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Modification of Norms on the Sequence Space ℓ^p

Rikha Syahda Salsabilla and Mochammad Idris*

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lambung Mangkurat, Banjarbaru 70714, Indonesia

Abstract

In this article, we explore sequence spaces by introducing a newly defined norm. We construct the Orlicz sequence space and demonstrate that the norm in this space is equivalent to the norm in the ℓ^p space. As a result, the fundamental properties of the ℓ^p space are carried over to the Orlicz sequence space through the equivalence of norms. One significant implication of this result is that Hölder's inequality, which holds in ℓ^p spaces, can also be applied to the Orlicz sequence space with different positif constan.

Keywords: ℓ^p space; Orlicz sequence space; Luxemburg norm;Young function; Hölder's inequality.

1. Introduction

A vector space equipped with a norm serves as a foundation in modern functional analysis, providing powerful tools for understanding various complex mathematical structures. A norm on a vector space functions as a measure that expresses the "length" of a vector within that space. Formally, a norm is a mapping denoted by $\|\cdot\|_X$, defined as $\|\cdot\|_X : X \to \mathbb{R}$. The norm must satisfy the following properties, as cited in [1, 2], for all $x, y \in X$ and $\alpha \in \mathbb{R}$

1. (Non-negativity) $\|x\|_X \geq 0$ and $\|x\|_X = 0 \iff x = 0$,
2. (Absolute homogeneity) $\|\alpha x\|_X = |\alpha|\|x\|_X$,
3. (Triangle inequality) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$.

If a vector space X is equipped with a norm $\|\cdot\|_X$, then the pair $(X, \|\cdot\|_X)$ is called a normed space.

One of the vector spaces frequently studied in functional analysis is the set consisting of sequences of real numbers $x = (x_k)$. According to [3], the membership definition of the space ℓ^p, for $1 \leq p < \infty$, is given by

$$\ell^p := \left\{ (x_k) \;\middle|\; \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}.$$

By [4] the norm commonly used in the space ℓ^p is defined as

$$\|x\|_{\ell^p} := \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}, \quad \text{for every } x \in \ell^p.$$

***Corresponding author**. E-mail: moch.idris@ulm.ac.id

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We observe that the structure of the sequence space ℓ^p is entirely determined by the analytic properties of the generating function $f(t) = t^p$. A sequence $x = (x_k)$ belongs to ℓ^p precisely when the series $\sum_{k=1}^{\infty} |x_k|^p = \sum_{k=1}^{\infty} f(|x_k|)$ is finite, so the membership criterion can be viewed as requiring that the total accumulated value of the transformed terms $f(|x_k|)$ remains bounded. The norm, $\|x\|_{\ell^p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$, is constructed directly from this generating function and thus inherits its essential features. The choice of $f(t) = t^p$ is crucial because it is both convex and p-homogeneous: convexity guarantees Minkowski's inequality and hence the triangle inequality, while homogeneity ensures the correct scaling rule $\|\lambda x\|_{\ell^p} = |\lambda| \, \|x\|_{\ell^p}$. Thus, the function $f(t) = t^p$ is not merely useful but is the fundamental element that generates and governs the entire structure of the space ℓ^p. In addition, $f(t)$ is recognized as a Young function (see [5, 6]), exhibiting the standard features of convexity, monotonic increase, and the limiting conditions $f(t) \to 0$ as $t \to 0$ and $f(t) \to \infty$ as $t \to \infty$. Thus, the analytic and geometric features of ℓ^p are determined by the behavior of the generating function $f(t)$.

The space ℓ^p, equipped with the norm $\|\cdot\|_{\ell^p}$, is a Banach space, as stated in [7] and [8], means that every Cauchy sequence in this space is convergent. Based on [7] and [9] with different values of the parameter p, the ℓ^p spaces satisfy an inclusion property, if $1 \le p_1 < p_2 < \infty$, then

$$\ell^{p_1} \subseteq \ell^{p_2}$$

and

$$\|x\|_{p_2} \le \|x\|_{p_1}, \quad \text{for every } x \in \ell^{p_1}.$$

We observe that the inequality $\|\cdot\|_{p_2} \le \|\cdot\|_{p_1}$ is not strict, as demonstrated by the sequence $x_a = (1, 0, 0, \cdots)$, which satisfies $\|x_a\|_{p_2} = \|x_a\|_{p_1} = 1$. Meanwhile, the inclusion is strict (proper), as illustrated by the sequence $y = (y_k)$ with $y_k := \frac{1}{k^{1/p_1}}$ where $k \in \mathbb{N}$. A direct computation shows that $\|y\|_{p_1} = \infty$, whereas $\|y\|_{p_2} < \infty$. Consequently, $y \notin \ell^{p_1}$ but $y \in \ell^{p_2}$. This demonstrates that there exist sequences belonging to ℓ^{p_2}, not to ℓ^{p_1}, and therefore the inclusion $\ell^{p_1} \subset \ell^{p_2}$ is proper (see [10, 11]).

In this article, we construct an Orlicz sequence space equipped with a norm generated by a specific Young function. We then examine its relationship with the classical space ℓ^p (for an appropriate choice of p) and analyze the connections between their corresponding norms. This investigation provides insight into which properties are shared by the two spaces.

2. Methods and Related Works

This article undertakes a comprehensive theoretical investigation with the objective of establishing a new norm on the space ℓ^p. In doing so, it incorporates an extensive literature review that examines various articles and books relevant to the topic, with a particular focus on the concepts of sequence spaces and discrete Orlicz spaces.

The study begins by modifying the well-known function $f(t) = t^p$ into a different function that retains the essential characteristics of a Young function. This modification is crucial as it allows us to explore the properties of the resulting sequence space, which is defined by a norm generated from this specific Young function. The resulting structure forms a Orlicz sequence space, which is significant in functional analysis. The primary aim of this research is to investigate whether this modified Young function can still be utilized to define a norm on the ℓ^p space, thereby expanding the theoretical framework surrounding these spaces. Additionally, the study seeks to analyze the implications of this modification on the properties of the space, including aspects such as completeness, convergence, and boundedness. By delving into these areas, the article aims to contribute to a deeper understanding of the interplay between Young functions and sequence spaces, ultimately enhancing the theoretical landscape of functional analysis.

3. Results and Discussion

A function $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is called a Young function if it satisfies the following conditions, as defined by Krasnosel'skii and Rutickii (1961) in the framework of Orlicz space theory.

1. Convexity with $\lambda \in [0, 1]$, it holds that

$$\psi(\lambda t_1 + (1-\lambda)t_2) \leq \lambda\psi(t_1) + (1-\lambda)\psi(t_2),$$

   for every $t_1, t_2 \in [0, \infty)$.
2. Monotonically increasing, if $t_1 \leq t_2$, then $\psi(t_1) \leq \psi(t_2)$.
3. When $t \to 0$, then it applies $\psi(t) \to 0$, and $\psi(0) = 0$.
4. When $t \to \infty$, then it applies $\psi(t) \to \infty$.

We also have the property in the following lemma.

Lemma 1. *For $0 < u < v$, it holds that $\frac{\psi(u)}{u} \leq \frac{\psi(v)}{v}$.*

Proof. Let $0 \leq s < u < v$, and define

$$\lambda := \frac{u-s}{v-s} \in (0, 1),$$

such that $u = (1-\lambda)s + \lambda v$. By the convexity of the function ψ, we have

$$\psi(u) \leq (1-\lambda)\psi(s) + \lambda\psi(v).$$

Subtracting $\psi(s)$ from both sides yields

$$\begin{aligned}\psi(u) - \psi(s) &\leq (1-\lambda)\psi(s) + \lambda\psi(v) - \psi(s)\\ &= \lambda(\psi(v) - \psi(s)).\end{aligned}$$

Dividing both sides by $u - s = \lambda(v - s)$, we obtain

$$\frac{\psi(u) - \psi(s)}{u-s} \leq \frac{\psi(v) - \psi(s)}{v-s}.$$

In particular, for $s = 0$, and since $\psi(0) = 0$, the inequality becomes

$$\frac{\psi(u)}{u} \leq \frac{\psi(v)}{v}.$$

□

Here, we propose to modify the classical Young function $f(t) = t^p$ into another Young function. Let $q \geq 1$, and define the function

$$\psi_q(t) := t^q \ln(t+1), \text{for every } 0 \leq t < \infty.$$

We will now show that the above function satisfies all the criteria of a Young function.

Lemma 2. *The function $\psi_q(t) := t^q \ln(t+1)$, for every $0 \leq t < \infty$ is a Young function.*

Proof. Observe that $\psi_q(t)$ is differentiable, which allows us to verify the Young function criteria using its derivatives. We compute

$$\psi_q'(t) = qt^{q-1}\ln(t+1) + \frac{t^q}{t+1} \tag{1}$$

$$\psi_q''(t) = q(q-1)t^{q-2}\ln(t+1) + 2\frac{qt^{q-1}}{t+1} - \frac{(q-1)t^q}{(t+1)^2}. \tag{2}$$

1. (Convex) to show that $\psi_q(t)$ is convex, we observe the second derivative in Eq. (2), since $q \geq 1$, it follows $q(q-1)t^{q-2}\ln(t+1) \geq 0$ and $2\frac{qt^{q-1}}{t+1} - \frac{(q-1)t^q}{(t+1)^2} \geq 0$ for every $t \geq 0$. Thus $\psi_q''(t) \geq 0$. According to [12], $\psi_q(t)$ is convex.
2. (Monotonically increasing) to prove that $\psi_q(t)$ is increasing, consider the first derivative in Eq. (1). For $t \geq 0$, we have $\psi_q'(t) \geq 0$. Therefore, $\psi_q(t)$ is monotonically increasing.
3. (Behavior near zero) as $t \to 0$, the function $\psi_q(t) = t^q \ln(1+t) \to 0$, since $t^q \to 0$ and $\ln(1+t) \to 0$. By the product of limits,

$$\lim_{t\to 0} t^q \ln(1+t) = \lim_{t\to 0} t^q \times \lim_{t\to 0} \ln(1+t) = 0.$$

4. (Behavior at infinity) since $\ln(t+1) > 0$ and grows unbounded as $t \to \infty$, and since $t^q \ln(t+1) \geq t^q$, we have $\psi_q(t) \to \infty$ as $t \to \infty$.

Therefore, the function $\psi_q(t)$ satisfies all the conditions of a Young function. □

3.1. The Orlicz Sequence Space ℓ^{ψ_q}

From the Young function $\psi_q(t)$, we can define an Orlicz sequence space as the set of real sequences $x = (x_k)_{k=1}^{\infty}$. This set forms a vector space and is called the Orlicz sequence space ℓ^{ψ_q}, defined by

$$\ell^{\psi_q} := \left\{ x = (x_k) \;\middle|\; \sum_{k=1}^{\infty} \psi_q\left(\frac{|x_k|}{A}\right) < \infty \right\},$$

for some $A > 0$. The space ℓ^{ψ_q} is equipped with a norm defined by

$$\|x\|_{\ell^{\psi_q}} := \inf\left\{ A > 0 \;\middle|\; \sum_{k=1}^{\infty} \psi_q\left(\frac{|x_k|}{A}\right) \leq 1 \right\},$$

for every $x \in \ell^{\psi_q}$. This type of norm is known as the Luxemburg norm, first introduced by W.A.J. Luxemburg in 1955. The pair $(\ell^{\psi_q}, \|\cdot\|_{\ell^{\psi_q}})$ becomes a normed space, whose structure is determined by the Young function $\psi_q(t) = t^q \ln(1+t)$. Further more, we will examine the relationship between the Orlicz sequence space and the classical sequence space ℓ^p.

Next, note that $\ln(t+1)$ as a component of the function $\psi_q(t)$, is a differentiable function. Therefore, we can consider the Taylor series expansion of $\ln(t+1)$ as follows

$$\ln(t+1) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots, \quad \text{for } 0 \leq t \leq 1. \tag{3}$$

Consequently,

$$\psi_q(t) = t^q \ln(1+t) = t^q\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\right) = t^q \sum_{k=1}^{\infty} (-1)^{k+1}\frac{t^k}{k}.$$

Next, the Expression (3) for $\psi_q(t)$ is useful in proving the following lemma.

Lemma 3. *For $0 \le t \le 1$, the following inequality holds*

$$\frac{t^{q+1}}{2} \le \psi_q(t) \le t^{q+1}.$$

Proof. Let $t \in [0,1]$ be arbitrary. Note that

$$\begin{aligned}\psi_q(t) = t^q \ln(t+1) &= t^{q+1}\left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \dots\right)\\ &= t^{q+1}M,\end{aligned}$$

with $M = 1 - \sum_{k=1}^{\infty} t^{2k-1}\left(\frac{1}{2k} - \frac{t}{2k+1}\right) > 0$. It can be verified that $M \le 1$, hence

$$\psi_q(t) \le t^{q+1}.$$

A similar analysis shows that $M = 1 - \frac{t}{2} + \left(\sum_{k=1}^{\infty} t^{2k}\left(\frac{1}{2k+1} - \frac{t}{2k+2}\right)\right)$. Each term $\left(\frac{1}{2k+1} - \frac{t}{2k+2}\right)$ is positive. Thus, $M \ge 1 - \frac{t}{2} \ge \frac{1}{2}$ which implies

$$\psi_q(t) \ge \frac{t^{q+1}}{2}.$$

Therefore, for every $t \in [0,1]$ we have $\frac{t^{q+1}}{2} \le \psi_q(t) \le t^{q+1}$. □

This lemma serves an important role in the equivalence of two norms, as our arguments will involve real numbers in the interval $[0,1]$. We note that for $x \in \ell^{q+1}$, written as $x = (x_1, x_2, \dots) = (x_k)_{k\in\mathbb{N}}$, and we have

$$|x_k|^{q+1} \le \sum_{k=1}^{\infty} |x_k|^{q+1} = \|x\|_{\ell^{q+1}}^{q+1} < \infty \qquad \text{for every } k \in \mathbb{N}.$$

Observe that not all terms $|x_k|$ can exceed 1, since the finiteness of

$$\sum_{k=1}^{\infty} |x_k|^{q+1} < \infty$$

ensures that $|x_k| \to 0$ while $k \to \infty$. Although there exists an index k_a for which $|x_{k_a}| > 1$, we still have

$$|x_{k_a}| \le \left(\sum_{k=1}^{\infty} |x_k|^{q+1}\right)^{1/(q+1)} = \|x\|_{\ell^{q+1}}.$$

Consequently, $0 \le \frac{|x_{k_a}|}{\|x\|_{\ell^{q+1}}} \le 1$ holds. In particular, this yields

$$0 \le \frac{|x_k|}{\|x\|_{\ell^{q+1}}} \le 1, \qquad \text{for every } k \in \mathbb{N}.$$

We label this condition as the normalized form. This will be discussed further in the next subsection.

Remark 1. Note that the above inequality cannot hold on L^p (a Lebesgue space).

3.2. Equivalence of Norms

A normed space may be equipped with more than one norm. It is important to investigate the relationship between these norms. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms defined on a vector space X. These norms are said to be equivalent, denoted by $\|\cdot\|_a \sim \|\cdot\|_b$, if there exist positive constants $C_1, C_2 > 0$ such that for all $x \in X$,

$$C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a.$$

In the space ℓ^p, the norm used is not always equivalent to other norms, as discussed in [13]. In ℓ^{ψ_q} we consider the norm $\|\cdot\|_{\ell^{\psi_q}}$ and seek to find an equivalent norm from the ℓ^p space. For $p = q+1$, we will examine the relationship between $\|\cdot\|_{\ell^{\psi_q}}$ and $\|\cdot\|_{\ell^{q+1}}$. Next, we will turn our attention to the structure of the space ℓ^{ψ_q}.

Theorem 1. *For $q \geq 1$, the following holds*

$$\frac{1}{2}\|x\|_{\ell^{q+1}} \leq \|x\|_{\ell^{\psi_q}} \leq \|x\|_{\ell^{q+1}}, \quad \text{for every } x \in \ell^{q+1}.$$

Proof. Take an arbitrary element $x = (x_k) \in \ell^{q+1}$. Define

$$y = (y_k) := \frac{x}{\|x\|_{\ell^{q+1}}}, \qquad y_k := \frac{x_k}{\|x\|_{\ell^{q+1}}}, \quad k \in \mathbb{N}.$$

Then $0 \leq |y_k| = \frac{|x_k|}{\|x\|_{\ell^{q+1}}} \leq 1$ for every $k \in \mathbb{N}$, and

$$\sum_{k=1}^{\infty} |y_k|^{q+1} = \sum_{k=1}^{\infty}\left(\frac{|x_k|}{\|x\|_{\ell^{q+1}}}\right)^{q+1} = 1 < \infty,$$

so that $\|y\|_{\ell^{q+1}} = 1$.
Using Lemma (3), since $0 \leq |y_k| \leq 1$, then we obtain

$$\frac{1}{2}|y_k|^{q+1} \leq \psi_q(|y_k|) \leq |y_k|^{q+1},$$

for every $k \in \mathbb{N}$ and summing over k,

$$\frac{1}{2} = \frac{1}{2}\sum_{k=1}^{\infty}|y_k|^{q+1} \leq \sum_{k=1}^{\infty}\psi_q(|y_k|) \leq \sum_{k=1}^{\infty}|y_k|^{q+1} = 1.$$

Thus $y \in \ell^{\psi_q}$, and in particular $\sum_{k=1}^{\infty}\psi_q\left(\frac{|y_k|}{1}\right) = \sum_{k=1}^{\infty}\psi_q(|y_k|) \leq 1$. Hence there exists $A > 0$ such that

$$\sum_{k=1}^{\infty}\psi_q\left(\frac{|y_k|}{1}\right) \leq \sum_{k=1}^{\infty}\psi_q\left(\frac{|y_k|}{A}\right) = 1.$$

By the Luxemburg norm on ℓ^{ψ_q}, this implies $\|y\|_{\ell^{\psi_q}} = A \leq 1$, because ψ_q is increasing. Since $\|y\|_{\ell^{\psi_q}} = \left\|\frac{x}{\|x\|_{\ell^{q+1}}}\right\|_{\ell^{\psi_q}}$, then we conclude that

$$\left\|\frac{x}{\|x\|_{\ell^{q+1}}}\right\|_{\ell^{\psi_q}} \leq 1, \qquad \text{or} \qquad \|x\|_{\ell^{\psi_q}} \leq \|x\|_{\ell^{q+1}}.$$

Next, from other part, we have

$$\frac{1}{2} \leq \sum_{k=1}^{\infty}\psi_q(|y_k|) \qquad \text{or} \qquad 1 \leq \sum_{k=1}^{\infty}2\psi_q(|y_k|).$$

We apply Lemma 1, which states that for $0 < u < v$,

$$\frac{\psi_q(u)}{u} \le \frac{\psi_q(v)}{v} \quad \Rightarrow \quad \psi_q(u) \le \frac{u}{v}\,\psi_q(v).$$

Taking $v = 2u$, we obtain

$$\psi_q(u) \le \tfrac{1}{2}\psi_q(2u) \quad \Rightarrow \quad 2\psi_q(u) \le \psi_q(2u).$$

Applying this to $u = |y_k|$, we get

$$1 \le \sum_{k=1}^{\infty} 2\psi_q(|y_k|) \le \sum_{k=1}^{\infty} \psi_q(2|y_k|) = \sum_{k=1}^{\infty} \psi_q\left(\frac{|y_k|}{1/2}\right).$$

Thus there exists $A > 0$ such that

$$1 = \sum_{k=1}^{\infty} \psi_q\left(\frac{y_k}{A}\right) \le \sum_{k=1}^{\infty} \psi_q\left(\frac{|y_k|}{1/2}\right).$$

Hence, by the Luxemburg norm definition and the increasing property of ψ_q, we get

$$\|y\|_{\ell^{\psi_q}} = A \ge \tfrac{1}{2}.$$

Since $\|y\|_{\ell^{\psi_q}} = \left\| \frac{x}{\|x\|_{\ell^{q+1}}} \right\|_{\ell^{\psi_q}}$, we conclude that

$$\frac{1}{2} \le \left\| \frac{x}{\|x\|_{\ell^{q+1}}} \right\|_{\ell^{\psi_q}} \qquad \text{or} \qquad \frac{1}{2}\,\|x\|_{\ell^{q+1}} \le \|x\|_{\ell^{\psi_q}}.$$

Combining the two inequalities, $\frac{1}{2}\|x\|_{\ell^{q+1}} \le \|x\|_{\ell^{\psi_q}} \le \|x\|_{\ell^{q+1}}$, for every $x \in \ell^{q+1}$. □

The above theorem asserts that the norms $\|\cdot\|_{\ell^{\psi_q}}$ and $\|\cdot\|_{\ell^{q+1}}$ are equivalent or $\|\cdot\|_{\ell^{\psi_q}} \sim \|\cdot\|_{\ell^{q+1}}$. This result plays a crucial role in showing that the underlying vector spaces are in fact identical. We now state this as an immediate corollary.

Corollary 1. *For $q \ge 1$ we have*

$$\ell^{q+1} = \ell^{\psi_q},$$

with $\|\cdot\|_{\ell^{\psi_q}} \sim \|\cdot\|_{\ell^{q+1}}$.

Proof. Let $x = (x_k) \in \ell^{q+1}$, then $\|x\|_{\ell^{q+1}} < \infty$. By the right inequality in Theorem 1 we obtain

$$\|x\|_{\ell^{\psi_q}} \le \|x\|_{\ell^{q+1}} < \infty.$$

Thus $x \in \ell^{\psi_q}$, so $\ell^{q+1} \subseteq \ell^{\psi_q}$.

Take $x = (x_k) \in \ell^{\psi_q}$. Then there exists $A > 0$ such that $\sum_{k=1}^{\infty} \psi_q\left(\frac{|x_k|}{A}\right) = 1$. It means $A = \|x\|_{\ell^{\psi_q}}$. From the left inequality in Theorem 1 it follows that

$$\frac{1}{2}\|x\|_{\ell^{q+1}} \le \|x\|_{\ell^{\psi_q}} < \infty.$$

Hence $x \in \ell^{q+1}$, which shows that $\ell^{\psi_q} \subseteq \ell^{q+1}$. From these two inclusions we conclude that

$$\ell^{q+1} = \ell^{\psi_q},$$

with equivalent norms. □

It is worth noting that the statement of the corollary could in fact be accepted without a separate proof. Indeed, once Theorem 1 establishes that the Luxemburg norm $\|\cdot\|_{\ell^{\psi_q}}$ is equivalent to the classical norm $\|\cdot\|_{\ell^{q+1}}$, it immediately follows that the sets ℓ^{q+1} and ℓ^{ψ_q} coincide as vector spaces. By [7] we know that ℓ^{q+1} is a Banach space. It means that every Cauchy sequence respect to usual norm in ℓ^{q+1} is a convergent sequence.

Corollary 2. *The space ℓ^{ψ_q} is a Banach space.*

Proof. From Theorem (1), we obtain

$$\frac{1}{2}\|x\|_{\ell^{q+1}} \leq \|x\|_{\ell^{\psi_q}} \leq \|x\|_{\ell^{q+1}} \qquad \text{for every } x \in \ell^{\psi_q}. \tag{4}$$

Take a Cauchy sequence $(x^{(n)})_{n\in\mathbb{N}}$ in the space $(\ell^{\psi_q}, \|\cdot\|_{\ell^{\psi_q}})$. That is, for every $\varepsilon > 0$ there exists N_ε such that for all $m, n \geq N_\varepsilon$ we have

$$\|x^{(n)} - x^{(m)}\|_{\ell^{\psi_q}} < \varepsilon.$$

Using the left-hand side of Inequality (4), we obtain

$$\frac{1}{2}\|x^{(n)} - x^{(m)}\|_{\ell^{q+1}} \leq \|x^{(n)} - x^{(m)}\|_{\ell^{\psi_q}} < \varepsilon,$$

so that

$$\|x^{(n)} - x^{(m)}\|_{\ell^{q+1}} < 2\varepsilon.$$

Thus, $(x^{(n)})$ is also a Cauchy sequence in the space $(\ell^{q+1}, \|\cdot\|_{\ell^{q+1}})$. Since ℓ^{q+1} is a Banach space, there exists $x \in \ell^{q+1}$ such that

$$x^{(n)} \to x \quad \text{in } \ell^{q+1}.$$

That is, for every $\varepsilon > 0$ there exists N_ε such that for every $n \geq N_\varepsilon$ we have

$$\|x^{(n)} - x\|_{\ell^{q+1}} < \varepsilon.$$

Next, from the right-hand side of Inequality (4) we obtain

$$\|x^{(n)} - x\|_{\ell^{\psi_q}} \leq \|x^{(n)} - x\|_{\ell^{q+1}}.$$

Hence, for every $\varepsilon > 0$ there exists N_ε such that for every $n \geq N_\varepsilon$ we have

$$\|x^{(n)} - x\|_{\ell^{\psi_q}} < \varepsilon.$$

Therefore, $x^{(n)} \to x$ in ℓ^{ψ_q}. This shows that every Cauchy sequence in $(\ell^{\psi_q}, \|\cdot\|_{\ell^{\psi_q}})$ converges in ℓ^{ψ_q}. Thus, ℓ^{ψ_q} is a Banach space. ∎

This means that the two norms induce the same topology on the respective sequence space. Consequently, the modification of the norm does not alter the underlying structure of the space.

We have shown that $\ell^{\psi_q} = \ell^p$ with $\|\cdot\|_{\ell^{\psi_q}} \sim \|\cdot\|_{\ell^p}$ where $p = q + 1$. This equivalence holds for $p \geq 2$. However, what happens when $p \in [1, 2)$? We can consider a new modification using a different Young function. Let $q \geq 1$ and $s \geq 0$. Now we define $\psi_q^s : [0, \infty) \to [0, \infty)$ using

$$\psi_q^s(t) := t^q \ln(1 + t^s)$$

for every $t \in [0, \infty)$. It is important to confirm that the function ψ_q^s is a Young function. This can be verified using a similar argument as in Lemma (1).

Lemma 4. *Let $q \geq 1$ and $s \geq 0$. We have ψ_q^s as a Young function on $[0, \infty)$.*

Proof. It is straightforward to verify that $\psi_q^s(0) = 0$, that $\lim_{t\to 0^+} \psi_q^s(t) = 0$, and that $\lim_{t\to\infty} \psi_q^s(t) = \infty$. We now proceed to show that ψ_q^s is an increasing function. For $t > 0$, compute the derivative:

$$(\psi_q^s)'(t) = \frac{d}{dt}(t^q \ln(1+t^s)) = t^{q-1}\left(q\ln(1+t^s) + \frac{st^s}{1+t^s}\right).$$

Since $t^s \geq 0$, we have $\ln(1+t^s) \geq 0$ and $\frac{st^s}{1+t^s} \geq 0$. Because $t^{q-1} \geq 0$ and $q \geq 1$, it follows that

$$\psi'(t) \geq 0 \qquad \text{for every } t > 0.$$

A direct limit computation shows that ψ' is non negative at $t = 0$. Thus ψ_q^s is increasing on $[0, \infty)$.

Next we prove convexity. Let

$$u := t^s, \qquad f(u) := q\ln(1+u) + \frac{su}{1+u}.$$

Then $(\psi_q^s)'(t) = t^{q-1} f(u)$. Differentiating again gives

$$(\psi_q^s)''(t) = (q-1)t^{q-2} f(u) + t^{q-1} f'(u)\,\frac{du}{dt}.$$

We compute

$$f'(u) = \frac{q}{1+u} + \frac{s}{(1+u)^2} > 0, \qquad \frac{du}{dt} = st^{s-1} \geq 0.$$

Since $f(u) \geq 0$, $q-1 \geq 0$, and $t^{q-2} \geq 0$, each term in $(\psi_q^s)''(t)$ is nonnegative. Thus

$$(\psi_q^s)''(t) \geq 0 \qquad \text{for every } t > 0.$$

Convexity on $(0, \infty)$ follows, and continuity extends it to the interval $[0, \infty)$. Hence, ψ_q^s is a Young function on $[0, \infty)$. □

We will examine the behavior of the function ψ_q^s near $t = 0$. This is described in the following lemma.

Lemma 5. *For $0 \leq t \leq 1$, we have*

$$\frac{1}{2}t^{q+s} \leq \psi_q^s(t) \leq t^{q+s}.$$

Proof. For $0 \leq t \leq 1$, we will show

$$\frac{1}{2}t^{q+s} \leq \psi_q^s(t) \leq t^{q+s}.$$

We start with the Taylor expansion of $\ln(1+t^s)$

$$\begin{aligned}\ln(1+t^s) &= t^s - \frac{t^{2s}}{2} + \frac{t^{3s}}{3} - \cdots \\ &= t^s - t^{2s}\Big(\frac{1}{2} - \frac{t^s}{3} + \cdots\Big) \qquad (5)\\ &= t^s - \frac{t^{2s}}{2} + t^{3s}\Big(\frac{1}{3} - \frac{t^s}{4} + \cdots\Big) \qquad (6)\end{aligned}$$

for $0 \le t \le 1$. From Eq. (5), take the first term, and from Eq. (6), take the first and second terms. Then we obtain

$$t^s - \frac{t^{2s}}{2} \le \ln(1+t^s) \le t^s.$$

for every $t \in [0,1]$. Multiplying this inequality by t^q, we get

$$\begin{aligned} t^q \cdot t^s \left(1 - \frac{t^s}{2}\right) \le \psi_q^s(t) \le t^q \cdot t^s \\ \frac{1}{2} t^{q+s} \le \psi_q^s(t) \le t^{q+s}, \end{aligned}$$

for every $0 \le t \le 1$. □

Using the function ψ_q^s, we define

$$\ell^{\psi_q^s} := \left\{ x = (x_k) \;\middle|\; \|x\|_{\ell^{\psi_q^s}} < \infty \right\},$$

with $\|x\|_{\ell^{\psi_q^s}} := \inf \left\{ A > 0 \;\middle|\; \sum_{k=1}^{\infty} \psi_q^s\left(\frac{|x_k|}{A}\right) \le 1 \right\}$, for every $x \in \ell^{\psi_q^s}$. We see that for the case $s = 1$, it has been discussed in the above. It also can be seen as a special case. Here, we generalize it for $0 \le s < \infty$.

Theorem 2. *For $1 \le q < \infty$ and $0 \le s < \infty$, we have*

$$\ell^{\psi_q^s} = \ell^{q+s},$$

with $\frac{1}{2}\|\cdot\|_{\ell^{q+s}} \le \|\cdot\|_{\ell^{\psi_q^s}} \le \|\cdot\|_{\ell^{q+s}}$.

Proof. We know that $\psi_q^s(t)$ is a Young function. It has convex property and satisfies $\psi_q^s(0) = 0$. From the previous results, for sufficiently small t (e.g., within the interval $[0,1]$), we have $\ln(1+t^s) \sim t^s$, so that $\psi_q^s(t) \sim t^{q+s}$. By Lemma (5), therefore

$$\frac{1}{2} t^{q+s} \le \psi_q^s(t) \le t^{q+s}$$

for $0 \le t \le 1$.

Suppose $x \in \ell^{q+s}$, then $\sum_{k=1}^{\infty} |x_k|^{q+s} < \infty$. We also have the fact that $x \in \ell^\infty$ since $\ell^{q+s} \subset \ell^\infty$, so $\|x\|_{\ell^\infty} := \sup_{k \in \mathbb{N}} |x_k| < \infty$. Consequently, for $E = \|x\|_{\ell^\infty}$, we have $0 \le \frac{|x_k|}{E} \le 1$ for every $k \in \mathbb{N}$. Since $\psi_q^s(t) \le t^{q+s}$ for every $t \in [0,1]$, it follows that

$$\sum_{k=1}^{\infty} \psi_q^s\left(\frac{|x_k|}{E}\right) \le \sum_{k=1}^{\infty} \left(\frac{|x_k|}{E}\right)^{q+s} < \infty.$$

It means $x \in \ell^{\psi_q^s}$.

Conversely, suppose $x \in \ell^{\psi_q^s}$, choose sufficiently large $D > 0$ such that $\sum_{k=1}^{\infty} \psi_q^s\left(\frac{|x_k|}{D}\right) \leq 1$ and $0 \leq \frac{|x_k|}{D} \leq 1$ for every $k \in \mathbb{N}$. Using Lemma (5), we have

$$\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{|x_k|}{D}\right)^{q+s} \leq \sum_{k=1}^{\infty}\psi_q^s\left(\frac{|x_k|}{D}\right) \leq 1.$$

So $\sum_{k=1}^{\infty} |x_k|^{q+s} \leq 2D^{q+s} < \infty$ and $x \in \ell^{q+s}$. Hence $\ell^{\psi_q^s} = \ell^{q+s}$. Next, to show equivalence between $\|\cdot\|_{\ell^{\psi_q^s}}$ and $\|\cdot\|_{\ell^{q+s}}$, one may use similar process in the proof of Theorem (1). Take $x = (x_k) \in \ell^{q+s}$ be arbitrary. Define the normalized sequence

$$z = (z_k) := \frac{x}{\|x\|_{\ell^{q+s}}}, \qquad z_k = \frac{x_k}{\|x\|_{\ell^{q+s}}},\ k \in \mathbb{N}.$$

We also have $0 \leq |z_k| = \frac{|x_k|}{\|x\|_{\ell^{q+s}}} \leq 1$ for very $k \in \mathbb{N}$. We compute to get

$$\sum_{k=1}^{\infty}|z_k|^{q+s} = \sum_{k=1}^{\infty}\left(\frac{|x_k|}{\|x\|_{\ell^{q+s}}}\right)^{q+s} = 1,$$

so that $\|z\|_{\ell^{q+s}} = 1$. By Lemma (5), the bound $0 \leq |z_k| \leq 1$ implies

$$\frac{1}{2}|z_k|^{q+s} \leq \psi_q^s(|z_k|) \leq |z_k|^{q+s}, \qquad k \in \mathbb{N}.$$

Summing these inequalities yields

$$\frac{1}{2} = \frac{1}{2}\sum_{k=1}^{\infty}|z_k|^{q+s} \leq \sum_{k=1}^{\infty}\psi_q^s(|z_k|) \leq \sum_{k=1}^{\infty}|z_k|^{q+s} = 1.$$

Hence $z \in \ell^{\psi_q^s}$, and in particular

$$\sum_{k=1}^{\infty}\psi_q^s\left(\frac{|z_k|}{1}\right) = \sum_{k=1}^{\infty}\psi_q^s(|z_k|) \leq 1.$$

Consequently, there exists $A > 0$ such that

$$\sum_{k=1}^{\infty}\psi_q^s\left(\frac{|z_k|}{1}\right) = \sum_{k=1}^{\infty}\psi_q^s(|z_k|) \leq 1 = \sum_{k=1}^{\infty}\psi_q^s\left(\frac{|z_k|}{A}\right).$$

By the definition of the Luxemburg norm, this gives $\|z\|_{\ell^{\psi_q^s}} = A \leq 1$, since ψ_q^s is increasing. Using $\|z\|_{\ell^{\psi_q^s}} = \left\|\frac{x}{\|x\|_{\ell^{q+s}}}\right\|_{\ell^{\psi_q^s}}$, we obtain

$$\left\|\frac{x}{\|x\|_{\ell^{q+s}}}\right\|_{\ell^{\psi_q^s}} \leq 1, \qquad \text{equivalently,} \qquad \|x\|_{\ell^{\psi_q^s}} \leq \|x\|_{\ell^{q+s}}.$$

Next, from the previous estimate we also have

$$\frac{1}{2} \leq \sum_{k=1}^{\infty}\psi_q^s(|z_k|) \qquad \text{equivalently} \qquad 1 \leq \sum_{k=1}^{\infty}2\,\psi_q^s(|z_k|).$$

We now apply Lemma (1), which asserts that for $0 < u < v$,

$$\frac{\psi_q^s(u)}{u} \leq \frac{\psi_q^s(v)}{v} \quad \Longrightarrow \quad \psi_q^s(u) \leq \frac{u}{v}\,\psi_q^s(v).$$

Choosing $v = 2u$ gives

$$\psi_q^s(u) \le \tfrac{1}{2}\,\psi_q^s(2u) \qquad \Longrightarrow \qquad 2\,\psi_q^s(u) \le \psi_q^s(2u).$$

Applying this with $u = |z_k|$, we obtain

$$1 \le \sum_{k=1}^{\infty} 2\,\psi_q^s(|z_k|) \le \sum_{k=1}^{\infty} \psi_q^s(2|z_k|) = \sum_{k=1}^{\infty} \psi_q^s\left(\frac{|z_k|}{1/2}\right).$$

Thus there exists $A > 0$ such that

$$1 = \sum_{k=1}^{\infty} \psi_q^s\left(\frac{|z_k|}{A}\right) \le \sum_{k=1}^{\infty} \psi_q^s\left(\frac{|z_k|}{1/2}\right).$$

By the definition of the Luxemburg norm, and using that ψ_q^s is increasing, this yields

$$\|z\|_{\ell^{\psi_q^s}} = A \ge \frac{1}{2}.$$

Since $\|z\|_{\ell^{\psi_q^s}} = \left\| \frac{x}{\|x\|_{\ell^{q+s}}} \right\|_{\ell^{\psi_q^s}}$, we conclude that

$$\frac{1}{2} \le \left\| \frac{x}{\|x\|_{\ell^{q+s}}} \right\|_{\ell^{\psi_q^s}} \qquad \Longleftrightarrow \qquad \frac{1}{2}\,\|x\|_{\ell^{q+s}} \le \|x\|_{\ell^{\psi_q^s}}.$$

Combining this with the upper estimate previously obtained, we arrive at

$$\frac{1}{2}\,\|x\|_{\ell^{q+s}} \;\le\; \|x\|_{\ell^{\psi_q^s}} \;\le\; \|x\|_{\ell^{q+s}},$$

for every $x \in \ell^{\,q+s}$. □

Corollary 3. *The space $\ell^{\psi_q^s}$ is a Banach space.*

Proof. By [7] we know that ℓ^{q+s} is a Banach space and from Theorem (2), we have $\ell^{\psi_q^s} = \ell^{q+s}$ with $\|\cdot\|_{\ell^{\psi_q^s}} \sim \|\cdot\|_{\ell^{q+s}}$. We conclude that $\ell^{\psi_q^s}$ is a Banach space. □

Previously, the special case when $s = 1$ was analyzed by the function $\psi_q(t) = t^q \ln(1+t)$, and it was shown that $\ell^{\psi_q} = \ell^{\psi_q^1}$. This result is now viewed as part of a broader generalization for all $s \ge 0$. On the interval $0 \le t \le 1$, by Lemma (5), we have

$$\frac{1}{2} t^{q+s} \le \psi_q^s(t) \le t^{q+s} = t^{q+s}$$

and

$$\frac{1}{2} t^{q+s} = \frac{1}{2} t^{1+(s+q-1)} \le \psi_1^{(s+q-1)}(t) \le t^{1+(s+q-1)} = t^{q+s}.$$

Hence $\frac{1}{2} t^{q+s} \le \psi_q^s(t) \le t^{q+s} = t^{1+(s+q-1)} \le 2\psi_1^{(s+q-1)}(t) \le 2t^{q+s}$.

Using the inequality established above, we may proceed exactly as in Theorem (1), Corollary (1), and Theorem (2) to obtain the chain of identifications

$$\ell^{\psi_q^s} = \ell^{\,q+s} = \ell^{\,1+(s+q-1)} = \ell^{\psi_1^{(s+q-1)}},$$

which shows that the corresponding Orlicz sequence spaces coincide with the classical ℓ^p spaces arising from these exponents. In particular, norms (induced by the Young functions ψ_q^s and

$\psi_1^{(s+q-1)}$) are equivalent, so although the defining expressions differ, they generate the same topological vector space.

It should be emphasized that the discussion above establishes the case $p = q + 1 \geq 2$. However, for the range $p \in [1, 2)$, the situation can be treated by considering the more general Young function $\psi_q^s(t) = t^q \ln(1 + t^s)$ with $q = 1$ and $0 \leq s < 1$. In particular, when $s = 0$ the corresponding space is simply $\ell^p = \ell^1$, equipped with the usual ℓ^p norm, not the Luxemburg norm. Thus, the general framework using $\psi_q^s(t)$ naturally covers the remaining case $p \in [1, 2)$, with the trivial instance $s = 0$ leading back to the classical ℓ^p space.

3.3. Some Remarks

It is important to note that in the previous results the parameter q does not need to vary. In fact, it is sufficient to take $q = 1$ and allow $s \geq 0$ to be arbitrary. For $p \geq 1$, the space ℓ^p can be viewed as an Orlicz sequence space with Young function ψ_1^{p-1}, and it can be equipped with Luxemburg norm $\|\cdot\|_{\ell^{\psi_1^{p-1}}}$. We also considered the Hölder inequality. For $\frac{1}{r_1} + \frac{1}{r_2} = 1$,

$$\left|\sum_k x_k y_k\right| \leq \|x\|_{\ell^{r_1}} \|y\|_{\ell^{r_2}}$$

holds for every $x \in \ell^{r_1}$ and $y \in \ell^{r_2}$. Assuming $r_1 \geq 2$ and $1 < r_2 \leq 2$, we may set $s_1 = r_1 - 1$ and $s_2 = r_2 - 1$. Under these identifications, each space ℓ^{r_i} for $i = 1, 2$ coincides with the Orlicz sequence space generated by the Young function $\psi_1^{s_i}(t) = t \ln(1 + t^{s_i})$, namely,

$$\ell^{r_i} = \ell^{\psi_1^{s_i}}.$$

Moreover, the corresponding norms satisfy $\frac{1}{2}\|\cdot\|_{\ell^{r_i}} \leq \|\cdot\|_{\ell^{\psi_1^{s_i}}} \leq \|\cdot\|_{\ell^{r_i}}$. Hence, Hölder inequality can be rewritten in the setting of Orlicz sequence spaces as

$$\left|\sum_k x_k y_k\right| \leq 4\|x\|_{\ell^{\psi_1^{s_1}}} \|y\|_{\ell^{\psi_1^{s_2}}}$$

for every $x \in \ell^{\psi_1^{s_1}}$ and $y \in \ell^{\psi_1^{s_2}}$.

4. Conclusion

This research shows that modifying the classical Young function $f(t) = t^p$ into more general forms, such as $\psi_q(t) = t^q \ln(1 + t)$ and $\psi_q^s(t) = t^q \ln(1 + t^s)$, gives rise to Orlicz sequence spaces whose norms are equivalent to those of the classical sequence spaces ℓ^{q+1} and ℓ^{q+s}, respectively. Thus, despite the differences in the generating Young functions, the resulting spaces share the same structural properties as ℓ^p. As a consequence, essential features of ℓ^p, such as completeness and Hölder's inequality, continue to hold in these Orlicz sequence spaces, with the difference lying only in the constants involved. Hence, the modification of norms enriches the theoretical framework without changing the underlying space, highlighting both the stability and adaptability of ℓ^p spaces in functional analysis. The present results provide a basis for further exploration within Lebesgue spaces L^p however, due to the nuanced differences in their properties, one cannot expect the conclusions to hold without rigorous verification. Another investigation has been undertaken from a geometric perspective, with particular emphasis on Orlicz spaces endowed with the Luxemburg norm, as discussed in [14–16].

CRediT Authorship Contribution Statement

Rikha Syahda Salsabilla: Conceptualization, methodology, software, resources, writing-original draft preparation, writing-review.
Mochammad Idris: Conceptualization, methodology, writing-review and editing, supervision, validation.

Declaration of Generative AI and AI-assisted technologies

In the preparation of this manuscript, the authors used ChatGPT Free to assist in paraphrasing sentences and improving grammar. The use of this tool was limited to language refinement, while the core ideas, mathematical results, and analyses were fully developed by the authors.

Declaration of Competing Interest

"The authors declare no competing interests."

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