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Notes on 2-Normed Spaces Through Their Quotient Spaces

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Abstract

In this paper, we defined new norms in 2-normed spaces derived from the 2-norm with respect to its quotient spaces. Moreover, these norms were used to observe some aspects of 2-normed spaces, namely a Convergent sequence, a Cauchy sequence, completeness, a closed set, and a bounded set. Furthermore, we used these aspects to prove the Fixed-Point Theorem in a 2-Banach Space.

Keywords: 2-Banach Space; 2-normed spaces; fixed point theorem; quotient space.

1 Introduction

A normed space is a vector space that is equipped with a norm. Let X be a vector space, then a function $\|\cdot\| : X \to \mathbb{R}$ that satisfies

1. $\|x\| \geq 0$, for all $x \in X$;
2. $\|x\| = 0$, if and only if $x = 0$ and $x \in X$;
3. $\|\alpha x\| = |\alpha|\|x\|$, for any $\alpha \in \mathbb{R}$ and $x \in X$;
4. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$;

is called a norm. A pair $(X, \|\cdot\|)$ is called a normed space [1].

Moreover, in the 1960s Gähler introduced a concept that is generalized from the concept of normed spaces [2], [3], [4], [5], [6], [7]. Since then, many researchers have developed some aspects in this space, see for instance [8], [9], [10], [11], [12], [13].

Let X be a vector space with $\dim(X) \geq 2$, a function $\|\cdot, \cdot\| : X \times X \to \mathbb{R}$ that satisfies:

1. $\|x, y\| \geq 0$;
2. $\|x, y\| = 0$ if and only if x, y are linearly dependent;
3. $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$;
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$;

is called 2-norm and the pair $(X, \|\cdot, \cdot\|)$ is called a 2-normed space [14].

For example, $\mathbb{R}^2$ is a 2-normed space with the 2-norm defined by

$$\|x_1, x_2\| = \left| \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right|.$$

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with $x_i = (x_{i1}, x_{i2})$ and $i = 1, 2$. Moreover, a function $\|x, y\|_g : \mathbb{R}^2 \to \mathbb{R}$ that is defined by

$$\|x, y\|_g = \left(\det \begin{pmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{pmatrix} \right)^{\frac{1}{2}},$$

is also a 2-norm on $\mathbb{R}^2$. This 2-norm is called a standard norm on $\mathbb{R}^2$. This 2-norm can be interpreted as an area that is spanned by vectors x_1, x_2 [15]. In [16], one can see that in a 2-normed space we have

$$\|x_1 + \alpha x_2, x_2\| = \|x_1, x_2\| \tag{1}$$

Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space. We can define a function on a 2-normed space, which is defined

$$\|x\|_* = \|x, y\| + \|x, z\|, \tag{2}$$

with y, z are linearly independent vectors on X. One can check that this function that defined on (2) is a norm derived from the 2-norm. Thus $(X, \|\cdot\|_*)$ is a normed space. The readers can also compare these results with those in [16].

Several previous researchers who have investigated aspects in 2-normed spaces have not employed the viewpoint of quotient spaces. They observed various aspects in 2-normed spaces using either the 2-norm itself or norms derived from the 2-norm. Furthermore, this study provides a new perspective by examining these aspects through a new tool, namely the quotient space. This new viewpoint offers a deeper understanding of the aspects that have been studied so far. Moreover, under this new perspective, a fixed point theorem in 2-normed spaces is established.

In the next section, we will observe the 2-normed space using norms of its quotient space. Quotient spaces in normed spaces play a central role in functional analysis because they allow us to simplify complex structures by "factoring out" subspaces while still retaining a meaningful normed structure. They provide a natural framework for understanding equivalence classes of functions or vectors, especially when analyzing kernels of linear operators and factor spaces.

Moreover, quotient constructions are essential in the development of Banach space theory, duality, and applications to differential equations and optimization.

2 Preliminaries

In this study, we conducted a literature-based investigation centered on the concept of 2-normed spaces. Our first step was to define quotient spaces that arise from a given 2-normed space. Within these quotient spaces, we introduced several norms derived from the original 2-norm. These newly defined norms provide a useful perspective for examining topological properties such as the behavior of sequences, completeness, closed sets, and finite sets. These insights form the groundwork for establishing and proving a Fixed Point Theorem within the framework of 2-normed spaces.

To construct the main tool, let us define quotient spaces on a 2-normed space. Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$ be a linearly independent set. Consider a set $A \setminus \{a_2\} = \{a_1\}$. Next, we define a subspace on X that is spanned by $A \setminus \{a_2\} = \{a_1\}$:

$$A_1 = \text{span}\{a_1\} = \{\alpha a_1 : \alpha \in \mathbb{R}\}.$$

Next, for any $u \in X$, the coset that corresponds to u in X with respect to A_1 is

$$\bar{u} = u + \text{span}\{a_1\} = \{u + \alpha a_1 : u \in X \text{ and } \alpha \in \mathbb{R}\}.$$

Thus, one can check that

1. $\bar{0} = \text{span}\{a_1\} = A_1$,
2. Let $u, w \in X$, $\bar{u} = \bar{w}$ if and only if $u - w \in A_1$.

Moreover, we have a quotient space $X_1^* = X/A_1 = \{\bar{u} : u \in X\}$. Next, one can see that for any $\bar{u}, \bar{w} \in X_1^*$ and $\alpha \in \mathbb{R}$, then

1. $\bar{u} + \bar{w} = \overline{u + w}$,
2. $\alpha\bar{u} = \overline{\alpha u}$.

In this space, we define a function $\|\cdot\|_1^* : X_1^* \to \mathbb{R}$ defined by

$$\|\bar{u}\|_1^* := \|u, a_1\|. \tag{3}$$

Remark 1. *Note that the function defined in (3) is well defined. For any $\overline{u}, \overline{u} \in X_1^*$ with $\overline{u}, \overline{u}$, we have $u - w \in A_1$, which means $u - w = \alpha a_1$ or $u = w + \alpha a_1$. Then we have $\|u, a_1\| = \|w + \alpha a_1, a_1\|$, by equation (1) we can write $\|u, a_1\| = \|w, a_1\|$. As a result, $\|\overline{u}\|_1^* = \|\overline{w}\|_1^*$.*

Based on this function, we have the following theorem.

Theorem 1. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, then $(X_1^*, \|\cdot\|_1^*)$ is a normed space.*

Proof. The proof is quite easy, we need to prove that the function defined in (3) is a norm. Let $A = \{a_1, a_2\}$ be a linearly independent set in X. Using properties of 2-norm, we have

i. If $\bar{u} = \bar{0}$, then it is clear that $\|\bar{u}\|_1^* = \|u, a_1\| = 0$. Conversely, if $\|\bar{u}\|_1^* = 0$, then $\|u, a_1\| = 0$, which means u, a_1 are linearly dependent. We can write $u = \alpha a_1$, $\alpha \in \mathbb{R}$. Thus $u \in A_1$, then $\bar{u} = \bar{0}$.

ii. For any $\alpha \in \mathbb{R}$,
$$\|\alpha\bar{u}\|_1^* = \|\alpha u, a_1\| = |\alpha|\|u, a_1\| = |\alpha|\|\bar{u}\|_1^*.$$

iii. Let $\bar{u}, \bar{w} \in X_1^*$, then
$$\|\bar{u} + \bar{w}\|_1^* = \|u + w, a_1\| \leq \|u, a_1\| + \|w, a_1\| = \|\bar{u}\|_1^* + \|\bar{w}\|_1^*.$$

Thus, $\|\cdot\|_1^*$ is a norm on X_1^*, as a consequence $(X_1^*, \|\cdot\|_1^*)$ is a normed space. □

We have defined $(X_1^*, \|\cdot\|_1^*)$ as a normed space, where X_1^* is a quotient space with respect to A_1. Next, we will construct another quotient set with respect to the following set. Now we consider the set

$$A \setminus \{a_1\} = \{a_2\}.$$

Using the same method to define the quotient space, we have a subspace that is spanned by A_2, namely

$$A_2 = \text{span}\{a_2\} = \{\beta a_2 : \beta \in \mathbb{R}\}.$$

We also have,

1. $\bar{0} = \text{span}\{a_2\} = A_2$,
2. Let $s, t \in X$, $\bar{s} = \bar{t}$ if and only if $s - t \in A_2$.

The quotient space is $X_2^* = X/A_2 = \{\bar{u} : u \in X\}$. One can see that for any $\bar{u}, \bar{w} \in X_2^*$ and $\alpha \in \mathbb{R}$, then

1. $\bar{u} + \bar{w} = \overline{u + w}$,
2. $\alpha\bar{u} = \overline{\alpha u}$.

In this space, we define a function $\|\cdot\|_2^* : X_2^* \to \mathbb{R}$ defined by

$$\|\bar{u}\|_2^* := \|u, a_2\|. \tag{4}$$

Note that the function defined in (4) is well defined. The reader can verify this by referring to Remark 1. By replacing the vector a_1 with a_2 the desired result can be obtained. We also have the following theorem.

Theorem 2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, then $(X_2^*, \|\cdot\|_2^*)$ is a normed space, where $\|\cdot\|_2^*$ is a function defined on (4)*

Proof. We prove this theorem by showing that $\|\cdot\|_2^*$ is a norm in X_2^*. In the proof Theorem 1, we use a linearly independent vector a_1. To prove this theorem, we can replace vector a_1 with a_2. By following the proof steps presented in Theorem 1, this theorem can be established. □

As we can see, we got two quotient spaces with this type of construction (by "eliminating" one vector from a linearly independent set $A = \{a_1, a_2\}$). We collect these two quotient spaces in a set and call it class-1.

Next, we construct another quotient space by "eliminating" two vectors from A using the same steps as the construction earlier. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$. Consider the set $A \setminus \{a_1, a_2\} = \emptyset$. We define a subspace of X that is spanned by $A \setminus \{a_1, a_2\}$, namely $A_{1,2} = \text{span}(A \setminus \{a_1, a_2\}) = \{0\}$.

The coset that correspond with $u \in X$ is $\bar{u} = \{u + \text{span}(A \setminus \{a_1, a_2\})\} = \{u\}$. This means:

1. $\bar{0} = \{0\}$,
2. Let $u, v \in X$, $\bar{u} = \bar{v}$ if and only if $u - v \in \bar{0}$ which means $u = v$.

Moreover, the quotient space in X with respect to $A_{1,2}$ is defined as

$$X_{1,2}^* = X/A_{1,2} = \{\bar{u} : u \in X\} = X.$$

We define a function $\|\cdot\|_{1,2}^* : X_{1,2}^* \to \mathbb{R}$ defined by

$$\|\bar{u}\|_{1,2}^* := \|u, a_1\| + \|u, a_2\|. \tag{5}$$

Theorem 3. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, then $(X_{1,2}^*, \|\cdot\|_{1,2}^*)$ is a normed space, where $\|\cdot\|_{1,2}^*$ is a function defined on (5).*

Proof. The proof is analogous to the proof of Theorems 1 and 2. A simple explanation will be elaborated in the following Remark. □

Remark 2. *One can see that using the last construction (by "eliminating" two vectors of Y). We have one quotient space. We put it in a set and called it class-2. Furthermore, one can also see that this quotient space we got from the construction is actually X itself. Thus, the norm that we defined on $X_{1,2}^*$ is the same norm we mentioned earlier in equation (2). The process of the last construction and Theorem 3 can be another simple way to explain how we got the norm defined in equation (2).*

There is a relation among norms of quotient spaces of class-1 and class-2, as we can see in equations (3), (4), and (5), we have

$$\|\bar{u}\|_{1,2}^* = \|u, a_1\| + \|u, a_2\| = \|\bar{u}\|_1^* + \|\bar{u}\|_2^*.$$

Using these norms of quotient spaces of class-1 and class-2, we will define some topological properties of 2-normed spaces.

3 Results and Discussion

Using the construction norms from the previous section, we will investigate some aspects of a 2-normed space. We start with sequences.

Definition 1. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$ be a linearly independent set in X. A sequence $\{x_n\} \subset X$ is said to converge to $x \in X$ with respect to class-1, if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, we have $\|\overline{x_n - x}\|_i^* < \varepsilon$, for each $i \in \{1, 2\}$.*

We use all norms of quotient spaces in class-1 in Definition 1. Next, we define a convergent sequence with respect to class-2 as follows.

Definition 2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$ be a linearly independent set in X. A sequence $\{x_n\} \subset X$ is said to converge to $x \in X$ with respect to class-2, if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ we have $\|\overline{x_n - x}\|^*_{1,2} < \varepsilon$.*

We have two definitions of the convergent sequence in a 2-normed space. Fortunately, these definitions are equivalent. Then we can use either of them to investigate the convergence. We show the equivalence in the following theorem.

Theorem 4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$ be a linearly independent set in X. A sequence $\{x_n\} \subset X$ is convergent with respect to class-1 if and only if it is convergent with respect to class-2.*

Proof. Let a sequence $\{x_n\} \subset X$ converges to $x \in X$ with respect to class-1. For any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$\|\overline{x_n - x}\|^*_1 = \|x_n - x, a_1\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\overline{x_n - x}\|^*_2 = \|x_n - x, a_2\| < \frac{\varepsilon}{2}.$$

Then,

$$\|\overline{x_n - x}\|^*_{1,2} = \|x_n - x, a_1\| + \|x_n - x, a_2\| < \varepsilon.$$

This means the sequence $\{x_n\} \subset X$ converges to $x \in X$ with respect to class-1.

Conversely, let a sequence $\{x_n\} \subset X$ converges to $x \in X$ with respect to class-2. For any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$\|\overline{x_n - x}\|^*_{1,2} < \varepsilon.$$

or

$$\|x_n - x, a_1\| + \|x_n - x, a_2\| < \varepsilon \tag{6}$$

This means each term of the left-hand side of equation (6) is less than ε. We write it as

$$\|x_n - x, a_1\| < \varepsilon \quad \text{and} \quad \|x_n - x, a_2\| < \varepsilon,$$

or

$$\|\overline{x_n - x}\|^*_i < \varepsilon \quad \text{for } i = 1, 2,$$

which means the sequence $\{x_n\} \subset X$ converges to $x \in X$ with respect to class-2. □

The above theorem states that to investigate a convergent sequence in a 2-normed space, we can use either definition 1 or 2. Since it is equivalent, from this point forward, we will only say a sequence is convergent. We will also say x is a limit point of $\{x_n\}$ (unless we need to specify the class). Next, we define a Cauchy sequence with respect to class-1 and class-2.

Definition 3. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$ be a linearly independent set in X. A sequence $\{x_n\} \subset X$ is called a Cauchy sequence with respect to class-1 if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, n \geq N$ we have*

$$\|\overline{x_n} - \overline{x_m}\|^*_i < \varepsilon, \quad \textit{for } i \in \{1, 2\}.$$

Definition 4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$ be a linearly independent set in X. A sequence $\{x_n\} \subset X$ is called a Cauchy sequence with respect to class-2 if for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, n \geq N$ we have*

$$\|\overline{x_n} - \overline{x_m}\|^*_{1,2} < \varepsilon.$$

We also have the following theorem, which states that Definitions 3 and 4 are equivalent.

Theorem 5. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $A = \{a_1, a_2\}$ be a linearly independent set in X. A sequence $\{x_n\} \subset X$ is Cauchy with respect to class-1 if and only if it is Cauchy with respect to class-2.*

Proof. This proof is quite straightforward. By following the proof of Theorem 4, we can transform the properties of a convergent sequence into those of a Cauchy sequence. The result will follow. □

Since it is equivalent, from this point forward, we will only say a sequence is convergent (unless we need to specify the class). Recall that in a normed space, if a sequence is convergent, then it is Cauchy. We still have this property in a 2-normed space, as stated below.

Theorem 6. *If a sequence is convergent in a 2-normed space, then it is Cauchy.*

Proof. Since both types (with respect to class-1 and 2) of convergent sequence and Cauchy sequence are equivalent, respectively, it is sufficient to prove the above theorem with respect to class-2. Let $\{x_n\} \subset X$ converges to $x \in X$, then for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $m, n \in \mathbb{N}$ we have

$$\|\overline{x_n} - \overline{x}\|^*_{1,2} \leq \tfrac{\varepsilon}{2} \quad \text{and} \quad \|\overline{x_m} - \overline{x}\|^*_{1,2} \leq \tfrac{\varepsilon}{2}.$$

Then we have

$$\|\overline{x_n} - \overline{x_m}\|^*_{1,2} \leq \|\overline{x_n} - \overline{x}\|^*_{1,2} + \|\overline{x_m} - \overline{x}\|^*_{1,2} < \varepsilon.$$

This proves the theorem. □

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, we say X is complete if all Cauchy sequences in X are convergent. Moreover, X is called a 2-Banach space. Furthermore, to prove the Fixed-Point Theorem, we define a closed set and a bounded set as follows.

Definition 5. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A set H is said to be closed if each point in H is a limit point of a sequence in H.*

Note that a closed set is defined in terms of limit points (convergent sequences). Since convergence with respect to class-1 and class-2 is equivalent, the notion of a closed set with respect to class-1 and class-2 is also equivalent. For this reason, in the above definition we simply use the term 'closed set' without specifying the class. Next, we give two definitions of a bounded set with respect to class-1 and class-2, respectively.

Definition 6. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A set B is said to be bounded with respect to class-1 if there exists a $K > 0$, for all $x \in B$ such that $\|\overline{x}\|^*_1 \leq K$ and $\|\overline{x}\|^*_2 \leq K$.*

Definition 7. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A set B is said to be bounded with respect to class-2 if there exists $K > 0$, for all $x \in B$ such that $\|\overline{x}\|^*_{1,2} \leq K$.*

Definitions 6 and 7 are also equivalent, as we show in the following theorem.

Theorem 7. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A set B is said to be bounded with respect to class-1 if and only if it is bounded with respect to class-2.*

Proof. Let B be bounded with respect to class-1 then there exists a $K > 0$, for all $x \in B$ such that $\|\overline{x}\|^*_1 \leq \frac{K}{2}$ and $\|\overline{x}\|^*_2 \leq \frac{K}{2}$. We have $\|\overline{x}\|^*_{1,2} = \|\overline{x}\|^*_1 + \|\overline{x}\|^*_2 \leq K$. This means B is bounded with respect to class-2. Conversely, let B be bounded with respect to class-2 then there exists a $K > 0$, for all $x \in B$ such that $\|\overline{x}\|^*_{1,2} = \|\overline{x}\|^*_1 + \|\overline{x}\|^*_2 \leq K$. Since the value of the norms will always be nonnegative, we have $\|\overline{x}\|^*_1 \leq K$ and $\|\overline{x}\|^*_2 \leq K$. This means B is bounded with respect to class-1. This ends the proof. □

Below, we give definitions of a contractive mapping and a continuous mapping, as well as a relation between both functions.

Definition 8. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A mapping $f : X \to X$ is said to be contractive with respect to class-1 if there exists a $C \in (0,1)$ such that for every $x, y \in X$, we have $\|\overline{f(x)} - \overline{f(y)}\|_1^* \leq C\,\|\overline{x} - \overline{y}\|_1^*$ and $\|\overline{f(x)} - \overline{f(y)}\|_2^* \leq C\,\|\overline{x} - \overline{y}\|_2^*$.*

Definition 9. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A mapping $f : X \to X$ is said to be contractive with respect to class-2 if there exists a $C \in (0,1)$ such that for every $x, y \in X$, we have $\|\overline{f(x)} - \overline{f(y)}\|_{1,2}^* \leq C\,\|\overline{x} - \overline{y}\|_{1,2}^*$.*

From the two definitions above, we have the following theorem

Theorem 8. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. If a mapping f is contractive with respect to class-1, then it is contractive with respect to class-2.*

Proof. Let a mapping f is contractive with respect to class-1, then there exists a $C \in (0,1)$ such that for any $x, y \in X$, we have

$$\|\overline{f(x)} - \overline{f(y)}\|_1^* \leq C\,\|\overline{x} - \overline{y}\|_1^* \quad \text{and} \quad \|\overline{f(x)} - \overline{f(y)}\|_2^* \leq C\,\|\overline{x} - \overline{y}\|_2^*.$$

This means

$$\|\overline{f(x)} - \overline{f(y)}\|_{1,2}^* \leq C\,\|\overline{x} - \overline{y}\|_1^* + C\,\|\overline{x} - \overline{y}\|_2^* = C\,\|\overline{x} - \overline{y}\|_{1,2}^*.$$

Then, f is contractive with respect to class-2. □

Note that, the constant C makes the converse of the theorem is a bit hard to prove. But this theorem is already sufficient for us to prove the Fixed Point Theorem. Next, we have the definition of a continuous mapping.

Definition 10. *Let $(X, \|\cdot,\cdot\|)$ and $(Z, \|\cdot,\cdot\|_2)$ be 2-normed spaces. A mapping $f : X \to Z$ is said to be continuous at a point $y \in X$, with respect to class-1 if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $y \in X$ that satisfies $\|\overline{x} - \overline{y}\|_{1_X}^* < \delta$ and $\|\overline{x} - \overline{y}\|_{2_X}^* < \delta$ we have $\|\overline{f(x)} - \overline{f(y)}\|_{1_Z}^* < \varepsilon$ and $\|\overline{f(x)} - \overline{f(y)}\|_{2_Z}^* < \varepsilon$.*

Definition 11. *Let $(X, \|\cdot,\cdot\|)$ dan $(Z, \|\cdot,\cdot\|_2)$ be 2-normed spaces. A mapping $f : X \to Z$ is said to be continuous at a point $y \in X$, with respect to class-2, if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in X$ that satisfies $\|x - \overline{y}\|_{(1,2)_X}^* < \delta$, we have $\|\overline{f(x)} - \overline{f(y)}\|_{(1,2)_Z}^* < \varepsilon$.*

Note that the norm $\|\cdot\|_{1_X}^*$ and $\|\cdot\|_{2_X}^*$ are norms of class-1, which are constructed with respect to a linearly independent set in X that contains two vectors. Meanwhile, the norms $\|\cdot\|_{1_Z}^*$ and $\|\cdot\|_{2_Z}^*$ are norms of class-1, which is constructed with respect to a linearly independent set in Z that contains two vectors. It also applies to the $\|\cdot\|_{(1,2)_X}^*$ and $\|\cdot\|_{(1,2)_Z}^*$ as norms on class-2. Moreover, we have the following theorem

Theorem 9. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. If a mapping f is a contractive mapping with respect to class-1, then it is continuous with respect to class-1.*

Proof. Let $f : X \to X$ be a contractive mapping with respect to class-1, this means there exists a $C \in (0,1)$ such that for every $x, y \in X$, we have $\|\overline{f(x)} - \overline{f(y)}\|_1^* \leq C\,\|\overline{x} - \overline{y}\|_1^*$ and $\|\overline{f(x)} - \overline{f(y)}\|_2^* \leq C\,\|\overline{x} - \overline{y}\|_2^*$. Given any $\varepsilon > 0$, choose $\delta < \varepsilon/C$, such that for any $x \in X$ that satisfies $\|\overline{x} - \overline{y}\|_1^* < \delta$ and $\|\overline{x} - \overline{y}\|_2^* < \delta$ we have $\|\overline{f(x)} - \overline{f(y)}\|_1^* < \varepsilon$ and $\|\overline{f(x)} - \overline{f(y)}\|_2^* < \varepsilon$. This means f is continuous with respect to class-1 □

Theorem 10. *Let $(X, \|\cdot,\cdot\|)$ and $(Z, \|\cdot,\cdot\|_2)$ be 2-normed spaces. A mapping f is continuous with respect to class-1 if and only if it is continuous with respect to class-2*

Proof. Let f be a continuous function with respect to class-1, then we have for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in X$ that satisfies $\|\overline{x} - \overline{y}\|_{1_X} < \frac{\delta}{2}$ and $\|\overline{x} - \overline{y}\|_{2_X} < \frac{\delta}{2}$ implies $\|\overline{f(x)} - \overline{f(y)}\|^*_{1_Z} < \frac{\varepsilon}{2}$ and $\|\overline{f(x)} - \overline{f(y)}\|^*_{2_Z} < \frac{\varepsilon}{2}$. From these, we have $\|\overline{x} - \overline{y}\|^*_{(1,2)_X} = \|\overline{x} - \overline{y}\|^*_{1_X} + \|\overline{x} - \overline{y}\|^*_{2_X} < \delta$ then $\|\overline{f(x)} - \overline{f(y)}\|^*_{(1,2)_Z} = \|\overline{f(x)} - \overline{f(y)}\|^*_{1_Z} + \|\overline{f(x)} - \overline{f(y)}\|^*_{2_Z} < \varepsilon$. This shows that f is continuous with respect to class-2. Conversely, let f be a continuous function with respect to class-2, then given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in X$ that satisfies $\|\overline{x} - \overline{y}\|^*_{1_X} + \|\overline{x} - \overline{y}\|^*_{2_X} = \|\overline{x} - \overline{y}\|^*_{(1,2)_X)} < \delta$ then $\|\overline{f(x)} - \overline{f(y)}\|^*_{1_Z} = \|\overline{f(x)} - \overline{f(y)}\|^*_{2_Z} = \|\overline{f(x)} - \overline{f(y)}\|^*_{(1,2)_Z} < \varepsilon$. One can see that for the first inequality, each term will be less than ε. We can write it as for any $y \in X$ that satisfies $\|\overline{x} - \overline{y}\|^*_{1_X} < \delta$ and $\|\overline{x} - \overline{y}\|^*_{2_X} < \delta$ imply $\|\overline{f(x)} - \overline{f(y)}\|^*_{1_Z} < \varepsilon$ and $\|\overline{f(x)} - \overline{f(y)}\|^*_{2_Z} < \varepsilon$. This means that f is continuous with respect to class-1. This ends the proof ☐

We need the theorem below before we prove the Fixed-Point Theorem.

Theorem 11. *Let $(X, \|\cdot,\cdot\|_X)$ and $(Z, \|\cdot,\cdot\|_Z)$ be 2-normed spaces and $\{x_n\} \subset X$ is convergent to an $x \in X$. If a mapping $f: X \to Z$ is continuous, then $f(x_n) \to f(x)$.*

Proof. We will prove the theorem using norm of class-2. Let $\{x_n\} \subset X$ is convergent to an $x \in X$, then for any $\delta > 0$ there exists an $N \in \mathbb{N}$ such that for any $n \geq N$ we have $\|\overline{x_n} - \overline{x}\|^*_{(1,2)_X} < \delta$. For any continuous mapping $f : X \to Z$ we have for all $\varepsilon > 0$, choose a $\delta > 0$ such that for $\|\overline{x_n} - \overline{x}\|^*_{(1,2)_X} < \delta$, we have $\|\overline{f(x_n)} - \overline{f(x)}\|^*_{(1,2)_Z} < \varepsilon$. This means $f(x_n) \to f(x)$ with respect to class-2. As mentioned before, since the convergence of a sequence and continuity of a mapping are equivalent with respect to class-1 and class-2, we say $f(x_n) \to f(x)$. ☐

Now, we will prove the Fixed-Point Theorem.

Theorem 12. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $H \subset X$ be a non-empty, and closed set. If $f: H \to H$ be a contractive mapping with respect to class-2, then f has a unique fixed point.*

Proof. Let $u_0 \in H$ and $\{u_k\}$ be a convergent sequence in K such that,

$$u_k = f(u_{k-1}) = f^k(u_0), \quad k = 1, 2, \ldots$$

Let $f: H \to H$ be a contractive mapping, for $u_0, u_1 \in K$, we have

$$\|\overline{f^2(u_0)} - \overline{f^2(u_1)}\|^*_{1,2} = \|\overline{f(f(u_0))} - \overline{f(f(u_1))}\|^*_{1,2}.$$

Since f is a contractive mapping, there exists a $C \in (0, 1)$ such that

$$\|\overline{f(f(u_0))} - \overline{f(f(u_1))}\|^*_{1,2} \leq C \, \|\overline{f(u_0)} - \overline{f(u_1)}\|^*_{1,2}.$$

Once again, because f is a contractive mapping, we can write

$$\|\overline{f(u_0)} - \overline{f(u_1)}\|^*_{1,2} \leq C \, \|\overline{u_0} - \overline{u_1}\|^*_{1,2}.$$

Then,

$$\|\overline{f^2(u_0)} - \overline{f^2(u_1)}\|^*_{i,2} \leq C^2 \, \|\overline{u_0} - \overline{u_1}\|^*_{1,2}.$$

Using mathematical induction, we have

$$\|\overline{f^k(u_0)} - \overline{f^k(u_1)}\|^*_{1,2} \leq C^k \, \|\overline{u_0} - \overline{u_1}\|^*_{1,2}.$$

Moreover, without losing of generality, choose $k > m$ with $m = k + p$; $k, m, p \in \mathbb{N}$. Then

$$\|\overline{u_k} - \overline{u_m}\|^*_{1,2} = \|\overline{u_k} - \overline{u_{k+1}}\|^*_{1,2} \leq \|\overline{u_k} - \overline{u_{k+1}}\|^*_{1,2} + \cdots + \|\overline{u_{k+p-1}} - \overline{u_{k+p}}\|^*_{1,2},$$

Using the property of sequence $\{x_n\}$, we have

$$\begin{aligned} \|\overline{u_k} - \overline{u_m}\|^*_{1,2} &\leq \|\overline{f^k(u_0)} - \overline{f^k(u_1)}\|^*_{1,2} + \cdots + \|\overline{f^{k+p-1}(u_0)} - \overline{f^{k+p-1}(u_1)}\|^*_{1,2} \\ &\leq (C^k + \cdots + C^{k+p-1})\|\overline{u_0} - \overline{u_1}\|^*_{1,2}. \\ &= (C^k + \cdots + C^{m-1})\|\overline{u_0} - \overline{u_1}\|^*_{1,2}. \end{aligned}$$

The constant $C \in (0,1)$ implies

$$\lim_{k,m\to\infty}(C^k + \cdots + C^{m-1})\|\overline{u_k} - \overline{u_m}\|^*_{1,2} = 0.$$

This means, $\{u_k\}$ is a Cauchy sequence. Since $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, $\{u_k\}$ is convergent. Let u be the limit point of the sequence, we write $u_k \to u$. Because f is a continuous mapping with respect to class-2, then it is continuous with respect to class-2. By Theorem 11, we have

$$f(u) = \lim_{k\to\infty} f(u_k) = \lim_{k\to\infty} u_{k+1} = u.$$

Now, for the uniqueness, let u and u' be two fixed points of f. We can see that,

$$\|\overline{u} - \overline{u'}\|^*_{1,2} = \|\overline{f(u)} - \overline{f(u')}\|^*_{1,2} \leq C\,\|\overline{u} - \overline{u'}\|^*_{1,2}.$$

with $C \in (0,1)$. The above inequality is only true if $\|\overline{u} - \overline{u'}\|^*_{1,2} = 0$. This means $u = u'$. It means the fixed point is unique. □

Note that, in Theorem 12, we used a contractive mapping with respect to class-2.

4 Conclusion

We used a new viewpoint to investigate some topological properties of 2-normed spaces. We got a new viewpoint by constructing some quotient classes in the 2-normed space with respect to a linearly independent set. Using the norm that is defined in each quotient space, we define several topological properties, namely convergent sequences, Cauchy sequences, closed sets, bounded sets, contractive mappings, and continuous mappings. These properties were defined with respect to class-1 and class-2. We showed that two types of definitions of each property are equivalent. In the end, using these properties, we proved a fixed point theorem in 2-normed spaces. The results presented here suggest several avenues for further research. One natural extension would be to investigate whether similar constructions can be applied to n-normed spaces with n>2, thereby generalizing the equivalence of topological properties beyond the 2-normed framework. Another promising direction is the exploration of quotient constructions in probabilistic or fuzzy 2-normed spaces, which could provide insights into uncertainty modeling and applications in applied sciences.

CRediT Authorship Contribution Statement

Francis Y. Rumlawang: Conceptualization, Writing – Original Draft, Funding Acquisition, Writing – Review and Editing, Correspondence. **Hernita**: Writing – Original Draft, Formal Analysis, Data Curation. **Yopi A. Lesnussa**: Writing – Review and Editing, Draft Preparation. **Harmanus Batkunde**: Conceptualization, Funding Acquisition, Writing – Review and Editing, Revision.

Declaration of Generative AI and AI-assisted technologies

No generative AI or AI-assisted technologies were used during the preparation of this manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

The data and code supporting the findings of this study are available from the corresponding author upon reasonable request and subject to confidentiality agreements.

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