of the payoff function and its derivatives. In contrast, as the asset price increases beyond the strike, the error grows significantly. The sharp rise in error for $S > 70$ suggests that the numerical method becomes less accurate in pricing deep in-the-money call options.

Meanwhile, for values where $S < K$, the absolute error remains consistently low and stable. This behavior is reasonable because the value of a European call option is close to zero in this region, making it easier for the FDM to approximate. Overall, the results confirm that while the FDM with $\alpha = 1$ performs well near the strike price, its accuracy deteriorates at the upper boundary of the domain. Improvements such as a finer grid, better boundary conditions, or higher-order discretization schemes may be required to reduce errors for larger asset prices.

4 Conclusion

This study has presented a comprehensive numerical approach for solving the time-fractional Black-Scholes equation by utilizing both explicit and implicit finite difference methods (FDM). The main problem addressed was the limitation of the classical Black-Scholes model in capturing memory effects and time-dependent volatility, which are commonly observed in real financial markets. By incorporating the Caputo fractional derivative, the proposed model offers a more flexible and realistic framework for option pricing under non-Markovian conditions.

The simulation results show that the fractional order parameter α significantly affects the option pricing dynamics. Lower values of α lead to higher option prices, reflecting stronger memory effects. The numerical solutions obtained are stable, accurate, and converge to the classical Black-Scholes results as α approaches 1. This confirms that the method not only generalizes the classical model but also retains its validity in the limiting case.

In practical terms, the proposed fractional model, combined with the finite difference method, can be implemented in real financial environments, especially in markets characterized by irregular volatility or long-memory features. Its clarity of formulation, ease of discretization, and strong convergence behavior make it a viable tool for both academic research and industry applications.

A key contribution of this work lies in bridging theoretical developments in fractional calculus with practical computational tools, thereby enhancing the understanding of fractional dynamics in financial modeling.

Looking forward, future research may focus on extending this numerical framework to handle more complex derivatives such as American options, integrating stochastic volatility components, or developing adaptive parameter estimation techniques based on market data. Additionally, empirical validation using historical financial datasets could further establish the practical relevance and robustness of the fractional Black-Scholes model in real-world scenarios.

CRediT Authorship Contribution Statement

Elza Rahma Dihna: Contributed to the conceptualization, development of methodology, software implementation, data management, formal analysis, initial drafting, manuscript editing, and funding acquisition. **Endang Rusyaman**: Involved in reviewing and refining the manuscript, as well as providing supervision. **Sukono**: Took part in the critical review and improvement of the manuscript, and contributed to the supervision of the research work.

Declaration of Generative AI and AI-assisted technologies

The author acknowledges the use of generative AI tools in the preparation of this manuscript. ChatGPT (OpenAI) was utilized to assist in refining technical language and structuring the text, while DeepL Translator was employed for translating selected sections. All content was carefully reviewed and edited by the author to maintain precision and uphold academic standards. The final version reflects the author's own intellectual input and professional judgment.

Declaration of Competing Interest

The authors declare no competing interests.

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Data Availability

The data underlying this study's findings can be obtained from the corresponding author upon reasonable request and in accordance with applicable confidentiality agreements.

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