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Function-Theoretic Operator Norm Inequalities: A Kosaki-type Generalization to Symmetric Probability Weights

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Abstract

Operator norm comparisons play a fundamental role in matrix analysis, yet existing proofs often depend on operator means or interpolation techniques. This study provides a function-theoretic approach to operator norm inequalities. It also extends the classical two-term Heinz comparison to multi-term averages with arbitrary symmetric probability weights. Our approach translates each operator norm comparison into a scalar condition. The condition is derived from functional calculus for the left and right multiplication operators. We examine positive-definiteness and infinite divisibility through Fourier-measure representations. We also use elementary closure properties. For positive operators and any unitarily invariant norm, the two-term Heinz symmetrization is dominated by the binomial average when the exponent differs from one-half by at most one divided by twice the number of terms. For general symmetric probability weights, domination occurs exactly when the exponent lies within a specific threshold. This threshold equals the smallest positive distance from the midpoint to any index carrying nonzero weight. The proposed function-theoretic framework yields necessary and sufficient thresholds to unify the binomial and general symmetric cases.

Keywords: infinitely divisible function; operator norm inequality; positive definite function; unitarily invariant norm

1 Introduction

The study of operator norm inequalities plays a pivotal role in functional analysis, particularly in the context of unitarily invariant norms and positive semidefinite operators. Among the foundational results in this area is the Heinz-type inequality

$$\left\| H^{1/2} X K^{1/2} \right\| \leq \frac{1}{2} \| HX + XK \|,$$

This inequality was first established in 1979 [1]. It was later generalized to arbitrary unitarily invariant norms, i.e., a norm satisfying $\|UX\| = \|X\| = \|XV\|$ for every operator X and unitary operators U, V. We denote such norms by $|||\cdot|||$ [2]. Such inequalities not only deepen the understanding of matrix and operator means but also connect deeply with other mathematical structures, such as the geometry of positive definite matrices, functional calculus, and the theory of positive-definite and infinitely divisible functions. These connections have been rigorously

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explored in a series of works [3], [4], [5], [6], [7], [8], [9] and this line of study has remained active and has continued to develop over the past decade [10], [11], [12], [13], [14], [15], [16], [17], [18].

Most existing approaches to Heinz-type operator-norm inequalities rely on interpolation and operator means. These tools are powerful, but they often conceal the mechanism behind norm domination and make sharp parameter thresholds difficult to read from the argument. They also lack a simple, checkable, and necessary-and-sufficient criterion that works uniformly for all unitarily invariant norms. When binomial weights are replaced by general symmetric probability weights—beyond Kosaki's binomial case—the literature offers no transparent test. In particular, it does not reveal exactly when a two-term Heinz symmetrization is dominated by an n-term symmetric average.

Our objective is to provide a *function-theoretic* route to these inequalities that (i) yields sharp ranges by a direct positive-definite test and (ii) extends the binomial result to *arbitrary* symmetric weights via a simple, computable threshold. Concretely:

- We introduce a translation principle that converts norm domination into positive-definiteness (and, where needed, infinite divisibility) of explicit scalar ratios. The proof uses Böchner's theorem and functional calculus.
- We give a new proof of the operator norm inequality in [9] with *sharp* range $\left|a - \frac{1}{2}\right| \leq \frac{1}{2n}$, valid for every unitarily invariant norm $|||\cdot|||$.
- We extend this to general symmetric probability weights $c = (c_0, \ldots, c_n)$ for positive c_i's and obtain a criterion $\left|a - \frac{1}{2}\right| \leq \delta_c$, where $\delta_c = \min\left\{\left|\frac{m}{n} - \frac{1}{2}\right| : c_m > 0\right\}$.

The present article contributes a novel perspective on a well-known result in [9] by introducing an alternative proof strategy grounded in the framework of positive definiteness and infinite divisibility. We adopt a function-theoretic approach inspired by our previous work [19], which characterizes norm inequalities through the lens of analytic function behavior. Specifically, our objective is to provide a new proof of Kosaki's operator inequality using a sequence of analytical tools: Böchner's theorem on positive-definite functions, the characterization of infinitely divisible functions. Our approach demonstrates that the operator inequality in question holds if and only if a corresponding function derived from operator parameters exhibits positive definiteness.

Here we give a concise mathematical formulation (the detailed arguments and formulations are deferred to Section 2). Let L_H and R_K denote left and right multiplication on matrices; for a two-variable function $M_f(s,t) = tf(s/t)$ we write $M_f(L_H, R_K)$ via joint functional calculus on the commuting pair (L_H, R_K). Lemma 1 shows that $f, g \in S_1^+(0,\infty)$,

$$|||M_f(L_H, R_K)X||| \leq |||M_g(L_H, R_K)X||| \qquad \text{for all } H, K > 0$$

holds if and only if $f(e^x)/g(e^x)$ is positive definite on $\mathbb{R}$. We apply this to the Heinz symmetrization $S_a(H,K)X = \frac{1}{2}\left(H^a X K^{1-a} + H^{1-a} X K^a\right)$ and to n-term averages $B_c(H,K)X = \sum_m c_m H^{m/n} X K^{(n-m)/n}$. The positive definite test then yields the sharp binomial range $\left|a - \frac{1}{2}\right| \leq \frac{1}{2n}$ and, in full generality, the criterion $\left|a - \frac{1}{2}\right| \leq \delta_c$. Full proofs appear in Section 3.

2 Analytical Preliminaries

We collect the foundational results and analytical tools required to rigorously give a new proof of Kosaki's operator norm inequality [9].

We first recall the classical Fourier–measure characterization, which furnishes the bridge from positive-definite functions to the integral representations used in our proofs.

Theorem 1 ([5]). *If f is a positive-definite function and continuous at 0, then there exists a finite positive measure μ on $\mathbb{R}$ such that $f(x) = \int_{-\infty}^{\infty} e^{ixt} d\mu(t)$.*

A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is called positive definite if, for every integer $N \in \mathbb{N}$, real numbers $x_1, x_2, \cdots, x_N$, and complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_N$, the inequality

$$\sum_{i,j=1}^{N} \alpha_i \overline{\alpha_j} f(x_i - x_j) \geq 0$$

holds. Equivalently, the matrix $[f(x_i - x_j)]_{i,j=1}^{N} \geq 0$.

The following are well-known properties of positive-definite functions.

Proposition 1. *1. For any positive definite functions f_1, f_2 and $\alpha > 0$, then $\alpha f_1, f_1 + f_2$ and $f_1 f_2$ are all positive definite functions.*

2. Let $f_1, f_2, \cdots$ be a sequence of positive definite functions. If $\lim_n f_n = f$ pointwise, then f is a positive definite function.

The ideas of Section 1 extend to two-variable functions $M(s,t)$. For example

$$\frac{\alpha - 1}{\alpha} \cdot \frac{s^{\alpha} - t^{\alpha}}{s^{\alpha-1} - t^{\alpha-1}} = M_{\alpha}(s,t), \quad (s, t > 0 \text{ and } \alpha \in \mathbb{R}).$$

The above two variable function is valid for all $\alpha \in \mathbb{R}$. For instance, if $\alpha = 0$, we can show using L'Hôpital's rule or Taylor—more precisely, the Maclaurin—expansion at $\alpha = 0$.

$$M_0(s,t) = \begin{cases} \dfrac{st \, \ln(s/t)}{s - t}, & s \neq t, \\ s, & s = t. \end{cases}$$

This scalar formulation provides the prototype for the operator setting. Each function $M_{\alpha}(s,t)$ induces an operator mean $M_{\alpha}(L_H, R_K)$ via functional calculus on the commuting pair (L_H, R_K) (see below). Consequently, the operator norm inequalities introduced in the beginning of Section 1 correspond to the positive definiteness of the ratio $\frac{M_{1/2}(s,t)}{M_2(s,t)}$ [5].

Let $M : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ be nondecreasing and satisfy the following properties:

- $M(s,t) = M(t,s)$
- $M(\alpha s, \alpha t) = \alpha M(s,t)$ for all $\alpha > 0$
- $\min\{s,t\} \leq M(s,t) \leq \max\{s,t\}$.

For such a function M, we define a new function f by

$$f(t) = M(t,1),$$

and thus obtain the identity

$$M(s,t) = tM(s/t, 1) = tf(s/t).$$

It is straightforward to verify that $f(t) = tf(1/t), f(1) = 1$ and for all $t \geq 1$ satisfies $f(t) \leq t$. This function f will serve as a key object in analyzing positive definiteness properties relevant to the main inequality under investigation.

A continuous scalar function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ that satisfies the symmetry and homogeneity conditions naturally extends to operators through the commuting pair (L_H, R_K), where

$$L_H(X) = HX \quad \text{and} \quad R_K(X) = XK.$$

Since L_H and R_K commute when $H, K > 0$, we may define

$$M(L_H, R_K)X$$

via the spectral functional calculus applied to the joint spectrum of (L_H, R_K). In this setting, the study of operator norm inequalities reduces to analyzing scalar identities of the form

$$M(s,t) = tf(s/t),$$

where f lies in the class

$$S_1^+(0,\infty) = \{f : (0,\infty) \to \mathbb{R} \ : \ \text{continuous}, \ f(1) = 1, \ f(t) = tf(1/t)\}.$$

Hence, functional calculus provides the precise mechanism that translates norm inequalities between operators into positivity questions about scalar functions, which subsequently test through positive definiteness and infinite divisibility [19].

For any $f, g \in S_1^+(0,\infty)$, we define a partial order $f \prec g$ if and only if the function

$$\varphi(x) = \frac{f(e^x)}{g(e^x)}$$

is positive definite on $\mathbb{R}$.

A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be infinitely divisible if, for every real number $\alpha > 0$, the function f^α remains a positive definite function.

The following are well-known properties of infinitely divisible functions.

Proposition 2. *1. For any infinitely divisible function f_1, f_2, then $f_1 f_2$ is an infinitely divisible function.*

2. Let $f_1, f_2, \cdots$ be a sequence of infinitely divisible functions. If $\lim_n f_n = f$ pointwise, then f is an infinitely divisible function.

The next theorem gives a sharp non–positive-definiteness criterion for specific hyperbolic-sine products, which we will use later to exclude certain parameter ranges.

Theorem 2 ([19])**.** *If $p = 2n + 1$ with $n > 1$, then the function*

$$\frac{\sinh\left(\frac{p+1}{p}x\right)\left(\sinh\frac{1}{p}x\right)^n}{\sinh x\left(\sinh\frac{p-1}{p}x\right)^n}$$

is not positive definite.

The following majorization-based test provides a sufficient condition ensuring that products of sinh-ratios are infinitely divisible.

Theorem 3 ([19])**.** *Let $n \in \mathbb{N}, \alpha = (a_1, a_2, \cdots, a_n), \ \beta = (b_1, b_2, \cdots, b_n) \in (0,\infty)^n$. If*

$$\sum_{i=1}^{k} a_{\sigma(i)} \leq \sum_{i=1}^{k} b_{\tau(i)}$$

for any $1 \leq k \leq n$ and for some permutations σ, τ on $\{1, 2, \cdots, n\}$ such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$ and $b_{\tau(i)} \geq b_{\tau(i+1)}$ for any indices i, then the function

$$\prod_{i=1}^{n} \frac{b_i \sinh(a_i x)}{a_i \sinh(b_i x)}$$

is an infinitely divisible function.

3 Main Results and Analytical Proofs

Our goal now is to give a new proof and approach that for any unitarily invariant norm $|||\cdot|||$, the norm inequality

$$\frac{1}{2}\left|\left|\left|H^aXK^{1-a}+H^{1-a}XK^a\right|\right|\right| \leq \frac{1}{2^n}\left|\left|\left|\sum_{m=0}^{n}\binom{n}{m}H^{\frac{m}{n}}XK^{\frac{n-m}{n}}\right|\right|\right|$$

holds if and only if $\frac{1}{2}\left(1-\frac{1}{n}\right) \leq a \leq \frac{1}{2}\left(1+\frac{1}{n}\right)$ for all unitarily invariant norms with $m, n \in \mathbb{N}, H, K$ and X are matrices with $H, K > 0$ [9].

To bridge operator inequalities and scalar function properties, we use functional calculus for the commuting pair (L_H, R_K). The next lemma is the translation principle: the norm domination between $M_f(L_H, R_K)$ and $M_g(L_H, R_K)$ is equivalent to the positive-definiteness of $\frac{f(e^x)}{g(e^x)}$ on $\mathbb{R}$.

Lemma 1. *Given $f, g \in S_1^+(0,\infty)$, for any unitarily invariant norm $|||\cdot|||$ and any square $N \times N$ matrices H, K, and X with $H, K > 0$, the norm inequality*

$$|||M_f(L_H, R_K)X||| \leq |||M_g(L_H, R_K)X|||$$

holds if and only if $f \prec g$.

Proof. At first, we will proceed for the "if" part, that is, we have $f \prec g$. By Theorem 1, we then have a finite positive (probability) measure μ on $\mathbb{R}$ such that

$$\frac{f\left(e^x\right)}{g\left(e^x\right)} = \int_{-\infty}^{\infty} e^{ixs}d\mu(s).$$

If $H, K > 0$, we let $H = \sum_{j=1}^{N} \lambda_i P_i$ and $K = \sum_{j=1}^{N} \zeta_i Q_i$ are the spectral decomposition, then $M_f(H,K)X = \sum_{j,k=1}^{N} M_f\left(\lambda_j, \zeta_k\right) P_j X Q_k$. Consider that

$$\begin{aligned}
M_f(H,K)X &= \sum_{j,k=1}^{N} M_f(\lambda_j, \zeta_k)P_jXQ_k \\
&= \sum_{j,k=1}^{N} \zeta_k f(\lambda_j/\zeta_k)P_jXQ_k \\
&= \sum_{j,k=1}^{N} \zeta_k g(\lambda_j/\zeta_k)\int_{-\infty}^{\infty} e^{is\log(\lambda_j/\zeta_k)}d\mu(s)P_jXQ_k \\
&= \int_{-\infty}^{\infty}\sum_{j,k=1}^{N} M_g(\lambda_j, \zeta_k)(\lambda_j/\zeta_k)^{is}P_jXQ_k\,d\mu(s) \\
&= \int_{-\infty}^{\infty}\sum_{j,k=1}^{N} (\lambda_j)^{is}M_g(\lambda_j, \zeta_k)(\zeta_k)^{-is}P_jXQ_k\,d\mu(s) \\
&= \int_{-\infty}^{\infty} H^{is}(M_g(H,K)X)K^{-is}\,d\mu(s).
\end{aligned}$$

Take a unitarily invariant norm on both sides and since the total mass of μ equals 1, we obtain

$$|||M_f(L_H, R_K)X||| \leq |||M_g(L_H, R_K)X|||.$$

For the "only if" part, since the usual operator norm is unitarily invariant norm, we have that

$$\|M_f(L_H, R_K)X\| \leq \|M_g(L_H, R_K)X\|$$

for any square $N \times N$ matrices H, K, and X with $H, K > 0$. Let $\lambda_1, \lambda_2, \cdots, \lambda_N > 0$ and $a_{ij} = \dfrac{M_f(\lambda_i, \lambda_j)}{M_g(\lambda_i, \lambda_j)}$. Since $M_\bullet(x, y) = M_\bullet(y, x)$ and $M_\bullet(1, 1) = 1$, then the matrix $A = [a_{ij}]_{i,j=1}^N$ is self-adjoint and the diagonal entries are all 1. Let

$$H = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \mathbf{0} & \\ & & \lambda_3 & & \\ & \mathbf{0} & & \ddots & \\ & & & & \lambda_N \end{pmatrix},$$

then by the assumption we have

$$\begin{aligned} \|A \circ X\| &= \left\| M_f(\lambda_i, \lambda_j) \circ \frac{1}{M_g(\lambda_i, \lambda_j)} \circ X \right\| \\ &= \left\| M_f(H, H) \circ \frac{1}{M_g(H, H)} \circ X \right\| \\ &\leq \left\| M_g(H, H) \circ \frac{1}{M_g(H, H)} \circ X \right\| \\ &= \left\| [1]_{i,j=1}^N \circ X \right\| \\ &= \|X\|, \end{aligned}$$

where $\circ$ is the Hadamard product of matrices and $[1]_{i,j=1}^N$ is the $N \times N$ matrices with all the entries are 1. The dual norm of the usual norm is a trace norm $\| \bullet \|_1$, which is equal to the sum of all absolute values of the eigenvalues. Since the trace norm is isometric under the Hadamard product, we obtain $\|A \circ X\|_1 \leq \|X\|_1$. Let $X = X_0 = [1]_{i,j=1}^N$, then $\|A \circ X_0\|_1 = \|A\|_1 \leq \|X_0\|_1 = N$. If $\alpha_1, \alpha_2, \cdots, \alpha_N$ are the eigenvalues of A, then

$$\sum_{i=1}^N |\alpha_i| = \|A\|_1 \leq N = \operatorname{Tr} A = \sum_{i=1}^N \alpha_i,$$

which means that every single α_i must be non-negative, which implies $A > 0$. From here,

$$\begin{aligned} 0 \leq A &= \left(\frac{M_f(\lambda_i, \lambda_j)}{M_g(\lambda_i, \lambda_j)} \right)_{i,j=1}^N \\ &= \left(\frac{f(\lambda_i/\lambda_j)}{g(\lambda_i/\lambda_j)} \right)_{i,j=1}^N \\ &= \left(\frac{f\left(e^{\log \lambda_i - \log \lambda_j}\right)}{g\left(e^{\log \lambda_i - \log \lambda_j}\right)} \right)_{i,j=1}^N. \end{aligned}$$

So, we conclude that $f \prec g$. □

We recall that for any $a, b \in \mathbb{R}$, the function

$$\frac{b \sinh(ax)}{a \sinh(bx)}$$

is positive definite whenever $b \geq a$. [19]

The next lemma supplies the negative counterpart to Theorem 3, identifying a parameter regime where positive-definiteness necessarily fails.

Lemma 2 ([19]). *Let $n \in \mathbb{N}, i = 1, 2, \cdots, n$ and $a_i, b_i \in \mathbb{R}$. If $\max\{|a_i|\} > \max\{|b_i|\}$, then*

$$\prod_{i=1}^{n} \frac{b_i \sinh(a_i x)}{a_i \sinh(b_i x)}$$

is not positive definite.

Proof. We may assume that $a_i, b_i > 0, a_1 > 1,\ a_1 \geq a_2 \geq \cdots \geq a_n$ and $1 = b_1 \geq b_2 \geq \cdots \geq b_n$. There exists a positive odd number p such that

$$a_1 > \frac{p+1}{p-1} \quad \text{and} \quad a_n > \frac{1}{p-1}.$$

Consider that

$$\frac{\sinh\left(\frac{p+1}{p}t\right)\left(\sinh\left(\frac{1}{p}t\right)\right)^{n-1}}{\sinh(t)\left(\sinh\left(\frac{p-1}{p}t\right)\right)^{n-1}} = \frac{\sinh\left(\frac{p+1}{p}t\right)\left(\sinh\left(\frac{1}{p}t\right)\right)^{n-1}}{\left(\sinh\left(\frac{p-1}{p}t\right)\right)^{n}} \times \frac{\sinh\left(\frac{p-1}{p}t\right)}{\sinh(t)}$$

and by remark prior to this lemma, Theorem 2 and Proposition 1 *1.* we conclude that

$$\frac{\sinh\left(\frac{p+1}{p}t\right)\left(\sinh\left(\frac{1}{p}t\right)\right)^{n-1}}{\left(\sinh\left(\frac{p-1}{p}t\right)\right)^{n}}$$

is not a positive definite function. By replacing the variable, $t = (p/p-1)x$, we have that the function

$$\frac{\sinh\left(\frac{p+1}{p-1}x\right)\left[\sinh\left(\frac{1}{p-1}x\right)\right]^{n-1}}{(\sinh x)^{n}}$$

also not positive definite. Since

$$a_1 > \frac{p+1}{p-1}, \quad a_2 \geq \cdots \geq a_n > \frac{1}{p-1} \quad \text{and} \quad 1 = b_1 \geq \cdots \geq b_n,$$

again by remark prior to this lemma, Proposition 1 *1.* and identity

$$\frac{\sinh\left(\frac{p+1}{p-1}x\right)\left(\sinh\left(\frac{1}{p-1}x\right)\right)^{n-1}}{(\sinh x)^{n}} = \prod_{i=1}^{n}\frac{\sinh(a_i x)}{\sinh(b_i x)} \times \frac{\sinh\left(\frac{p+1}{p-1}x\right)}{\sinh(a_1 x)} \times \frac{\left(\sinh\left(\frac{1}{p-1}x\right)\right)^{n-1}}{\prod_{i=2}^{n}\sinh(a_i x)} \times \frac{\prod_{i=1}^{n}\sinh(b_i x)}{(\sinh x)^{n}}$$

we conclude that $\prod_{i=1}^{n}\frac{\sinh(a_i x)}{\sinh(b_i x)}$ is not positive definite function. □

Now we are ready to give a new proof of the following theorem.

Theorem 4 ([9]). *For any unitarily invariant norm $|||\cdot|||$, the norm inequality*

$$\frac{1}{2}\left|\left|\left|H^a X K^{1-a} + H^{1-a} X K^a\right|\right|\right| \leq \frac{1}{2^n}\left|\left|\left|\sum_{m=0}^{n}\binom{n}{m} H^{\frac{m}{n}} X K^{\frac{n-m}{n}}\right|\right|\right|$$

holds if and only if $\frac{1}{2}\left(1 - \frac{1}{n}\right) \leq a \leq \frac{1}{2}\left(1 + \frac{1}{n}\right)$ for all unitarily invariant norms with $m, n \in \mathbb{N}, H, K$ and X are matrices with $H, K > 0$.

Proof. We consider the function

$$g_{n,\alpha}(x) = 2^{n-1}\frac{\sinh\left(2\left|\alpha\right|x\right)\left(\sinh\left(\frac{1}{n}x\right)\right)^n}{\sinh\left(\left|\alpha\right|x\right)\left(\sinh\left(\frac{2}{n}x\right)\right)^n}.$$

By Lemma 2, $g_{n,\alpha}$ is not a positive definite function when $|\alpha| > \dfrac{1}{n}$. Next, we consider for $0 \le |\alpha| \le \dfrac{1}{n}$ and we have

$$\begin{aligned}
\frac{1}{n} &\le \frac{2}{n}\\
\frac{1}{n}+\frac{1}{n} &\le \frac{2}{n}+\frac{2}{n}\\
&\vdots\\
\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n} &\le \frac{2}{n}+\frac{2}{n}+\cdots+\frac{2}{n}\\
\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}+2|\alpha| &\le \frac{2}{n}+\frac{2}{n}+\cdots+\frac{2}{n}+|\alpha|.
\end{aligned}$$

So, by Theorem 3 the function $g_{n,\alpha}$ is infinitely divisible function, hence positive definite function. This is equivalent to saying that $g_{n,\alpha}$ is a positive definite function if and only if $|\alpha| \le \dfrac{1}{n}$.

We set

$$f_{\mathbf{a},\mathbf{b}}\left(e^{2x}\right) = e^x\prod_{i=1}^{n}\frac{\beta_i\sinh(\alpha_i x)}{\alpha_i\sinh(\beta_i x)} \quad \text{where} \quad \mathbf{a} = (\alpha_1,\alpha_2,\cdots,\alpha_n) \text{ and } \mathbf{b} = (\beta_1,\beta_2,\cdots,\beta_n),$$

and since $g_{n,\alpha} = g_{n,-\alpha}$, we have

$$g_{n,\alpha}(x) = \frac{f_{(2\alpha),(\alpha)}\left(e^{2x}\right)}{f_{(2/n,2/n,\cdots,2/n),(1/n,1/n,\cdots,1/n)}\left(e^{2x}\right)}.$$

Now, consider that by putting $e^{2x} = \dfrac{s}{t}$ for some $s,t>0$, we have

$$\begin{aligned}
t\,f_{\mathbf{a},\mathbf{b}}\left(\frac{s}{t}\right) &= t\left(\frac{s}{t}\right)^{1/2}\prod_{i=1}^{n}\frac{\beta_i\sinh\left(\frac{\alpha_i}{2}\ln\frac{s}{t}\right)}{\alpha_i\sinh\left(\frac{\beta_i}{2}\ln\frac{s}{t}\right)}\\
&= (st)^{1/2}\prod_{i=1}^{n}\frac{\beta_i\sinh\left(\frac{\alpha_i}{2}\ln\frac{s}{t}\right)}{\alpha_i\sinh\left(\frac{\beta_i}{2}\ln\frac{s}{t}\right)}
\end{aligned}$$

and by using the identity $\sinh z = \dfrac{1}{2}\left(e^z - e^{-z}\right)$, for each i, satisfies

$$\begin{aligned}
\frac{\sinh\left(\frac{\alpha_i}{2}\ln\frac{s}{t}\right)}{\sinh\left(\frac{\beta_i}{2}\ln\frac{s}{t}\right)} &= \frac{\left(\frac{s}{t}\right)^{\alpha_i/2}-\left(\frac{t}{s}\right)^{\alpha_i/2}}{\left(\frac{s}{t}\right)^{\beta_i/2}-\left(\frac{t}{s}\right)^{\beta_i/2}}\\
&= \frac{s^{\alpha_i/2}t^{-\alpha_i/2}\left(1-\left(\frac{t}{s}\right)^{\alpha_i}\right)}{s^{\beta_i/2}t^{-\beta_i/2}\left(1-\left(\frac{t}{s}\right)^{\beta_i}\right)}\\
&= \frac{s^{\alpha_i/2}t^{-\alpha_i/2}\left(s^{\beta_i-\alpha_i}\right)\left(s^{\alpha_i}-t^{\alpha_i}\right)}{s^{\beta_i/2}t^{-\beta_i/2}\left(s^{\beta_i}-t^{\beta_i}\right)}\\
&= (st)^{-(\alpha_i-\beta_i)/2}\cdot\frac{s^{\alpha_i}-t^{\alpha_i}}{s^{\beta_i}-t^{\beta_i}}.
\end{aligned}$$

Applying that identity to the product

$$\prod_{i=1}^{n} \frac{\sinh\left(\frac{\alpha_i}{2}\ln\frac{s}{t}\right)}{\sinh\left(\frac{\beta_i}{2}\ln\frac{s}{t}\right)},$$

we then have

$$tf_{\mathbf{a},\mathbf{b}}\left(\frac{s}{t}\right) = (st)^{\frac{1-\sum_{i=1}^{n}(\alpha_i-\beta_i)}{2}} \prod_{i=1}^{n} \frac{\beta_i(s^{\alpha_i}-t^{\alpha_i})}{\alpha_i(s^{\beta_i}-t^{\beta_i})}.$$

By our definition $M(x,y) = yf(x/y)$ in Section 2, we have that

$$M_{\mathbf{a},\mathbf{b}}(s,t) = (st)^{\frac{1-\sum_{i=1}^{n}(\alpha_i-\beta_i)}{2}} \prod_{i=1}^{n} \frac{\beta_i(s^{\alpha_i}-t^{\alpha_i})}{\alpha_i(s^{\beta_i}-t^{\beta_i})}.$$

Therefore

$$M_{(2\alpha),(\alpha)}(s,t) = \frac{1}{2}(st)^{\frac{1-\alpha}{2}}(s^{\alpha}+t^{\alpha})$$

and

$$\begin{aligned} M_{\underbrace{(2/n,2/n,\cdots,2/n)}_{\text{n-term}},\underbrace{(1/n,1/n,\cdots,1/n)}_{\text{n-term}}}(s,t) &= \frac{1}{2^n}\left(s^{1/n}+t^{1/n}\right)^n \\ &= \frac{1}{2^n}\sum_{m=0}^{n}\binom{n}{m} s^{m/n}t^{(n-m)/n}. \end{aligned}$$

Since $g_{n,\alpha}$ is a positive definite function if and only if $|\alpha| \le \frac{1}{n}$, then by Lemma 1

$$\left\| M_{(2a),(a)}(L_H,R_K)X \right\| \le \left\| M_{\underbrace{(2/n,2/n,\cdots,2/n)}_{\text{n-term}},\underbrace{(1/n,1/n,\cdots,1/n)}_{\text{n-term}}}(L_H,R_K)X \right\|$$

$$\Rightarrow \quad \frac{1}{2}\left\| H^{(1+\alpha)/2}XK^{(1-\alpha)/2} + H^{(1-\alpha)/2}XK^{(1+\alpha)/2} \right\| \le \frac{1}{2^n}\left\| \sum_{m=0}^{n}\binom{n}{m} H^{m/n}XK^{(n-m)/n} \right\|.$$

Setting $\alpha = 2a-1$ gives

$$\frac{1}{2}\left\| H^{a}XK^{1-a} + H^{1-a}XK^{a} \right\| \le \frac{1}{2^n}\left\| \sum_{m=0}^{n}\binom{n}{m} H^{m/n}XK^{(n-m)/n} \right\|$$

if and only if $\frac{1}{2}\left(1-\frac{1}{n}\right) \le a \le \frac{1}{2}\left(1+\frac{1}{n}\right)$. □

Having settled the binomial case in Theorem 4, we now extend the domination criterion to arbitrary symmetric probability weights, with the sharp threshold captured by δ_c.

Theorem 5. *Let $n \ge 1$ be an integer, $c = (c_0, c_1, \cdots, c_n) \in (0,\infty)^n$ not identically zero with $c_m = c_{n-m}$ for all m, $\sum_{m=0}^{n} c_m = 1$,*

$$\delta_c = \min\left\{\left|\frac{m}{n}-\frac{1}{2}\right| : c_m > 0\right\} \quad \textit{and} \quad B_c(H,K)X = \sum_{m=0}^{n} c_m H^{m/n}XK^{(n-m)/n}.$$

For $a \in [0,1]$, let $S_a(H,K)X = \frac{1}{2}\left(H^aXK^{1-a} + H^{1-a}XK^a\right)$. Then, for any unitarily invariant norm $\|\cdot\|$, the inequality $\|S_a(H,K)X\| \le \|B_c(H,K)X\|$ holds for all matrices H, K and X with $H, K > 0$ if and only if

$$\left|a-\frac{1}{2}\right| \le \delta_c.$$

Proof. For $t > 0$, let

$$f_a(t) = \frac{1}{2}\left(t^a + t^{1-a}\right) \quad \text{and} \quad g_c(t) = \sum_{m=0}^{n} c_m t^{\frac{m}{n}}.$$

Given any $H, K > 0$, since $tf_a\left(\frac{s}{t}\right) = \frac{1}{2}\left(s^a t^{1-a} + s^{1-a}t^a\right)$ and $tg_c\left(\frac{s}{t}\right) = \sum_{m=0}^{n} c_m s^{\frac{m}{n}} t^{\frac{n-m}{n}}$, then

$$M_{f_a}\left(L_H, R_K\right) = S_a(H, K) \quad \text{and} \quad M_{g_c}\left(L_H, R_K\right) = B_c(H, K).$$

Set $t = e^{2x}$ for some $x \in \mathbb{R}$, then

$$\begin{aligned} f_a(e^{2x}) &= \frac{1}{2}\left(e^{2ax} + e^{2(1-a)x}\right) \\ &= \frac{1}{2}\left(e^x e^{(2a-1)x} + e^x e^{(1-2a)x}\right) \\ &= e^x \frac{1}{2}\left(e^{(2a-1)x} + e^{(1-2a)x}\right) \\ &= e^x \cosh\left((2a-1)x\right) \\ &= e^x \cosh\left(2\beta x\right) \quad \text{with } \beta = \left|a - \frac{1}{2}\right| \end{aligned}$$

and

$$\begin{aligned} g_c(e^{2x}) &= \sum_{m=0}^{n} c_m e^{\frac{2m}{n}x} \\ &= e^x \sum_{m=0}^{n} e^{2d_m x} \quad \left(d_m = \frac{m}{n} - \frac{1}{2}\right). \end{aligned}$$

Since $d_{n-m} = -d_m$ and by reindexing $m \mapsto n - m$, we have the function

$$\psi_{a,c}(x) = \frac{\cosh\left(2\beta x\right)}{\sum_{m=0}^{n} c_m e^{2d_m x}}$$

is an even function. By exploiting the symmetry $c_m = c_{n-m}$ and $d_{n-m} = -d_m$, for a fixed m we have

$$c_m e^{2d_m x} + c_{n-m} e^{2d_{(n-m)}x} = c_m\left(e^{2d_m x} + e^{-2d_m x}\right) = 2c_m \cosh(2d_m x).$$

Define a finite, positive discrete symmetrised measure

$$\mu_c = \sum_{m=0}^{n} c_m \frac{\delta_{d_m} + \delta_{-d_m}}{2}$$

where δ_{d_m} is the Dirac measure at the point d_m. By considering Borel function e^{2sx} we have

$$\sum_{m=0}^{n} c_m e^{2d_m x} = 2\sum_{m=0}^{n} c_m \cosh\left(2d_m x\right) = 2\int_{\mathbb{R}} e^{2sx} d\mu_c(s),$$

likewise

$$\cosh(2\beta x) = \int_{\mathbb{R}} e^{2sx} d\nu_\beta(s), \quad \text{with} \quad \nu_\beta = \frac{1}{2}\left(\delta_\beta + \delta_{-\beta}\right).$$

Hence,

$$\psi_{a,c}(x) = \frac{1}{2}\int_{\mathbb{R}} e^{2sx} d\gamma(s)$$

with $d\gamma(s) = \dfrac{d\nu_\beta(s)}{\int_{\mathbb{R}} e^{2sx} d\mu_c(s)}$. Because both ν_β and μ_c are even measures, positivity of $d\gamma$ reduces to a support-inclusion condition, that is

$$\operatorname{supp}\nu_\beta \subseteq \operatorname{supp}\mu_c \iff \{\pm\beta\} \subseteq \{\pm d_m : c_m > 0\} \iff \left|a - \frac{1}{2}\right| \leq \delta_c$$

and so by Theorem 1 and Lemma 1, we have

$$|||S_a(H,K)X||| \leq |||B_c(H,K)X|||\,.$$

□

The quantity $\delta_c = \min\{\,|\frac{m}{n} - \frac{1}{2}| : c_m > 0\,\}$ measures how close the weighting grid $\{m/n\}$ that actually appears in B_c places nonzero mass to the balanced exponent $1/2$. Operationally, it is the smallest *positive* distance from $1/2$ to an index with $c_m > 0$. Thus, Theorem 5 asserts that the Heinz symmetrization S_a is dominated by B_c in every unitarily invariant norm exactly when the averaging exponent lies within this distance of $1/2$, i.e.

$$|a - \tfrac{1}{2}| \leq \delta_c.$$

In the classical binomial choice $c_m = 2^{-n}\binom{n}{m}$, the nearest non-central index satisfies $|m - \frac{n}{2}| = 1$, so $\delta_c = \frac{1}{2n}$, and the condition reduces to $|a - \frac{1}{2}| \leq \frac{1}{2n}$, which is precisely Kosaki's interval recovered in Theorem 4. More generally, if all indices with $|m - \frac{n}{2}| < k$ are absent (that is, $c_m = 0$ there), then $\delta_c = \frac{k}{n}$ and the admissible range widens to $|a - \frac{1}{2}| \leq \frac{k}{n}$ (for instance, $c_0 = c_n = \frac{1}{2}$ gives $\delta_c = \frac{1}{2}$ and allows all $a \in [0,1]$).

This characterization unifies and extends Kosaki's binomial result. In Kosaki's approach [9], the admissible interval $\left|a - \frac{1}{2}\right| \leq \frac{1}{2n}$ arises via interpolation and operator means, whereas here it emerges from the positive-definiteness of $f_a(e^x)/g_c(e^x)$ and the support condition underlying δ_c. Thus, the present framework not only recovers Kosaki's sharp bound as the binomial case $c_m = 2^{-n}\binom{n}{m}$, but also clarifies its structural origin for arbitrary symmetric weights.

4 Conclusion

We have presented a function-theoretic approach—grounded in positive-definiteness and infinite divisibility via Böchner's characterization and spectral functional calculus—that yields a streamlined proof of Kosaki-type norm domination without appealing to interpolation or operator means. Specifically, for positive operators H, K and any unitarily invariant norm, the Heinz symmetrization S_a is dominated by the binomial n-term average exactly when

$$\left|a - \tfrac{1}{2}\right| \leq \tfrac{1}{2n} \quad \text{(equivalently, } a \in \tfrac{1}{2}\left(1 \pm \tfrac{1}{n}\right)\text{)},$$

and this range is sharp. Moreover, the result extends to arbitrary symmetric probability weights $c = (c_0, \ldots, c_n)$. Domination holds if and only if

$$\left|a - \tfrac{1}{2}\right| \leq \delta_c, \text{ where } \delta_c = \min\left\{\left|\tfrac{m}{n} - \tfrac{1}{2}\right| : c_m > 0\right\}.$$

This unifies and generalizes the binomial case.

Promising directions include extending the present framework to *non-symmetric* or *continuous* weight distributions and investigating quantitative stability of the domination thresholds. Further applications to refined matrix inequalities and operator-function bounds—particularly those arising in quantum information theory and numerical linear algebra—also appear within reach of this positive-definite and infinitely divisible function framework.

CRediT Authorship Contribution Statement

Imam Nugraha Albania: Conceptualization, Formal Analysis, Methodology, Writing draft.
Rizky Rosjanuardi: Writing – Review & Editing.

Declaration of Generative AI and AI-assisted technologies

ChatGPT and Writefull-integrated Overleaf technologies were used during the preparation of the manuscript, respectively, for the mathematical statement OCR, the fine-tuning of English and the proofreading.

Declaration of Competing Interest

The authors declare no conflict of interest.

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Data Availability

Data supporting the findings of this study are available from the corresponding author on a reasonable request and subject to confidentiality agreements.

References

[1] A. McIntosh, "Heinz inequalities and perturbation of spectral families," *Macquarie Mathematical Reports*, vol. 79-0006, 1979.

[2] R. Bhatia and C. Davies, "More matrix forms of the arithmetic-geometric mean inequality," *SIAM Journal on Matrix Analysis and Applications*, vol. 14, no. 1, pp. 132–136, 1993. DOI: https://doi.org/10.1137/0614012.

[3] X. Zhan, "Inequalities for unitarily invariant norms," *SIAM Journal on Matrix Analysis and Applications*, vol. 20, no. 2, pp. 466–470, 1998.

[4] F. Hiai and H. Kosaki, "Comparison of various means for operators," *Journal of Functional Analysis*, vol. 163, no. 2, pp. 300–323, 1999.

[5] F. Hiai and H. Kosaki, "Means for matrices and comparison of their norms," *Indiana University Mathematics Journal*, vol. 48, no. 3, pp. 899–936, 1999.

[6] R. Bhatia, "Infinitely divisible matrices," *The American Mathematical Monthly*, vol. 113, no. 3, pp. 221–235, 2006. DOI: https://doi.org/10.1080/00029890.2006.11920300.

[7] X. Xiao, F. Zhang, Y. Cao, and C. Zhang, "Some matrix inequalities related to norm and singular values," *AIMS Mathematics*, vol. 9, no. 2, pp. 4205–4210, 2023. DOI: https://doi.org/10.3934/math.2024207.

[8] H. Kosaki, "On infinite divisibility of positive definite functions arising from operator means," *Journal of Functional Analysis*, vol. 254, no. 1, pp. 84–108, 2008. DOI: https://doi.org/10.1016/j.jfa.2007.09.021.

[9] H. Kosaki, “A certain generalization of the heinz inequality for positive operators,” *International Journal of Mathematics*, vol. 27, no. 2, p. 2 650 013, 2016. DOI: https://doi.org/10.1142/S0129167X16500087.

[10] M. Bakherad, “Unitarily invariant norm inequalities involving G_1 operators,” *Communications of the Korean Mathematical Society*, vol. 33, no. 3, pp. 889–899, 2018. DOI: https://doi.org/10.4134/CKMS.c170329.

[11] T. Hayashi, “On a norm inequality for a positive block-matrix,” *Linear Algebra and its Applications*, vol. 576, pp. 273–284, 2019. DOI: https://doi.org/10.1016/j.laa.2018.12.027.

[12] M. W. Alomari, “Operator popoviciu’s inequality for superquadratic and convex functions of selfadjoint operators in hilbert spaces,” *Advances in Pure and Applied Mathematics*, vol. 10, no. 4, pp. 313–324, 2019. DOI: https://doi.org/10.1515/apam-2018-0154.

[13] M. Buhmann and Y. Xu, “Positive definite functions on a regular domain,” *IMA Journal of Numerical Analysis*, vol. 45, pp. 2824–2843, 2025. DOI: https://doi.org/10.1093/imanum/drae074.

[14] P. Massey, D. Stojanoff, and S. Zarate, “Norm inequalities for the spectral spread of hermitian operators,” *Mathematische Nachrichten*, vol. 296, no. 9, pp. 4335–4356, 2023. DOI: https://doi.org/10.1002/mana.202200350.

[15] A. Al-Natoor, S. Benzamia, and F. Kittaneh, “Unitarily invariant norm inequalities for positive semidefinite matrices,” *Linear Algebra and its Applications*, vol. 633, pp. 303–315, 2022. DOI: https://doi.org/10.1016/j.laa.2021.10.012.

[16] Z. Lin, H. G. Müller, and B. U. Park, “Additive models for symmetric positive-definite matrices and lie groups,” *Biometrika*, vol. 110, no. 2, pp. 361–379, 2023. DOI: https://doi.org/10.1093/biomet/asac055.

[17] A. Chavez, “Norms on complex matrices induced by random vectors ii: Extension of weakly unitarily invariant norms,” *Canadian Mathematical Bulletin*, vol. 67, no. 2, pp. 447–457, 2023. DOI: https://doi.org/10.4153/S0008439523000875.

[18] S. Chen and H. Hu, “Unitarily invariant norms on dual quaternion matrices,” *Pacific Journal of Optimization*, vol. 20, no. 2, pp. 371–387, 2024. DOI: http://www.yokohamapublishers.jp/online-p/PJO/vol20/pjov20n2p371.pdf.

[19] I. N. Albania and M. Nagisa, “Some families of operator norm inequalities,” *Linear Algebra and its Applications*, vol. 534, pp. 102–121, 2017. DOI: https://doi.org/10.1016/j.laa.2017.06.030.