A Super (A,D)-Bm-Antimagic Total Covering Of A Generalized Amalgamation Of Fan Graphs

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ABSTRACT

We assume finite, simple and undirected graphs in this study. Let G, H be two graphs. By an (a,d)-H-antimagic total graph, we mean any obtained bijective function \( f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., |V(G)| + |E(G)|\} \) such that for each subgraph \( H' \) which is isomorphic to \( H \), their total \( H \)-weights \( w(H) = \sum_{v \in E(H')} f(v) + \sum_{e \in E(H')} f(e) \) show an arithmetic sequence \( \{a, a + d, a + 2d, ..., a + (m - 1)d\} \) where \( a, d > 0 \) are integers and \( m \) is the cardinality of all subgraphs \( H' \) isomorphic to \( H \). An \( (a, d) \)-H-antimagic total labeling \( f \) is called super if the smallest labels are assigned in the vertices. In this paper, we will study a super \( (a, d) \)-Bm-antimagicness of a connected and disconnected generalized amalgamation of fan graphs in which a path is a terminal.

Keywords: Super \( (a, d) \)-Bm-antimagic total covering, generalized amalgamation of fan graphs, connected and disconnected

INTRODUCTION

In [1], Dafik et al. defined an amalgamation of graphs as follows: Let \( G_i \) be a finite collection of graphs and suppose each \( G_i \) has a fixed vertex \( v_i \) called a terminal. The amalgamation \( G' \) where \( v_i \) as a terminal is formed by taking all the \( G_i \)'s and identifying their terminal. When \( G_i \) are all isomorphic connected graphs, for any positive integer \( m \), we denote such amalgamation by \( \text{Amal}(G, m) \), where \( m \) denotes the number of copies of \( G \). If we replace the terminal vertex \( v_i \) by a subgraph \( P \subset G \) then such amalgamation is said to be a generalized amalgamation of \( G \) and denoted by \( \text{amal}(G, P, m) \).

Furthermore, Baca et al. in [2] and Dafik et al. [3] defined an \( (a, d) \)-edge-antimagic total labeling of \( G \) as a mapping \( f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., |V(G)| + |E(G)|\} \) such that the set of edge-weights \( \{f(u) + f(uv) + f(v) | uv \in E(G)\} \) is equal to the set \( \{a, a + d, a + 2d, ..., a + (|E(G)| - 1)d\} \) for some positive integers \( a \) and \( d \). Combining the two previous labelings, [1], [4], [5], [6], [7] introduced the \( (a,d) \)-H-antimagic total labeling. A graph \( G \) is said to be an \( (a, d) \)-H-antimagic total graph if there exist a bijective function \( f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., |V(G)| + |E(G)|\} \) such that for all subgraphs \( H' \) isomorphic to \( H \), the total \( H \)-weights \( w(H) = \sum_{v \in E(H')} f(v) + \sum_{e \in E(H')} f(e) = \gamma \) form an arithmetic sequence.
progression \( \{a, a + d, a + 2d, \ldots, a + (m - 1)d\} \), where \( a, d > 0 \) are integers and \( m \) is the number of all subgraphs \( H' \) isomorphic to \( H \). An \((a, d)\)-\(H\)-antimagic total labeling is called super if the smallest labels are assigned in the vertices.

There are many results show the existence of the \((a, d)\)-\(H\)-antimagic total labeling, see [1], [4], [7], [8], [9], and [10]. In this paper, we will study a super \((a, d)\)-\(B_m\)-antimagicness of an amalgamation of fans of order \( m \) when a path of order \( n \) is a terminal, denoted by \( \text{Amal}(F_n, P_n, m) \) as well as the disjoint union of multiple \( s \) copies of \( \text{Amal}(F_n, P_n, m) \). The cover \( H' \) is a book of order two, thus \( H = B_m \). In other word, we will show the existence of super \((a, d)\)-\(B_m\)-antimagic total labeling of \( \text{Amal}(F_n, P_n, m) \) and disjoint union of multiple \( s \) copies of \( \text{Amal}(F_n, P_n, m) \) denoted by \( s\text{Amal}(F_n, P_n, m) \).

**LITERATURE REVIEW**

Prior to showing the research result on the existence of super \((a,d)\)-\(B_m\)-antimagic total labeling \( s\text{Amal}(F_n, P_n, m) \), we will rewrite a known lemma excluding the proof that will be useful for determining the necessary condition for a graph to be super \((a,d)\)-\(B_m\)-antimagic total labeling. This lemma proved by [2] provides an upper bound for feasible value of \( d \), and it is a sharp.

**Lemma 1.** [2] Let \( G \) be a simple graph of order \( p_G \) and size \( q_G \). If \( G \) is super \((a, d)\)-\(H\)-antimagic total labeling then \( d \leq \frac{(p_G - q_G)}{p_G} \). If \( H' \) are subgraphs isomorphic to \( H \).

\[ |V(G)| = p_G, |E(G)| = q_G, |V(H')| = p_{H'}, |E(H')| = q_{H'}, \text{and } t = |H'_{j}|. \]

**RESULTS AND DISCUSSIONS**

**The Connected Graph.** An amalgamation of fan graphs, denoted by \( \text{Amal}(F_n, P_n, m) \), is a connected graph with vertex set \( V(\text{Amal}(F_n, P_n, m)) = \{A_j, x_i : 1 \leq j \leq m, 1 \leq i \leq n\} \) and edge set \( E(\text{Amal}(F_n, P_n, m)) = \{A_j, x_i : 1 \leq j \leq m, 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n - 1\} \). Since we study a super \((a, d)\)-\(H\)-antimagic total labeling for \( H' = B_m \) isomorphic to \( H \), thus \( p_G = |V(\text{Amal}(F_n, P_n, m))| = m + n, q_G = |E(\text{Amal}(F_n, P_n, m))| = mn + n - 1, p_{H'} = |V(B_m)| = m + 2, q_{H'} = |E(B_m)| = 2m + 1, t = |H'_{j}| = |B_m| = n - 1. \)

If amalgamation of fan graphs \( \text{Amal}(F_n, P_n, m) \) has a super \((a, d)\)-\(B_m\)-antimagic total labeling then for \( p_G = |V(\text{Amal}(F_n, P_n, m))| = m + n, q_G = |E(\text{Amal}(F_n, P_n, m))| = mn + n - 1, p_{H'} = |V(F_n, P_n, m)| = m + 2, q_{H'} = |E(F_n, P_n, m)| = 2m + 1, t = |H'_{j}| = n - 1, it follows from Lemma 1.1 the upper bound of \( d \leq 2m^2 + 4m + 3 \).

Now we start to describe the result of the super \((a,d)\)-\(H\)-antimagic total labeling of amalgamation of fan graph with the following theorems. Figure 1 shows an illustration of graph \( \text{Amal}(F_n, P_n, m) \).

![Figure 1 illustration of graph Amal(F_n, P_n, m)](image-url)
Theorem 2.1. For \( m, n \geq 2 \), the graph \( \text{Amal}(F_n, P_n, m) \) admits a super \( \left( \left( n + \frac{5}{2} \right) m^2 + \left( 2n + \frac{9}{2} \right) m + n + 2m + 3 + 1, 2m + 3 \right) \cdot B_m \)-antimagic total labeling.

Proof. For \( G = \text{Amal}(F_n, P_n, m) \), define the vertex labeling \( f_1 \), as follow: \( f_1(A_j) = j \) and \( f_1(x_i) = m + i; 1 \leq j \leq m, 1 \leq i \leq n \), and the edge labeling as follows:

\[
f_1(A_jx_i) = m + n + (j - 1)n + i; 1 \leq j \leq m, 1 \leq i \leq n
\]

\[
f_1(x_ix_{i+1}) = m + n + nm + i + 1; 1 \leq i \leq n - 1
\]

The vertex and edge labelings \( f_1 \) are a bijective function \( f_1: V(G) \cup E(G) \to \{1, 2, 3, ..., 3mn - m + 1\} \). The \( H \)-weights of \( \text{Amal}(F_n, P_n, m) \), for \( 1 \leq j \leq m, 1 \leq i \leq n \) under the labeling \( f_1 \), constitute the following sets \( w_{f_1} = \bigcup_{i=1}^{n-1} \{f_1(A_j) + f_1(x_i)\} = \bigcup_{i=1}^{n-1}\{(2m + 2i + 1 + \left( \frac{m^2 + m}{2} \right))\} \), and the total \( H \)-weights of \( \text{Amal}(F_n, P_n, m) \) constitute the following sets

\[
W_{f_1} = \bigcup_{i=1}^{n-1} \{w_{f_1} + \sum_{j=1}^{m} f_1(A_jx_i) + f_1(x_ix_{i+1})\} = \bigcup_{i=1}^{n-1}\left( \left( n + \frac{5}{2} \right) m^2 + \left( 2n + \frac{9}{2} \right) m + n + (2m + 3)i + 1 \right).
\]

It is easy to observe that the set \( W_{f_1} = \{\left( n + \frac{5}{2} \right) m^2 + \left( 2n + \frac{9}{2} \right) m + n + (2m + 4), \left( n + \frac{5}{2} \right) m^2 + \left( 2n + \frac{9}{2} \right) m + n + 4m + 7, \left( n + \frac{5}{2} \right) m^2 + \left( 2n + \frac{9}{2} \right) m + n + 6m + 10, ..., \left( n + \frac{5}{2} \right) m^2 + \left( 4n + \frac{5}{2} \right) m + 4n - 2 \} \). It gives the desired proof.

\( \blacksquare \)

Theorem 2.2. For \( m, n \geq 2 \), the graph \( \text{Amal}(F_n, P_n, m) \) admits a super \( \left( \left( n + \frac{5}{2} \right) m^2 + \left( 2n + \frac{5}{2} \right) m + 2n + 2m + 1 \right) \cdot B_m \)-antimagic total labeling.

Proof. For \( G = \text{Amal}(F_n, P_n, m) \), define the vertex labeling \( f_2 \), as follow: \( f_2(A_j) = \{n + j; 1 \leq j \leq m\} \) and \( f_2(x_i) = i; 1 \leq i \leq n \), and the edge labeling as follows:

\[
f_2(A_jx_i) = 2n + m - 1 + (j - 1)n + i; 1 \leq j \leq m, 1 \leq i \leq n
\]

\[
f_2(x_ix_{i+1}) = 2n + m - i; 1 \leq i \leq n - 1
\]

The vertex and edge labelings \( f_2 \) are a bijective function \( f_2: V(G) \cup E(G) \to \{1, 2, 3, ..., 3mn - m + 1\} \). The \( H \)-weights of \( \text{Amal}(F_n, P_n, m) \), for \( 1 \leq j \leq m, 1 \leq i \leq n \) under the labeling \( f_2 \), constitute the following sets \( w_{f_2} = \bigcup_{i=1}^{n-1} \{f_2(x_i) + f_2(x_{i+1}) + \sum_{j=1}^{m} f_2(A_j)\} = \bigcup_{i=1}^{n-1}\left( \left( n + \frac{5}{2} \right) m^2 + 4nm + \frac{1}{2} m + 2n + 1 + i(2m + 1) \right) \), and the total \( H \)-weights of \( \text{Amal}(F_n, P_n, m) \) constitute the following sets \( W_{f_2} = \bigcup_{i=1}^{n-1} \{w_{f_2} + \sum_{j=1}^{m} f_2(A_j) + f_2(x_ix_{i+1})\} = \bigcup_{i=1}^{n-1}\left( \left( n + \frac{5}{2} \right) m^2 + \left( 4n + \frac{5}{2} \right) m + 2n + 2\left( n + \frac{5}{2} \right) m^2 + \left( 4n + \frac{9}{2} \right) m + 2n + 3, \left( n + \frac{5}{2} \right) m^2 + \left( 4n + \frac{13}{2} \right) m + 2n + 4, ..., \left( n + \frac{5}{2} \right) m^2 + \left( 6n - \frac{3}{2} \right) m + 3n \} \). Therefore, the graph \( \text{Amal}(F_n, P_n, m) \) admits a super \( \left( \left( n + \frac{5}{2} \right) m^2 + \left( 2n + \frac{5}{2} \right) m + 2n + 2, 2m + 1 \right) \cdot B_m \)-antimagic total labeling. For \( m, n \geq 2 \)

\( \blacksquare \)

Theorem 2.3. For \( m, n \geq 2 \), the graph \( \text{Amal}(F_n, P_n, m) \) admits a super \( \left( \left( n + \frac{5}{2} \right) m^2 + m + 4nm + 6 + 2m^2, 2m^2 + 3 \right) \cdot B_m \)-antimagic total labeling.
Proof. For \( G = Amal(F_n, P_n, m) \), define the vertex labeling \( f_3 \) as follows: \( f_3(A_1) = 1, f_3(x_i) = i + 1; 1 \leq i \leq n \) and \( f_3(xA_j) = n + j; 2 \leq j \leq m \) and the edge labeling as follows:
\[
f_3(A_j x_i) = n + mi + j; 1 \leq j \leq m, 1 \leq i \leq n \\
f_3(x_i x_{i+1}) = m + n + nm + i; 1 \leq i \leq n - 1
\]
The vertex and edge labelings \( f_3 \) are a bijective function \( f_3 : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 3mn - m + 1\} \). The \( H \)-weights of \( Amal(F_n, P_n, m) \), for \( 1 \leq j \leq m, 1 \leq i \leq n \) under the labeling \( f_3 \), constitute the following sets \( W_{f_3} = U_{i=1}^{n-1} \{\sum_{j=2}^{m} f_3(A_j) + f_3(x_i) + f_3(x_{i+1}) + f_3(A_1)\} = U_{i=1}^{n-1} \{\frac{1}{2}m^2 + \frac{1}{2}m + (m - 1)n + 2i + 3\} \), and the total \( H \)-weights of \( Amal(F_n, P_n, m) \) constitute the following sets \( W_{f_3} = U_{i=1}^{n-1} \{w_{f_3} + f_3(x_i x_{i+1}) + \sum_{j=1}^{m} f_3(A_j x_i) + f_3(A_j x_{i+1})\} = U_{i=1}^{n-1} \{\frac{5}{2}m^2 + \frac{5}{2}m + 4nm + 3 + (2m^2 + 3)i\} \). It is easy to observe that the set \( W_{f_3} = \{\frac{5}{2}(m^2 + m) + 4nm + 2m^2 + 6\frac{5}{2}(m^2 + m) + 4nm + 4m^2 + 9, \ldots, 3n^2(2m - \frac{1}{2}) + n(\frac{15}{2} - 6m) + 5m - 5\} \) gives the desired proof.

**Theorem 2.4.** For \( n \geq 2 \), the graph \( Amal(F_n, P_n, 2) \) admits a super \( (\frac{29n+32}{2}, 0) \)-antimagic total labeling for \( n \) even and super \( (\frac{29n+32}{2}, 0) - B_2 \)-antimagic total labeling for \( n \) odd.

**Proof.** Define the vertex and edge labeling \( f_4 \) as follows:
\[
f_4(a) = 1; f_4(b) = 2 \\
f_4(x_i) = \begin{cases} 
\frac{i + 5}{2}, & \text{for } 1 \leq i \leq n, i \text{ odd} \\
\frac{n + i + 4}{2}, & \text{for } 1 \leq i \leq n, i \text{ even, even} \\
\frac{n + i + 5}{2}, & \text{for } 1 \leq i \leq n, i \text{ even, odd} 
\end{cases} \\
f_4(x_i x_{i+1}) = 2n - i + 2, \text{for } 1 \leq i \leq n - 1
\]
The vertex and edge labelings \( f_4 \) are a bijective function \( f_4 : V(\text{Amal}(F_n, P_n, 2)) \cup E(\text{Amal}(F_n, P_n, 2)) \to \{1, 2, 3, \ldots, 4n + 1\} \). The \( H \)-weights of \( \text{Amal}(F_n, P_n, 2) \), for \( 1 \leq i \leq n \) under the labeling \( f_4 \), constitute the following sets \( W_{f_4} = f_4(a) + f_4(b) + f_4(x_i) + f_4(x_{i+1}) = \frac{n + 2i + 17}{2} \) for \( n \) odd and the total \( H \)-weights of \( \text{Amal}(F_n, P_n, 2) \) constitute the following sets \( W_{f_4} = w_{f_4} + f_4(x_i x_{i+1}) + f_4(bx_i) + f_4(bx_{i+1}) + f_4(ax_i) + f_4(ax_{i+1}) = \frac{29n+32}{2} \) for \( n \) even and \( W_{f_4} = w_{f_4} + f_4(x_i x_{i+1}) + f_4(bx_i) + f_4(bx_{i+1}) + f_4(ax_i) + f_4(ax_{i+1}) = \frac{29n+25}{2} \) for \( n \) odd. It is easy to observe that the set \( W_{f_4} = \{\frac{29n+32}{2}, \frac{29n+32}{2}, \ldots, \frac{29n+32}{2}\} \) for \( n \) even and \( W_{f_4} = \{\frac{29n+25}{2}, \frac{29n+25}{2}, \ldots, \frac{29n+25}{2}\} \) for \( n \) odd. Therefore, the graph \( Amal(F_n, P_n, 2) \) admits a super \( (\frac{29n+25}{2}, 0) - B_2 \)-antimagic total labeling for \( n \geq 2 \) for \( n \) even, and the graph \( Amal(F_n, P_n, 2) \) admits a super \( (\frac{29n+25}{2}, 0) - B_2 \)-antimagic total labeling for \( n \geq 2 \) for \( n \) odd. It gives the desired proof.
Theorem 2.5. For \( n \geq 2 \), the graph \( \text{Amal}(F_n, P_n, 2) \) admits a super \((13n + 19, 1)\)-\( B_2 \)-antimagic total labeling.

Proof. Define the vertex and edge labeling \( f_5 \) as follows:

\[
f_5(a) = 1; f_5(b) = n + 2
\]

\[
f_5(bx_i) = 2n - i + 3, \text{ for } 1 \leq i \leq n
\]

\[
f_5(ax_i) = 2n + i + 2, \text{ for } 1 \leq i \leq n
\]

\[
f_5(x_i x_{i+1}) = 4n - i + 2, \text{ for } 1 \leq i \leq n - 1
\]

The vertex and edge labelings \( f_5 \) are a bijective function \( f_5\) : \( V(\text{Amal}(F_n, P_n, 2)) \cup E(\text{Amal}(F_n, P_n, 2)) \rightarrow \{1, 2, 3, \ldots, 4n + 1\} \). The \( H \)-weights of \( \text{Amal}(F_n, P_n, 2) \), for \( 1 \leq i \leq n \) under the labeling \( f_5 \), constitute the following sets \( w_f = f_5(a) + f_5(b) + f_5(x_i) + f_5(x_{i+1}) = n + 2i + 6 \), and the total \( H \)-weights of \( \text{Amal}(F_n, P_n, 2) \) constitute the following sets \( W_f = w_f + f_5(x_i x_{i+1}) + f_5(bx_i) + f_5(bx_{i+1}) + f_5(ax_i) + f_5(ax_{i+1}) = 13n + i + 18 \).

It is easy to observe that the set \( W_f = \{29n + 32, 29n + 32, \ldots, 29n + 32\} \) for \( n \) even and \( W_f = \{13n + 19, 13n + 20, \ldots, 14n + 18\} \). Therefore, the graph \( \text{Amal}(F_n, P_n, 2) \) admits a super \((13n + i + 18, 1) - B_2 \)-antimagic total labeling for \( n \geq 2 \). It gives the desired proof \( \blacksquare \)

The Disconnected Graph. A disjoint union of amalgamation of fan graphs, denoted by \( \text{sAmal}(F_n, P_n, m) \), is a disconnected graph with vertex set \( V(\text{sAmal}(F_n, P_n, m)) = \{x_i^k; 1 \leq j \leq m, 1 \leq i \leq n; 1 \leq k \leq s\} \) and \( E(\text{sAmal}(F_n, P_n, m)) = \{A_j^k, x_i^k; 1 \leq j \leq m, 1 \leq i \leq n; 1 \leq k \leq s\} \). Since we study a super \((a, d)\)-\( H \)-antimagic total labeling for \( H = B_m \) isomorphic to \( H \), thus \( p_G = |V(\text{sAmal}(F_n, P_n, m))| = s(m + n), q_G = |E(\text{sAmal}(F_n, P_n, m))| = s(mn + n - 1), p_H = |V(B_m)| = m + 2, q_H = |E(B_m)| = 2m + 1, t = |H'| = |B_m| = s(n - 1) \).

If amalgamation of fan graphs \( \text{sAmal}(F_n, P_n, m) \) has a super \((a, d)\)-\( B_m \)-antimagic total labeling then for \( p_G = s(m + n), q_G = s(mn + n - 1), p_H = m + 2, q_H = 2m + 1, t = s(n - 1) \), it follows from Lemma 1.1 the upper bound of

\[
d \leq \frac{m^2(2sn + s - 5) + 4sm + 3sn - 8m - s - 5}{s(n - 1) - 1}
\]

Theorem 2.6. For \( m, n \geq 2 \), \( s \geq 2 \) and \( m \) is even integer, the \( \text{sAmal}(F_n, P_n, m) \) admits a super \((3 + n)m^2s + (2m + 1)ns - 2s + \frac{m}{2} + (2m + 3)(s + 1), 2m + 3\)-\( B_m \)-antimagic total labeling.

Proof. For \( G = \text{sAmal}(F_n, P_n, m) \), define the vertex labeling \( f_6 \) for \( 1 \leq j \leq m, 1 \leq i \leq n \) \( (m \) is even integer), \( 1 \leq k \leq s \) as follow:

\[
f_6(x_i^k) = s(m + i - 1) + k
\]

\[
f_6(A_j^k) = \begin{cases} k + (j - 1)s; & \text{for } 1 \leq k \leq s, 1 \leq j \leq m, j \text{ odd} \\ (m - 4)s + 1 + js - k & \text{for } 1 \leq k \leq s, 1 \leq j \leq m, j \text{ even} \end{cases}
\]

and edge labeling as follow:

for \( 1 \leq j \leq m, 1 \leq i \leq n \) \((m \) is even integer), \( 1 \leq k \leq s \)

\[
f_6(A_j^k x_i^k) = s(m + nj + i - 1) + k
\]

for \( 1 \leq i \leq n - 1, 1 \leq k \leq s \)

\[
f_6(x_i^k x_{i+1}^k) = s(m + n + nm + i - 1) + k
\]
The vertex and edge labelings $f_0$ are a bijective function $f_0: V(G) \cup E(G) \to \{1, 2, 3, ... 3mnts - ms + s\}$. The $H$-weights of $sAmal(F_0, P_0, m)$, for $1 \leq j \leq m, 1 \leq i \leq n (m$ is even integer), $1 \leq k \leq s$ under the labeling $f_0$, constitute the following sets $w_{f_0} = \cup_{i=1}^{n-1} \cup_{k=1}^{s} f_0(x_i^k) + f_0(x_{i+1}^k) \cup \sum_{j=1}^{m} (A_j^k) = \cup_{i=1}^{n-1} \cup_{k=1}^{s} s(2m + 2i - 1) + 2k + \frac{m}{2}(2ms - 4s + 1))$, and the total $H$-weights of $sAmal(F_0, P_0, m)$ constitute the following sets: $W_{f_0} = \cup_{i=1}^{n-1} \cup_{k=1}^{s} W_{f_0} = \cup_{i=1}^{n-1} \cup_{k=1}^{s} s(3m + n + mm + 3i - 2) + 3k + \frac{m}{2}(2ms - 4s + 1) + \sum_{j=1}^{m} s(m + jn + i - 1) + k + s(m + jn + i + k)] = \cup_{i=1}^{n-1} \cup_{k=1}^{s} s(3 + n)m^2s + (2m + 1)ns - 2s + \frac{m}{2}(2m + 3)(si + k)). \) It is easy to observe that the set $W_{f_0} = [(3 + n)m^2s + (2m + 1)ns - 2s + \frac{m}{2}(2m + 3)(s + 1), (3 + n)m^2s + (2m + 1)ns - 2s + \frac{m}{2}(2m + 3)(s + 2), (3 + n)m^2s + (2m + 1)ns - 2s + \frac{m}{2}(2m + 3)(s + 3), ... 2ms(2n^2 - 2n + 1) - s\left(n^2 - n \cdot \frac{n}{2}\right) - \frac{1}{2}(n^2 - n - 3) + (n^2 + 2n - 3)(ms + s))]. It gives the desired proof. 

Theorem 2.7. For $m, n \geq 2, s \geq 2$ and $m$ is even integer, the $sAmal(F_0, P_0, m)$ admits a super $\left(\frac{m^2s}{2}(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)(s + 1), 2m + 1\right)$-antimagic total labeling.

Proof. For $G = sAmal(F_0, P_0, m)$, define the vertex labeling $f_0$, for $1 \leq j \leq m, 1 \leq i \leq n, 1 \leq k \leq s$ as follow:

$\begin{align*}
& f_0(x_i^k) = si + k - s \\
& f_0(A_j^k) = \begin{cases} 
\frac{s(j - 1) + sn + k}{sn + 1 + js - k} & \text{for } 1 \leq k \leq s, 1 \leq j \leq m, j \text{ odd} \\
\frac{(j - 1)n + i + k}{sn + 1 + js - k} & \text{for } 1 \leq k \leq s, 1 \leq j \leq m, j \text{ even}
\end{cases}
\end{align*}$

and edge labeling as follow:

for $1 \leq j \leq m, 1 \leq i \leq n, 1 \leq k \leq s$

$\begin{align*}
& f_0(A_j^k) = s(2n + m) + 1 - si - k \\
& f_0(x_i^k x_{i+1}^k) = s(2n + m + 2 - (j - 1)n + i + k)
\end{align*}$

The vertex and edge labelings $f_0$ are a bijective function $f_0: V(G) \cup E(G) \to \{1, 2, 3, ... 3mnts - ms + s\}$. The $H$-weights of $sAmal(F_0, P_0, m)$, for $1 \leq j \leq m, 1 \leq i \leq n (m$ is even integer), $1 \leq k \leq s$ under the labeling $f_0$, constitute the following sets $w_{f_0} = \cup_{i=1}^{n-1} \cup_{k=1}^{s} f_0(x_i^k) + f_0(x_{i+1}^k) \cup \sum_{j=1}^{m} (A_j^k) = \cup_{i=1}^{n-1} \cup_{k=1}^{s} s(2n + 5) + (2sn - s)(2m + 1) + \frac{m}{2} + 1 + (2m + 1)(s + 1), (3 + n)m^2s + (2m + 1)ns - 2s + \frac{m}{2}(2m + 3)(s + 1), m^2s + (2m + 1)(s + 2), ... 2ms(2n^2 - 2n + 1) - s\left(n^2 - n \cdot \frac{n}{2}\right) - \frac{1}{2}(n^2 - n - 3) + (n^2 + 2n - 3)(ms + s))]. It gives the desired proof. 

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Theorem 2.8. For $m, n \geq 2$, $s \geq 2$ and $m$ is even integer, the $sAmal(F_n, P_n, m)$ admits a super $(\frac{s}{4}m^2s + sn(4m + 2) + 2s + \frac{m}{2} + 2 + (2m^2 - 1)s + 2m - 1, 2m - 1)$-B$_m$- antimagic total labeling.

Proof. For $G = sAmal(F_n, P_n, m)$, define the vertex labeling $f_8$ for $1 \leq j \leq m, 1 \leq i \leq n$ ($m$ is even integer), $1 \leq k \leq s$ as follow:

$$f_8(A_j^k) = k$$
$$f_8(x_j^k) = s(n + 2) + 1 - si - k$$
$$f_8(A_j^k) = \begin{cases} \frac{s}{n} + 1 + js - k; & \text{for } 1 \leq k \leq s, 1 \leq j \leq m, j \text{ odd} \\ s(n + j - 1) + k; & \text{for } 1 \leq k \leq s, 1 \leq j \leq m, j \text{ even} \end{cases}$$

and edge labeling as follow:

for $1 \leq j \leq m, 1 \leq i \leq n$ ($m$ is even integer), $1 \leq k \leq s$

$$f_8(A_j^k x_i^k) = s(n + mi + j - 1) + k$$

for $1 \leq i \leq n - 1, 1 \leq k \leq s$

$$f_8(x_i^k x_{i+1}^k) = s(n + m + m + i - 1) + k$$

The vertex and edge labelings $f_8$ are a bijective function $f_8: V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., 3mn - ms + s\}$. The $H$-weights of $sAmal(F_n, P_n, m)$, for $1 \leq j \leq m, 1 \leq i \leq n$ ($m$ is even integer), $1 \leq k \leq s$ under the labeling $f_8$ constitute the following sets

$$W_{f_8} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{n} \{f_8(A_j^k) + f_8(x_j^k) + f_8(x_j^k) + \sum_{j=2}^{m} f_8(A_j^k) + \sum_{j=1}^{m} \{f_8(A_j^k x_i^k) + f_8(A_j^k x_i^k) + f_8(A_j^k x_i^k)\}\} = \bigcup_{i=1}^{n-1} \bigcup_{k=1}^{n} \left\{s \left(2m^2 + sn(4m + 2) + 2s + \frac{m}{2} + 2 + (2m^2 - 1)si + (2m - 1)k\right)\right\}$$

It is easy to observe that the set $W_{f_8}$ admits a super $(12sn + 16s + 5, 1)$-B$_2$- antimagic total labeling.

Theorem 2.9. For $n \geq 2$, the graph $sAmal(F_n, P_n, 2)$ admits a super $(12sn + 16s + 5, 1)$-B$_2$- antimagic total labeling.

Proof. Define the vertex and edge labeling $f_9$ as follows:

$$f_9(a_i^j) = s - j + 1, \text{ for } 1 \leq j \leq s$$
$$f_9(b_i^j) = s + j, \text{ for } 1 \leq j \leq s$$
$$f_9(x_i^j) = si + s + j, \text{ for } 1 \leq i \leq n, 1 \leq j \leq s$$
$$f_9(a_i^j x_i^j) = 2sn + 3s - si - j + 1, \text{ for } 1 \leq i \leq n, 1 \leq j \leq s$$
$$f_9(b_i^j x_i^j) = si + 2sn + s + j, \text{ for } 1 \leq i \leq n, 1 \leq j \leq s$$
$$f_9(x_i^j x_{i+1}^j) = 4sn - si + 2s - j + 1, \text{ for } 1 \leq i \leq n - 1, 1 \leq j \leq s$$

The vertex and edge labelings $f_9$ are a bijective function $f_9: V(sAmal(F_n, P_n, 2)) \cup E(sAmal(F_n, P_n, 2)) \rightarrow \{1, 2, 3, ..., 4sn + s\}$. The $H$-weights of $sAmal(F_n, P_n, 2)$, for $1 \leq i \leq n$ and $1 \leq j \leq s$ under the labeling $f_9$ constitute the following sets

$$W_{f_9} = f_9(a_i^j) + f_9(b_i^j) + f_9(x_i^j) + f_9(x_{i+1}^j) = 5s + 2j + 2si + 1$$

and gives the desired proof.

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2) constitute the following sets $W_{f_9} = w_{f_9} + f_9(a^jx^1) + f_9(a^{j+1}x^1) + f_9(b^jx^j) + f_9(b^jx^{j+1}) + f_9(x^jx^{j+1}) = 15s + j + si + 4 + 12sn$. It is easy to observe that the set $W_{f_9} = \{12sn + 16s + 5, 12sn + 16s + 6, ..., 13sn + 16s + 4\}$. Therefore, the graph $sA_{m}(F_n, P_n, 2)$ admits a super $(12sn + 16s + 5, 1) - B_2$- antimagic total labeling for $m, n \geq 2$ It gives the desired proof. 

**Theorem 2.10.** For $n \geq 2$, the graph $sA_{m}(F_n, P_n, 2)$ admits a super $(11sn + 17s + 6, 3) - B_2$- antimagic total labeling.

**Proof.** Define the vertex and edge labeling $f_{10}$ as follows:

$$f_{10}(a^j) = s - j + 1, \text{for } 1 \leq j \leq 5$$

$$f_{10}(a^{j+1}) = 2sn + 3s - si - j + 1, \text{for } 1 \leq i \leq n, 1 \leq j \leq s$$

$$f_{10}(b^j) = si + s + j, \text{for } 1 \leq i \leq n, 1 \leq j \leq s$$

$$f_{10}(b^{j+1}) = si + 2sn + s + j, \text{for } 1 \leq i \leq n, 1 \leq j \leq s$$

$$f_{10}(x^jx^{j+1}) = si + s + 3sn + j, \text{for } 1 \leq i \leq n - 1, 1 \leq j \leq s$$

The vertex and edge labelings $f_{10}$ are a bijective function $f_{10} : V(sA_{m}(F_n, P_n, 2)) \cup E(sA_{m}(F_n, P_n, 2)) \rightarrow \{1, 2, 3, ..., 4sn + s\}$. The $H$-weights of $sA_{m}(F_n, P_n, 2)$, for $1 \leq i \leq n$ and $1 \leq j \leq s$ under the labeling $f_{10}$, constitute the following sets $W_{f_{10}} = f_{10}(a^j) + f_{10}(a^{j+1}) + f_{10}(b^j) + f_{10}(b^{j+1}) + f_{10}(x^jx^{j+1}) = 5s + 2j + 2si + 1$, and the total $H$-weights of $sA_{m}(F_n, P_n, 2)$ constitute the following sets $W_{f_{10}} = f_{10}(a^jx^1) + f_{10}(a^{j+1}x^1) + f_{10}(b^jx^j) + f_{10}(b^{j+1}x^{j+1}) + f_{10}(x^jx^{j+1}) = 3mi + 14m + 3j + 3 + 11sn$. It is easy to observe that the set $W_{f_{10}} = \{11sn + 17s + 6, 11sn + 17s + 9, ..., 14sn + 17s + 3\}$. Therefore, the graph $sA_{m}(F_n, P_n, 2)$ admits a super $(11sn + 17s + 6, 3) - B_2$- antimagic total labeling for $m, n \geq 2$ It gives the desired proof.

**CONCLUSIONS**

In this paper, the result show that super $(a, d)$-$B_{m}$-antimagic total labeling of $Amal(F_n, P_n, m)$ and $sA_{m}(F_n, P_n, m)$ for some feasible $d$ are respectively $d \in \{2m + 1, 2m + 3, 2m^2 + 3\}$ and $d \in \{2m + 3, 2m + 1, 2m - 1\}$. Apart from obtained $d$ above, we haven’t found any result yet, so we propose the following open problem:

Let $sG = sA_{m}(F_n, P_n, m)$, for $m, n \geq 2$, $s \geq 2$, and $s$ odd, does $sG$ admit a super $(a, d)$-$B_{m}$-antimagic total labeling for feasible $d$?

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**REFERENCES**


