CAUCHY – Jurnal Matematika Murni dan Aplikasi
Volume 10 (2) (2025), Pages 1015-1025
p-ISSN: 2086-0382; e-ISSN: 2477-3344

A Remark on Resonance and Beat in a Homogeneous Linear Delay System

Robby*

Center for Mathematics and Society, Faculty of Science, Parahyangan Catholic University, Bandung 40141, Indonesia

Abstract

We study a linear system of delay differential equations with internal coupling. Analytical derivation of the characteristic equation and stability range shows that even without external forcing, delay interaction can cause resonance growth in one variable. This occurs when the internal feedback frequency equals the system's natural oscillation. When the two delays differ slightly, a beat pattern appears because of the small mismatch between their feedback frequencies. The analytical findings are confirmed through numerical simulation using the method of steps implemented in Python. The study explains how delay and coupling act as a self-exciting source of oscillation in linear systems.

Keywords: Beat; Hopf bifurcation; Resonance; System of delay differential equations.

1 Introduction

Resonance is a classical phenomenon in oscillatory systems, most notably observed in second-order ordinary differential equations (ODEs) subject to periodic external forcing. For example,

$$x''(t) + \omega^2 x(t) = \sin(\omega_0 t), \tag{1}$$

exhibits resonance when $\omega_0 = \omega$, leading to a particular solution of the form $t\sin(\omega t)$ and linear amplitude growth over time. When $\omega_0 \neq \omega$, a beat phenomenon with modulated amplitude occurs due to interference between nearby frequencies.

Delay differential equations (DDEs) continue to play a central role in modeling oscillatory and memory-dependent systems across physics, biology, and control engineering. Classical studies on scalar linear DDEs established that time delays can generate periodic behavior through Hopf bifurcations [1], [2], [3]. Building on these foundations, recent works have uncovered diverse phenomena ranging from delay-induced resonance and beat modulation to multistability and spectral complexity.

For instance, Ohira and Ohta [4] demonstrated resonant transients in scalar DDEs by tuning internal delay, while Wang et al. [5] analytically characterized phase and amplitude responses under harmonic forcing. Closed-form oscillatory solutions in homogeneous linear DDEs were obtained via Laplace transforms by Kerr et al. [6], and resonance behaviors due to state-dependent

*****Corresponding author**. E-mail: robby@unpar.ac.id

Submitted: July 05, 2025 Reviewed: August 31, 2025 Accepted: October 17, 2025
DOI: https://doi.org/10.18860/cauchy.v10i2.35283

delay were reported by Calleja et al. [7]. In higher-order systems, oscillation criteria have been developed for third-order or even-order DDEs [8], [9], [10], and comparison theorems for neutral and mixed-type delays have been advanced in [11], [12].

Further analytic frameworks include connections between resonance and the Lambert W function [13], delay-induced vibrational resonance in neuron models [14], and efficient computation of weakly coupled systems with delay [15]. Inverse problem formulations for impulsive DDEs were addressed by Ruhil and Malik [16]. On the spectral side, Breda et al. [17] proposed a unified characterization of stability and bifurcation behavior from the geometry of characteristic root curves, while Wang et al. [18] studied universal Hopf bifurcations in scalar systems.

At the network and control levels, Zhang et al. [19] analyzed delay-induced multistability in coupled oscillators, Singh and Rotea [20] investigated resonance via frequency response theory, and Liu et al. [21] provided a rigorous analytical approach to resonance in feedback control delay systems. Phase–amplitude response patterns in periodically modulated DDEs have been refined through harmonic balance analysis in [22].

While these developments reveal rich structures in delay-driven systems, ranging from scalar to networked, forced, or nonlinear, the specific case of resonance and beat phenomena arising solely from internal delayed coupling in a linear, homogeneous system remains analytically underrepresented. In this work, we show that such phenomena can emerge purely through internal dynamics without external forcing, using exact analytical derivations in a two-dimensional homogeneous DDE model. Although resonance in DDEs has been studied under external or parametric forcing, little attention has been given to self-excited resonance arising solely from internal delay coupling. This paper isolates that mechanism in a minimal two-dimensional linear system.

Motivated by these developments, we turn our attention to a simple yet illustrative model: a two-dimensional, linear, and homogeneous delay differential system with internal coupling

$$\begin{cases} x_1'(t) = a\,x_1(t-T_1), \\ x_2'(t) = a\,x_2(t-T_2) + b\,x_1(t-T_1). \end{cases} \tag{2}$$

This system is fully homogeneous and contains no external input; yet, as we show, it is capable of reproducing two hallmark behaviors typically associated with externally forced second-order ODEs: resonance and beat phenomena. To rigorously establish this claim, we begin by revisiting the stability structure of linear DDEs in the next section.

2 Linear Delay Differential Equation and Its Stability

Consider the simplest scalar case of a delay differential equation (DDE):

$$x'(t) = -\alpha x(t-T), \quad \text{for } t \geq 0, \tag{3}$$

where $\alpha > 0$ is a constant and $T > 0$ is the delay. Unlike ordinary differential equations (ODEs), the initial condition for a DDE is given by a function $x(t) = \phi(t)$ defined on the interval $t \in [-T, 0]$, commonly referred to as a history function. Although existence, uniqueness, and even explicit solutions can be studied, we will focus solely on the stability of the zero solution.

To facilitate stability analysis, introduce the scaled variables $\tau = t/T$, $\beta = \alpha T$, and define $u(\tau) = x(t)$. Then, equation (3) becomes

$$\frac{du}{d\tau} = \frac{1}{T} \cdot \frac{dx}{dt} = -\alpha T \cdot u\left(\frac{t}{T} - 1\right) = -\beta u(\tau - 1). \tag{4}$$

To examine the stability, we look for exponential solutions of the form $u(\tau) = Ce^{\lambda\tau}$ with $C \neq 0$. Substituting into (4), we obtain the characteristic equation:

$$C\lambda e^{\lambda\tau} = -\beta C e^{\lambda(\tau-1)} \iff e^{\lambda\tau}(\lambda + \beta e^{-\lambda}) = 0 \iff \lambda = -\beta e^{-\lambda}. \tag{5}$$

Details of the stability analysis for equation (5) can be found in [3] and [23]. It is established that the solution is asymptotically stable if and only if all roots λ of the characteristic equation satisfy $\Re(\lambda) < 0$. This condition is equivalent to $\beta \in (0, \pi/2)$, because outside this range one or more roots acquire positive real parts, destabilizing the equilibrium.

A Hopf bifurcation occurs when a pair of complex conjugate characteristic roots of a dynamical system crosses the imaginary axis, causing the steady state to lose stability and giving rise to periodic oscillations. In this system, the bifurcation occurs at the critical value $\beta = \pi/2$. At this point, the characteristic equation admits a pair of purely imaginary simple roots $\lambda = \pm i\pi/2$. Consequently, the solutions of (4) include oscillatory functions such as

$$x(t) = \sin\left(\frac{\pi t}{2T}\right), \quad \text{and} \quad x(t) = \cos\left(\frac{\pi t}{2T}\right).$$

We now generalize the analysis to a homogeneous linear system of delay differential equations of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t - T), \tag{6}$$

where A is a constant matrix. Using the same time-scaling transformation $\tau = t/T$, and defining $\mathbf{u}(\tau) = \mathbf{x}(t)$, equation (6) becomes

$$\frac{d\mathbf{u}}{d\tau} = B\mathbf{u}(\tau - 1), \quad \text{where } B = TA. \tag{7}$$

We nondimensionalize time by the dominant delay T so that the normalized matrix $B = TA$ fully determines stability independent of absolute delay scale. To analyze the stability of the system, we again look for exponential solutions of the form $\mathbf{u}(\tau) = e^{\lambda\tau}\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$. Substituting into (7) yields the nonlinear eigenvalue problem

$$\lambda e^{\lambda\tau}\mathbf{v} = Be^{\lambda(\tau-1)}\mathbf{v} \iff Be^{-\lambda}\mathbf{v} = \lambda\mathbf{v}. \tag{8}$$

The corresponding characteristic equation is

$$\det\left(\lambda I - Be^{-\lambda}\right) = 0. \tag{9}$$

As in the scalar case, the system is asymptotically stable if all characteristic roots λ satisfy $\Re(\lambda) < 0$. Particular attention is given to the Hopf bifurcation scenario, which occurs when a pair of complex conjugate roots crosses the imaginary axis.

As an example, reconsider the system given in (2). Applying the same scaling procedure, the system can be rewritten in the normalized form (7), with $B = \begin{bmatrix} aT & 0 \\ bT & aT \end{bmatrix}$.

Substituting into the characteristic equation (9), we obtain

$$\begin{aligned} 0 &= \det(\lambda I - Be^{-\lambda}) \\ &= \det\left(\begin{bmatrix} \lambda - aTe^{-\lambda} & 0 \\ -bTe^{-\lambda} & \lambda - aTe^{-\lambda} \end{bmatrix}\right) \\ &= \left(\lambda - aTe^{-\lambda}\right)^2. \end{aligned}$$

Thus, both eigenvalues satisfy the same scalar transcendental equation:

$$\lambda = aTe^{-\lambda},$$

which is identical in form to equation (5).

As previously established, the solution is asymptotically stable if and only if all roots satisfy $\Re(\lambda) < 0$, which is equivalent to

$$-\frac{\pi}{2} < aT < 0 \quad \iff \quad -\frac{\pi}{2T} < a < 0. \tag{10}$$

3 The Resonance and Beat Phenomena

Consider again the system defined in (2). From the previous section, we know that for $a < 0$ and $|aT_1| = \pi/2$, the solution of the first equation undergoes Hopf bifurcation, and admits a periodic solution

$$x_1(t) = \sin\left(\frac{\pi t}{2T_1}\right).$$

This expression will be used as the input to the second equation. We will examine two cases based on the relation between T_1 and T_2.

3.1 Case I: Resonance Phenomenon for $T_1 = T_2$

Assume $T_1 = T_2 = T$, then the equation for x_2 becomes

$$x_2'(t) = a\,x_2(t-T) + b\,\sin\left(\frac{\pi t}{2T}\right).$$

To find a particular solution, we propose

$$x_{2,p}(t) = A\,t\,\cos\left(\frac{\pi t}{2T}\right) + B\,t\,\sin\left(\frac{\pi t}{2T}\right),$$

where A and B are constants. Differentiating this yields

$$x_{2,p}'(t) = A\,\cos\left(\frac{\pi t}{2T}\right) - A\,\frac{\pi t}{2T}\sin\left(\frac{\pi t}{2T}\right) + B\,\sin\left(\frac{\pi t}{2T}\right) + B\,\frac{\pi t}{2T}\cos\left(\frac{\pi t}{2T}\right).$$

This derivative is substituted back into the original equation to verify that the form matches. On the right side of the equation, by using trigonometric identites, we obtain

$$a\left[A(t-T)\sin\left(\frac{\pi t}{2T}\right) - B(t-T)\cos\left(\frac{\pi t}{2T}\right)\right] + b\sin\left(\frac{\pi t}{2T}\right).$$

By collecting each term and defining $\theta := \pi t/(2T)$ for simplicity, we obtain

$$aAt\sin(\theta) - aBt\cos(\theta) + (b - aAT)\sin(\theta) + aBT\cos(\theta).$$

By matching each coefficients from two sides, we obtain a system of equation

$$\begin{cases} -A\pi/(2T) & = aA, \\ B\pi/(2T) & = -aB, \\ B & = b - aAT, \\ A & = aBT. \end{cases}$$

From the first two equations, for nontrivial A and B, we get $a = -\pi/(2T)$, which is exactly the Hopf threshold that we obtain before. Using $aT = -\pi/2$ in the last two equations, we obtain $A = -B\pi/2$ and $B = \pi A/2 + b$. Solving them yields

$$B = \frac{\pi}{2}\cdot\left(-\frac{\pi}{2}B\right) + b = \frac{4b}{4+\pi^2} \quad \text{and} \quad A = -\frac{2\pi b}{4+\pi^2}.$$

Hence, the particular solution

$$x_{2,p}(t) = \frac{2b}{4+\pi^2}\,t\,\left(-\pi\cos\left(\frac{\pi t}{2T}\right) + 2\sin\left(\frac{\pi t}{2T}\right)\right)$$

exhibits linear growth.

The result confirms that the solution grows linearly in amplitude, a signature of resonance, due to the matching frequency between the homogeneous part and the forcing.

3.2 Case II: Beat Phenomenon for $T_1 \neq T_2$

Consider the modified system

$$\begin{cases} x_1'(t) = a_1\, x_1(t - T_1), \\ x_2'(t) = a_2\, x_2(t - T_2) + b\, x_1(t - T_1). \end{cases} \tag{11}$$

Now suppose $T_1 \neq T_2$. The solution of the first equation remains

$$x_1(t) = \sin(\omega_1 t)\,, \qquad \text{where } \omega_1 = \frac{\pi}{2T_1}.$$

Substituting into the second equation yields

$$x_2'(t) = a_2\, x_2(t - T_2) + b\, \sin(\omega_1 t).$$

To find a particular solution, we consider the ansatz

$$x_{2,p}(t) = A\, \cos(\omega_1 t) + B\, \sin(\omega_1 t),$$

where A and B are constants. Then

$$\begin{aligned} x_{2,p}'(t) &= -A\, \omega_1 \sin(\omega_1 t) + B\, \omega_1 \cos(\omega_1 t), \text{and} \\ x_{2,p}(t - T_2) &= A \cos(\omega_1 (t - T_2)) + B \sin(\omega_1 (t - T_2)). \end{aligned}$$

Using the trigonometric identities, we substitute and group by $\cos(\omega_1 t)$ and $\sin(\omega_1 t)$, yielding the system

$$\begin{cases} B\, \omega_1 = a_2 \left(A \cos(\omega_1 T_2) - B \sin(\omega_1 T_2) \right), \\ -A\, \omega_1 = a_2 \left(A \sin(\omega_1 T_2) + B \cos(\omega_1 T_2) \right) + b. \end{cases}$$

The solution for this system is

$$A = -\frac{b(\omega_1 + a_2 \sin(\omega_1 T_2))}{(\omega_1 + a_2 \sin(\omega_1 T_2))^2 + (a_2 \cos(\omega_1 T_2))^2} \text{ and } B = -\frac{a_2 b \cos(\omega_1 T_2)}{(\omega_1 + a_2 \sin(\omega_1 T_2))^2 + (a_2 \cos(\omega_1 T_2))^2}.$$

This gives the particular solution

$$x_{2,p}(t) = -\frac{b(\omega_1 + a_2 \sin(\omega_1 T_2)) \cos(\omega_1 t) + a_2 b \cos(\omega_1 T_2) \sin(\omega_1 t)}{(\omega_1 + a_2 \sin(\omega_1 T_2))^2 + (a_2 \cos(\omega_1 T_2))^2}.$$

To interpret these oscillations more clearly, it is useful to recall how the system parameters relate to the Hopf threshold for purely trigonometric behavior. Consider the second equation in (11):

$$x_2'(t) = a_2\, x_2(t - T_2).$$

Its characteristic equation is $\lambda = a_2 e^{-\lambda T_2}$, whose roots become purely imaginary, $\lambda = \pm i\omega_2$, when

$$\omega_2 T_2 = \frac{\pi}{2}, \qquad a_2 = -\frac{\pi}{2T_2}.$$

At this Hopf threshold, the homogeneous solution of x_2 is a pure oscillation

$$x_2(t) = A \sin\left(\frac{\pi t}{2T_2}\right) + B \cos\left(\frac{\pi t}{2T_2}\right)$$

with no exponential growth or decay. By setting $a_1 = a_2 = -\pi/(2T_i)$ for both components, x_1 and x_2 each oscillate with their own natural frequencies

$$\omega_1 = \frac{\pi}{2T_1} \quad \text{and} \quad \omega_2 = \frac{\pi}{2T_2}.$$

When $T_1 \neq T_2$, these frequencies are slightly different, and the coupling term $b\,x_1(t - T_1)$ induces a superposition of two close oscillations in x_2, producing a slow envelope modulation,

$$x_2(t) \approx 2C \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t\right),$$

which corresponds to the beat phenomenon. If $T_1 = T_2$, the two frequencies coincide and the modulation disappears, giving the resonance case discussed earlier.

The resulting solution $x_2(t)$ thus consists of a particular component with frequency ω_1 and a homogeneous component with frequency ω_2. Their superposition produces amplitude modulation when $\omega_1 \neq \omega_2$, giving rise to the classical beat phenomenon

$$x_2(t) \sim \cos(\Delta\omega\, t) \cos(\bar{\omega}\, t), \quad \Delta\omega = \frac{\omega_1 - \omega_2}{2}, \quad \bar{\omega} = \frac{\omega_1 + \omega_2}{2}.$$

This beat produces envelope period $T_{\text{beat}} = 2\pi/|\omega_1 - \omega_2|$.

Having derived the analytical forms of the resonance and beat responses, we next verify these results through direct numerical simulation. The next chapter presents the implementation of the method of steps in Python, the selected parameter sets, and the resulting time-series plots demonstrating the predicted resonance amplification and beat modulation.

4 Numerical Simulation

4.1 Method

The delay differential system studied in this paper cannot, in general, be solved in closed form once numerical parameters and distinct delays are introduced. To compute approximate solutions, we apply the method of steps combined with a fourth–order Runge-Kutta integration scheme.

The method of steps converts a DDE into a sequence of ODEs on consecutive time intervals. If the delay is τ and a continuous history function $x(t) = \phi(t)$ is known for $t \in [-\tau, 0]$, then

1. on the first interval $[0, \tau]$, the delayed term $x(t - \tau)$ is known from $\phi(t)$, so the equation reduces to an ODE that can be solved numerically,
2. the obtained solution over $[0, \tau]$ becomes the new history function used to compute the next interval $[\tau, 2\tau]$, and so on.

This recursive construction ensures continuity of $x(t)$ and allows the solution to be built step by step over the integration range $[0, T_{\max}]$.

Each ODE segment generated by the method of steps is integrated using the classical fourth–order Runge-Kutta (RK4) method. For an ODE $x' = f(t, x)$ with time step h, where t_n and x_n denote the current time and numerical solution, the classical RK4 update reads

$$\begin{aligned} k_1 &= f(t_n,\, x_n), \\ k_2 &= f\Big(t_n + \tfrac{h}{2},\, x_n + \tfrac{h}{2} k_1\Big), \\ k_3 &= f\Big(t_n + \tfrac{h}{2},\, x_n + \tfrac{h}{2} k_2\Big), \\ k_4 &= f(t_n + h,\, x_n + h\, k_3), \\ x_{n+1} &= x_n + \tfrac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4), \end{aligned}$$

where $t_{n+1} = t_n + h$ and x_{n+1} denotes the numerical approximation of $x(t_{n+1})$. For DDEs, the function f depends on both the current value $x(t)$ and the delayed value $x(t - \tau)$. The delayed term is obtained from the previously computed segment, either by direct lookup or (usually) by linear interpolation.

4.2 Simulation Results

Numerical experiments are implemented in Python (version 3.10) in Google Colab using:

- `numpy` for array operations,
- `matplotlib` for plotting, and
- `scipy.interpolate` for linear interpolation of delayed values.

The integration follows the method of steps with a classical RK4 update on each subinterval. Delayed terms $x(t-\tau)$ are obtained from previously computed segments by linear interpolation.

The history on $[-\max\{T_1, T_2\}, 0]$ is prescribed as

$$x_1(t) = x_2(t) = \phi(t) = \sin(t).$$

We integrate on $[0, T_{\max}]$ with uniform step size h; $t_n = nh$, $n \in \{0, 1, \ldots, \lfloor T_{\max}/h \rfloor\}$.

For the parameters, consider two scenarios:

- Resonance case ($T_1 = T_2$): $b = 0.2, \quad T_1 = T_2 = 1.$
- Beat case ($T_1 \neq T_2$): $b = 0.2, \quad T_1 = 1, \quad T_2 = 1.1.$

For both cases, we use $h = 10^{-2}$, $T_{\max} = 150$, a, a_1, and a_2 are all Hopf threshold, also the same RK4 scheme and interpolation rule are used. The resulting trajectories $x_1(t)$ and $x_2(t)$ are plotted over $[0, T_{\max}]$.

Fig. 1 shows the numerical trajectories of $x_1(t)$ and $x_2(t)$ for the resonance case where $T_1 = T_2 = 1$. Both variables oscillate with the same frequency, but the amplitude of $x_2(t)$ increases approximately linearly with time, confirming the predicted resonance amplification due to constructive feedback between the two delayed components.

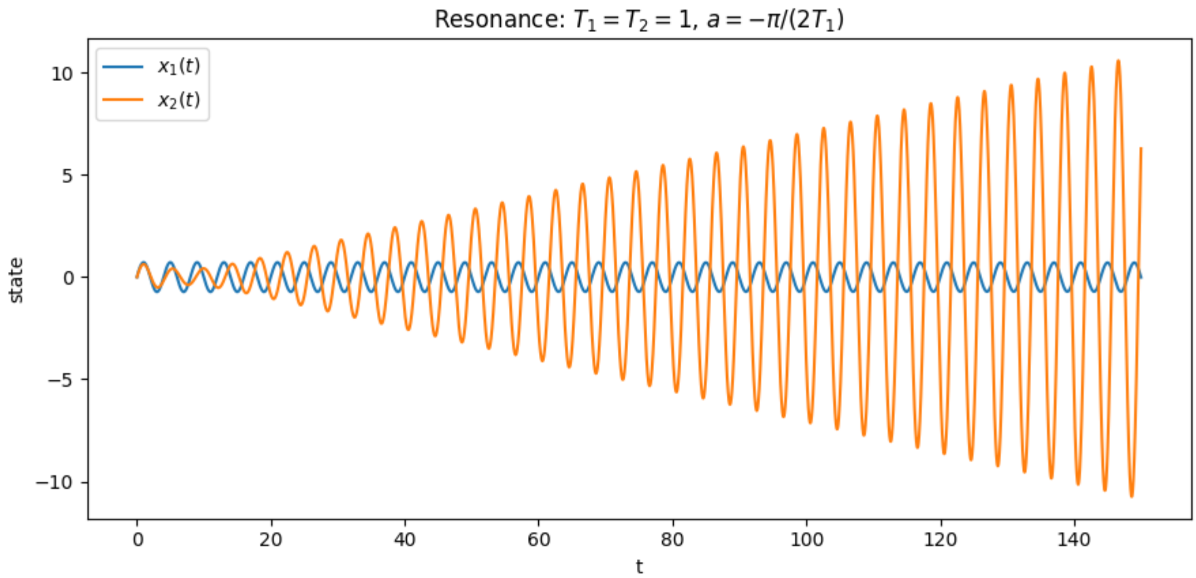

Figure 1: Time evolution of $x_1(t)$ and $x_2(t)$ under the resonance condition $T_1 = T_2 = 1$.

Fig. 2 presents the beat case with slightly different delays, $T_1 = 1$ and $T_2 = 1.05$. The small detuning between the internal feedback frequencies produces slow amplitude modulation: the two signals alternate between constructive and destructive interference, generating a characteristic beat envelope.

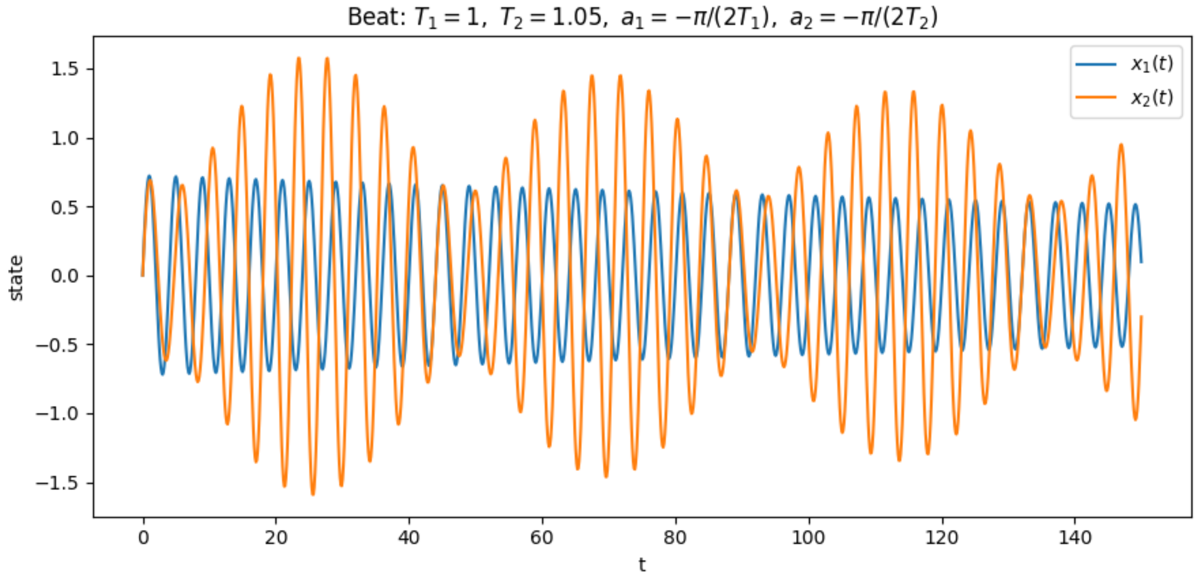

Figure 2: Time evolution of $x_1(t)$ and $x_2(t)$ under the beat condition $T_1 = 1$ and $T_2 = 1.05$.

The numerical trajectories in Fig. 1 and Fig. 2 reproduce the behaviors predicted analytically in Chapter 3. In the resonance case, the in–phase feedback between delayed components causes the effective forcing to coincide with the system's natural frequency, producing linear growth in amplitude. In the beat case, the small mismatch between the two feedback delays introduces detuning, which manifests as slow modulation of the oscillation envelope. These results confirm that the derived coefficients and parameter relations accurately capture the transition from steady oscillation to resonant amplification and beat modulation.

5 Conclusion and Suggestion

5.1 Conclusion

This paper has demonstrated that resonance amplification and beat modulation can arise within a purely linear and homogeneous delay differential system, without the presence of any external forcing. Unlike classical forced oscillators, where such phenomena are externally induced, here they emerge solely through internal delayed coupling and feedback.

These findings highlight a structural insight: time delays, even in the absence of nonlinearity or external inputs, can fundamentally enrich system dynamics. Aligned delay times induce resonance, while mismatched delays yield beat patterns, all of which are captured analytically.

Beyond offering a new perspective on delay-induced dynamics, the model provides an analytically tractable example that may serve as a benchmark for future investigations in delay systems, both theoretical and applied. This remark, albeit simple, illustrates the subtle dynamical richness that can arise in delay systems, inviting further exploration beyond classical ODE intuition. The analysis reveals that even a minimal two-dimensional delay system reproduces key signatures of classical forced oscillators purely through internal feedback.

5.2 Suggestion

While the system analyzed here is intentionally minimal, it raises several directions for future investigation. One natural extension is to explore the structural stability of the resonance and beat phenomena under perturbations, such as small deviations in delay values, parameter changes, or variations in initial conditions. Another promising direction involves higher-dimensional delay-coupled systems, where network topology may influence the emergence or suppression of internal frequency interactions. Finally, the inclusion of weak nonlinearities could reveal new bifurcation structures or amplitude saturation mechanisms, providing a richer dynamical landscape beyond the linear case. Overall, the study highlights delay as a self-contained excitation mechanism capable of producing complex oscillatory behavior even in linear homogeneous systems.

CRediT Authorship Contribution Statement

The author solely conducted all aspects of this work by the CRediT (Contributor Roles Taxonomy): Conceptualization, Methodology, Software, Validation, Formal Analysis, Investigation, Writing - Original Draft Preparation, Writing - Review & Editing, Supervision, Project Administration, and Funding Acquisition.

Declaration of Generative AI and AI-assisted technologies

The author used ChatGPT (version 5, developed by OpenAI) solely to refine language and grammar during the preparation of this manuscript. All ideas, methodologies, results, and interpretations are entirely the original work of the author. No AI-generated content contributed to the conceptual, analytical, or substantive aspects of the research.

Declaration of Competing Interest

The authors declare no competing interests.

Funding and Acknowledgments

This research received no external funding and was fully supported by the author's personal resources. The author would also like to acknowledge the use of institutional facilities and academic support provided informally during the course of this work.

Data and Code Availability

No external data were used in this research. All results are based on original theoretical analysis conducted by the author. Numerical simulations were implemented in Python using self-developed scripts. These codes are not publicly released but can be provided upon reasonable request to the author.

References

[1] T. Erneux and H. Smith, *Delay Differential Equations with Applications in Population Dynamics* (Applied Mathematical Sciences). Springer, 1998, vol. 57.

[2] Y. Kuang, *Delay Differential Equations: With Applications in Population Dynamics*. Academic Press, 1993, p. 398.

[3] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences* (Texts in Applied Mathematics). Springer, 2010, vol. 57.

[4] K. Ohira and T. Ohta, "Resonating delay equation: Delay-tuned resonant transients in a simple dde," *Journal of Mathematical Analysis and Applications*, vol. 507, no. 1, pp. 125–140, 2022. DOI: 10.1016/j.jmaa.2021.125140.

[5] Y. Wang, J. Cao, and J. Kurths, "Phase and amplitude responses for delay equations using harmonic analysis," *Physical Review E*, vol. 110, no. 1, p. 012 202, 2024. DOI: 10.1103/PhysRevE.110.012202.

[6] G. Kerr, N. Lopez, and G. González-Parra, "Analytical solutions of systems of linear delay differential equations by the laplace transform: Featuring limit cycles," *Mathematical and Computational Applications*, vol. 29, no. 1, p. 11, 2024. DOI: 10.3390/mca29010011.

[7] R. C. Calleja, A. R. Humphries, and B. Krauskopf, "Resonance phenomena in a scalar delay differential equation with two state-dependent delays," *SIAM Journal on Applied Dynamical Systems*, vol. 16, no. 3, pp. 1474–1513, 2017. DOI: 10.1137/16M1083212.

[8] A. Al-Jaser, I. F. B. Saoud, B. Qaraad, and H. Ramos, "Investigation of the oscillatory behavior of the solutions of a class of third-order delay differential equations with several terms," *Axioms*, vol. 13, no. 10, p. 703, 2024. DOI: 10.3390/axioms13100703.

[9] Z. Alqahtani, I. F. B. Saoud, A. Almuneef, B. Qaraad, and H. Ramos, "New criteria for the oscillation of a class of third-order quasilinear delay differential equations," *AIMS Mathematics*, vol. 10, no. 2, pp. 4205–4225, 2025. DOI: 10.3934/math.2025195.

[10] O. Moaaz and H. Ramos, "An improved oscillation result for a class of higher order non-canonical delay differential equations," *Mediterranean Journal of Mathematics*, vol. 20, no. 3, p. 166, 2023. DOI: 10.1007/s00009-023-02446-1.

[11] B. Almarri, F. Masood, A. Muhib, and O. Moaaz, "New comparison results for oscillation of even-order delay differential equations," *Symmetry*, vol. 14, no. 5, p. 946, 2022. DOI: 10.3390/sym14050946.

[12] A. K. Alsharidi and A. Muhib, "Oscillation criteria for mixed neutral differential equations," *AIMS Mathematics*, vol. 9, no. 1, pp. 14 473–14 489, 2024. DOI: 10.3934/math.202414473.

[13] K. Ohira and T. Ohira, "Delay, resonance and the lambert w function," *Physica D: Nonlinear Phenomena*, vol. 454, p. 133 825, 2023. DOI: 10.1016/j.physd.2023.133825.

[14] M. S. Siewe, S. Rajasekar, M. Coccolo, and M. A. F. Sanjuán, "Vibrational resonance in the fitzhugh–nagumo neuron model under state-dependent time delay," *Physical Review E*, vol. 110, no. 1, p. 012 203, 2024. DOI: 10.1103/PhysRevE.110.012203.

[15] V. Kumar, "A numerical scheme for a weakly coupled system of singularly perturbed delay differential equations," *Journal of Applied & Computational Mathematics*, vol. 11, no. 5, pp. 1–8, 2022. DOI: 10.37421/jacm.2022.11.488.

[16] S. Ruhil and M. Malik, "Inverse problem for abstract delay differential equations with impulsive effects," *AIMS Mathematics*, vol. 9, no. 4, pp. 8412–8432, 2024. DOI: 10.3934/math.2024384.

[17] D. Breda, G. Menegon, and M. Nonino, "Delay equations and characteristic roots: Stability and more from a single curve," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 89, pp. 1–22, 2018. DOI: 10.14232/ejqtde.2018.1.89.

[18] Y. Wang, J. Cao, J. Kurths, and S. Yanchuk, "Universal bifurcation scenarios in delay-differential equations with one delay," *Journal of Dynamics and Differential Equations*, 2023. DOI: 10.1007/s10884-023-10275-w.

[19] L. Zhang, X. Chen, and Q. Meng, "Delay-induced multistability and resonance in time-delay coupled networks," *Chaos*, vol. 33, no. 6, p. 063 116, 2023. DOI: 10.1063/5.0138724.

[20] R. Singh and M. T. Rotea, "Frequency response and resonance in linear time-delay systems," *International Journal of Control*, vol. 95, no. 12, pp. 2802–2815, 2022. DOI: 10.1080/00207179.2022.2032497.

[21] W. Liu, P. Shi, and J. M. Hill, "Analytical characterization of delay-induced resonance in feedback control systems," *IEEE Transactions on Cybernetics*, vol. 54, no. 2, pp. 1130–1141, 2024. DOI: 10.1109/TCYB.2023.3234567.

[22] R. Nicks, R. Allen, and S. Coombes, "Phase and amplitude responses for delay equations using harmonic balance," *Physical Review E*, vol. 110, no. 1, p. 012 202, 2024. DOI: 10.1103/PhysRevE.110.012202.

[23] Robby and J. Widjaja, "Stability analysis of mixing problem with cascading tanks," in *Proceedings of The Second International Conference on Mathematical Analysis and Its Applications (ICONMAA 2024)*, ser. ITM Web of Conferences, Published online 21 February 2025, vol. 75, 2025, p. 02 007. DOI: 10.1051/itmconf/20257502007.