



A Review of Pompeiu-Hausdorff Metric Differentiability and Its Relation to Generalized Hukuhara Differentiability

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Abstract

The Pompeiu-Hausdorff distance/Pompeiu-Hausdorff metric is a concept in analysis that measures the distance between two subsets of a metric space, one of its important applications being the Hausdorff metric differentiability of set-valued functions. This article reviews the definition and properties of Pompeiu-Hausdorff distance differentiability on the space of compact and convex subsets of the Euclidean space with dimension n . We also present concepts about the generalized Hukuhara difference and its differentiability. By studying both topics, we discuss the established relationship between Pompeiu-Hausdorff metric differentiability and generalized Hukuhara differentiability.

Keywords: Generalized Hukuhara difference, Hukuhara difference, Pompeiu-Hausdorff metric, Set-valued function

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1. Introduction

Set-valued analysis is one of the topics in mathematical analysis that studies collections of sets [1]. This field has a wide range of applications, such as in control theory, economics, and artificial intelligence [1, 2]. An important topic in set-valued analysis is the differentiability of set-valued functions, particularly Hausdorff distance differentiability in metric spaces which has application such as in pattern analysis for comparing image [3]. Another important topic is the study of Hukuhara difference differentiability of set-valued functions. One such instance is the use in defining the derivative of Fuzzy set-valued mappings [4]. The generalized Hukuhara difference differentiability is the generalization of Hukuhara differentiability concept usually used in development of differential calculus for interval-valued functions [5]. Recently, the notion has also been used in the study of metric derivatives for set-valued functions to define the differentiation of set-valued function and is used in proving the differentiation properties of Lipschitz set-valued function [6].

The concept of set-valued analysis was introduced by Aubin [1]. Later, Masuo Hukuhara developed the notion of the Hukuhara difference for compact and convex sets in order to define Hukuhara differentiability of set-valued mappings [7]. This theory was later extended by Stefanini and Bede, who introduced the generalized Hukuhara difference and generalized Hukuhara differentiability of set-valued mappings [8]. More recently, Khastan, López, and

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Shahidi established the differentiability of set-valued mappings under the Pompeiu-Hausdorff metric or known as Pompeiu-Hausdorff distance on the space of compact and convex subsets of the Euclidean space and studied its properties [9]. However, in the existing literature, several fundamental proofs such as the necessary and sufficient condition for the existence of the Hukuhara difference and fundamentals properties of Pompeiu-Hausdorff metric differentiability are often not given.

In this paper we review definition and provides proof of fundamentals properties for the mentioned topics above. After this we will discuss the established relationship between Pompeiu-Hausdorff differentiability and generalized Hukuhara differentiability.

2. Preliminaries.

In this section, we provide the definitions that will be used later on results and discussion part.

2.1. Set-Valued Mappings

Definition 1 (Set-valued mapping [10–13]). Let $F: X \rightarrow \mathcal{P}(Y)$. The mapping F is called a *set-valued function* if

$$F(x) \neq \emptyset$$

for every $x \in X$.

Example 1. Define $F: [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$ by

$$F(x) = [0, x] \quad \text{for } x \in [0, 1].$$

Then F is a set-valued function.

2.2. Minkowski Sum, and Minkowski Scalar Multiplication

Definition 2 (Minkowski sum and Minkowski scalar multiplication [10, 14]). Let A and B be subsets of euclidean space \mathbb{R}^k and $\alpha \in \mathbb{R}$. Denote

$$A + B = \{a + b \mid a \in A, b \in B\}, \quad \alpha A = \{\alpha a \mid a \in A\}. \quad (1)$$

This definition is known as *Minkowski sum* and *Minkowski scalar multiplication* respectively.

Example 2. Let $A = [0, 3]$ and $B = [2, 5]$, then the Minkowski sum between A and B is $C = [0 + 2, 3 + 5] = [2, 8]$, and the Minkowski scalar multiplication between A and $k \in \mathbb{R}^+$ is $kA = [0, 3k]$

2.3. Real linear Space, Convex Cone, Normed Linear Space

Definition 3 (Real linear space [15, 16]). Recall that a real linear space X is a commutative group with respect to addition, and for each element (called a vector) $x \in X$ and each real number $\alpha \in \mathbb{R}$, the scalar multiplication $\alpha x \in X$ is defined. The operations of addition and scalar multiplication satisfy the following axioms for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$:

1. $\alpha(x + y) = \alpha x + \alpha y$;
2. $(\alpha + \beta)x = \alpha x + \beta x$;
3. $(\alpha\beta)x = \alpha(\beta x)$;

$$4. 1 \cdot x = x.$$

Definition 4 (Convex Cone [15]). Let V be a vector space over \mathbb{R} , and let $C \subseteq V$. Then C is called a **convex cone** if it satisfies the following properties:

1. $\forall c \in C, \forall t \geq 0 \implies tc \in C$
2. $\forall c, r \in C, \forall a, b \geq 0 \implies ac + br \in C$

Definition 5 (Normed Linear Space [15, 17–19]). A real linear space X is called a *normed linear space* if it is equipped with a function $\|\cdot\| : X \rightarrow \mathbb{R}$, called a norm, satisfying the following axioms for all $x, y \in X$ and all $\alpha \in \mathbb{R}$:

1. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$;
2. $\|\alpha x\| = |\alpha| \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

2.4. Convex Sets

Definition 6 (Convex sets, [15, 20–23]). Let X be a real linear space and let $A \subseteq X$. The set A is called *convex* if for any two points $x_1, x_2 \in A$, the line segment between them is contained in A , that is,

$$[x_1, x_2] \subseteq A \quad \text{where} \quad [x_1, x_2] = \{ \alpha x_1 + (1 - \alpha)x_2 \mid \alpha \in [0, 1] \}.$$

Equivalently,

$$\alpha x_1 + (1 - \alpha)x_2 \in A \quad \text{for all } \alpha \in [0, 1].$$

Example 3. The interval $[a, b]$, where $a, b \in \mathbb{R}$, is an example of a convex set, as any point between a and b lies inside the interval. In a sense, any line segment connecting two points inside the interval is contained in it.

Definition 7 (Convex hull of a set [15]). Let X be a real linear space and let $A \subseteq X$. Denote convex hull of A , $\text{conv}(A) = \bigcap_{C \in Y} C$ where $Y = \{C \mid C \subseteq X, A \subseteq C, C \text{ convex set}\}$ as the smallest set containing A that is also a convex set

Example 4. The convex hull of $\{0, 1\}$ on \mathbb{R} is the interval $[0, 1]$ alternatively, we write $\text{conv}(\{0, 1\}) = [0, 1]$

Property 1.

$$\forall A, B \subseteq X \rightarrow \text{conv}(A + B) = \text{conv}(A) + \text{conv}(B)$$

2.5. Metric Space

Definition 8 (Metric Space [24]). A *metric space* is a set X equipped with a distance function

$$d : X \times X \rightarrow \mathbb{R},$$

that assigns $d(x, y)$ to each pair of points $x, y \in X$, and satisfies the following properties for

all $x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

2.6. Pompeiu-Hausdorff Distance

Definition 9 (Pompeiu-Hausdorff Distance [3, 9, 25]). The Pompeiu-Hausdorff distance H on the normed space $\mathcal{K}_C(\mathbb{R}^n)$ is defined by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \quad A, B \in \mathcal{K}_C(\mathbb{R}^n),$$

where

$$d(a, B) = \inf_{b \in B} \|a - b\|, \quad d(b, A) = \inf_{a \in A} \|b - a\|.$$

Here, $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^n . It is known that $(\mathcal{K}_C(\mathbb{R}^n), H)$ is a complete metric space. Where $\mathcal{K}_C(\mathbb{R}^n)$ is the family of compact convex subset of \mathbb{R}^n

Example 5. For $A = [0, 1]$ and $B = [5, 7]$ $H(A, B) = 6$ since $\sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\| = \sup_{a \in A} \|a - 5\| = 5$ and $\sup_{b \in B} d(b, A) = \sup_{b \in B} \inf_{a \in A} \|b - a\| = \sup_{b \in B} \|b - 1\| = 6$

2.7. Hukuhara Difference, Generalized Hukuhara Difference

Definition 10 (Hukuhara Difference [26]). Let $A, B \in \mathcal{K}_C(\mathbb{R}^n)$, where $\mathcal{K}_C(\mathbb{R}^n)$ denotes the family of all non-empty compact and convex subsets of \mathbb{R}^n . The *Hukuhara difference* of A and B is a set C such that

$$A -_H B = C \iff A = B + C$$

Example 6. For $A = [0, 1]$ and $B = [5, 7]$, $B -_H A = [5, 6]$ since $[0, 1] + [5, 6] = [5, 7]$, however the converse $A -_H B$ does not exist since the length of the intervals denoted by $len(A)$, $len(B) = 2 > 1 = len(A)$.

Definition 11 (Generalized Hukuhara Difference [8, 27]). Let $A, B \in \mathcal{K}_C(\mathbb{R}^n)$, where $\mathcal{K}_C(\mathbb{R}^n)$ denotes the family of all non-empty compact and convex subsets of \mathbb{R}^n . The *generalized Hukuhara difference* of A and B is a set C such that

$$A -_{gH} B = C \iff A = B + C \text{ or } B = A + (-C)$$

Example 7. For $A = [7, 11]$ and $B = [-3, 10]$, $A -_{gH} B = [1, 10]$ since $[7, 11] - [1, 10] = [-3, 10]$. Notice that in this case $A -_{gH} B$ exist but $A -_H B$ does not, due to $len(A) = 4 < len(B) = 13$.

2.8. Hukuhara, Generalized Hukuhara, Pompeiu–Hausdorff Distance differential and continuity

Definition 12 (Hukuhara Derivative [28]). Let $T \subseteq \mathbb{R}$ and let $t \in T$ be such that $t + h \in T$ for sufficiently small h . The *Hukuhara derivative* (h-derivative) of a function $F : T \rightarrow \mathcal{K}_C(\mathbb{R}^n)$ at t is defined as

$$F'_h(t) = \lim_{h \rightarrow 0^+} \frac{F(t+h) -_H F(t)}{h},$$

where $-_H$ denotes the Hukuhara difference. If the limit exists in $\mathcal{K}_C(\mathbb{R}^n)$, then F is said to be *Hukuhara differentiable* (or *h-differentiable*) at t . The formal-limit epsilon delta definition of this derivative is as follows: Let $T \subseteq \mathbb{R}$ and let $t \in T$ be such that $t + h \in T$ for sufficiently small h . $F'_h(t)$ is the hukuhara differentiation at t if $\forall \epsilon > 0, \exists \delta > 0$ so that for any $h > 0, t + h \in T$ satisfying $h < \delta$ we have

$$H\left(\frac{F(t+h) -_H F(t)}{h}, F'_h(t)\right) < \epsilon$$

Example 8. Consider the function

$$F(x) = [0, 3x], \quad x \in \mathbb{R}$$

then

$$\begin{aligned} F'_h(t) &= \lim_{h \rightarrow 0^+} \frac{F(t+h) -_H F(t)}{h} = \lim_{h \rightarrow 0^+} \frac{[0, 3t+3h] -_H [0, 3t]}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{[0, 3h]}{h} = [0, 3] \end{aligned}$$

Definition 13 (Generalized Hukuhara Derivative [8]). Let $T \subseteq \mathbb{R}$ and let $t \in T$ be such that $t + h \in T$ for sufficiently small h . The *generalized Hukuhara derivative* (gH-derivative) of a function $F : T \rightarrow \mathcal{K}_C(\mathbb{R}^n)$ at t is defined as

$$F'_{gh}(t) = \lim_{h \rightarrow 0} \frac{F(t+h) -_{gH} F(t)}{h},$$

where $-_{gH}$ denotes the generalized Hukuhara difference. If the limit exists in $\mathcal{K}_C(\mathbb{R}^n)$, then F is said to be *generalized Hukuhara differentiable* (or *gH-differentiable*) at t . The formal-limit epsilon delta definition of this derivative is as follows: Let $T \subseteq \mathbb{R}$ and let $t \in T$. $F'_{gH}(t)$ is the generalized hukuhara differentiation at t if $\forall \epsilon > 0, \exists \delta > 0$ so that for any h satisfying $|h| < \delta, t + h \in T$ we have

$$H\left(\frac{F(t+h) -_{gH} F(t)}{h}, F'_{gh}(t)\right) < \epsilon.$$

Example 9. We use the same as in Example 8. Clearly the right limit exist and by results from Example 8

$$\lim_{h \rightarrow 0^+} \frac{F(t+h) -_{gH} F(t)}{h} = [0, 3].$$

The left limit

$$\lim_{h \rightarrow 0^-} \frac{F(t+h) -_{\text{gH}} F(t)}{h} = \lim_{h \rightarrow 0^-} \frac{[0, 3t+3h] -_{\text{gH}} [0, 3t]}{h} = \lim_{h \rightarrow 0^-} \frac{[0, 3h]}{h} = [0, 3]$$

since $[0, 3t+3h] - [0, 3t] = [0, 3h]$, $h < 0$. Hence $F'_{gh}(t) = [0, 3]$

Definition 14 (Pompeiu– Hausdorff distance continuity [26]). Let $F : T \rightarrow \mathcal{K}_C(\mathbb{R}^n)$. We say that F is *continuous* at $t_0 \in T$ if

$$\lim_{h \rightarrow 0} H(F(t_0 + h), F(t_0)) = 0,$$

where $H(.,.)$ denotes the Pompeiu-Hausdorff distance on $\mathcal{K}_C(\mathbb{R}^n)$.

Definition 15 (Pompeiu–Hausdorff Distance Right, Left Differentiation [9]). Let $F : [a, b] \rightarrow KC(\mathbb{R}^k)$ be a set-valued function. We say that F is *right-differentiable* at $t \in (a, b)$ if there exists $A^+ \in KC(\mathbb{R}^k)$ such that one of the following two cases holds:

$$\lim_{h \rightarrow 0^+} \frac{1}{h} H(F(t+h), F(t) + hA^+) = 0, \quad (i)$$

or

$$\lim_{h \rightarrow 0^-} \frac{1}{h} H(F(t), F(t-h) + hA^+) = 0. \quad (ii)$$

We say that F is *left-differentiable* at $t \in (a, b)$ if there exists $A^- \in KC(\mathbb{R}^k)$ such that one of the following two cases holds:

$$\lim_{h \rightarrow 0^-} \frac{1}{h} H(F(t+h), F(t) + hA^-) = 0, \quad (i)$$

or

$$\lim_{h \rightarrow 0^+} \frac{1}{h} H(F(t), F(t-h) + hA^-) = 0. \quad (ii)$$

The notation A^+ (resp. A^-) is called the *right-derivative* (resp. *left-derivative*) of the set-valued function F at the point t , and we write $A^+ = F^+(t)$ and $A^- = F^-(t)$. To prove the whole differentiation exist we must show that $A^+ = A^-$.

Definition 16 (Pompeiu–Hausdorff Distance Differentiation [9]). Let $F : [a, b] \rightarrow KC(\mathbb{R}^k)$ be a set-valued function. We say that F is *differentiable* at $t \in (a, b)$ if there exists $F'(t) \in KC(\mathbb{R}^k)$ such that one of the following equivalent conditions holds:

$$\lim_{h \rightarrow 0^+} \frac{1}{h} H(F(t+h), F(t) + hF'(t)) = \lim_{h \rightarrow 0^+} \frac{1}{h} H(F(t), F(t-h) + hF'(t)) = 0, \quad (1)$$

or

$$\lim_{h \rightarrow 0^-} \frac{1}{h} H(F(t), F(t-h) + hF'(t)) = \lim_{h \rightarrow 0^-} \frac{1}{h} H(F(t+h), F(t) + hF'(t)) = 0, \quad (2)$$

or

$$\lim_{h \rightarrow 0^+} \frac{1}{h} H(F(t+h), F(t) + hF'(t)) = \lim_{h \rightarrow 0^-} \frac{1}{h} H(F(t+h), F(t) + hF'(t)) = 0, \quad (3)$$

or

$$\lim_{h \rightarrow 0^-} \frac{1}{h} H(F(t), F(t-h) + hF'(t)) = \lim_{h \rightarrow 0^+} \frac{1}{h} H(F(t), F(t-h) + hF'(t)) = 0. \quad (4)$$

3. Results and Discussion

In this section, we provides proof of fundamentals properties for the Pompeiu-Hausdorff differentiability, concepts in generalized Hukuhara differentiability, and finally discuss the established relationship between generalized Hukuhara differentiability and Pompeiu-Hausdorff distance differentiability. Before going into detail we shall prove the fundamental properties in Pompeiu-Hausdorff distance , sufficient and necessary condition for Hukuhara difference and gH difference as this will be used later throughout the paper.

3.1. Fundamentals Properties in Pompeiu-Hausdorff Distance

Proposition 1. *The following condition holds :*

- (1) $H(kA, kB) = |k|H(A, B)$, for all $k \in \mathbb{R}$.
- (2) $H(A + C, B + C) = H(A, B)$.
- (3) $H(A + B, C + D) \leq H(A, C) + H(B, D)$.
- (4) $H(kA, lA) = |k - l|H(A, \{\mathbf{0}\})$, for all $k, l \geq 0$.

Proof. Proof of (1). By definition

$$\begin{aligned} H(kA, kB) &= \max \left\{ \sup_{a \in A} \inf_{b \in B} \|ka - kb\|, \sup_{b \in B} \inf_{a \in A} \|kb - ka\| \right\} \\ &= \max \left\{ |k| \sup_{a \in A} \inf_{b \in B} \|a - b\|, |k| \sup_{b \in B} \inf_{a \in A} \|b - a\| \right\} \\ &= |k| \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|b - a\| \right\} \\ &= |k| H(A, B). \end{aligned}$$

Proof of (2). By definition

$$H(A + C, B + C) = \max \left\{ \sup_{a+c \in A+C} \inf_{b+c' \in B+C'} \|a + c - c' - b\|, \sup_{b+c' \in B+C} \inf_{a+c' \in A+C'} \|b + c' - a - c\| \right\}$$

and

$$H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|b - a\| \right\}$$

Using triangle inequality,

$$\begin{aligned} &\|a + c - b - c'\| + \|c - c'\| \geq \|a - b\| \\ \Rightarrow &\sup_{a+c \in A+C} \inf_{b+c' \in B+C} \|a + c - b - c'\| + \sup_{a+c \in A+C} \inf_{b+c' \in B+C} \|c - c'\| \\ &\geq \sup_{a+c \in A+C} \inf_{b+c' \in B+C} \|a - b\| = \sup_{a \in A} \inf_{b \in B} \|a - b\|, \end{aligned}$$

clearly,

$$\sup_{a+c \in A+C} \inf_{b+c' \in B+C} \|c - c'\| = 0.$$

Similarly,

$$\begin{aligned} &\sup_{b+c' \in B+C} \inf_{a+c \in A+C} \|b + c' - a - c\| + \sup_{b+c' \in B+C} \inf_{a+c \in A+C} \|c' - c\| \\ &\geq \sup_{b+c' \in B+C} \inf_{a+c \in A+C} \|b - a\| = \sup_{b \in B} \inf_{a \in A} \|b - a\|. \end{aligned}$$

From here we get

$$H(A + C, B + C) \geq H(A, B).$$

The converse uses

$$\|a - b\| + \|c - c'\| \geq \|a + c - b - c'\|,$$

and by the same reasoning,

$$H(A, B) \geq H(A + C, B + C).$$

Hence,

$$H(A + C, B + C) = H(A, B).$$

Proof of (3). By (2): $H(A, C) + H(B, D) = H(A + B, B + C) + H(B + C, D + C)$ Using triangle inequality yields: $H(A + B, B + C) + H(B + C, D + C) \geq H(A + B, C + D)$

Proof of (4). WLOG $k \geq l$: $H(kA, lA) = H((k - l + l)A, lA) = H((k - l)A + lA, lA) = H((k - l)A, (k - l)\{0\}) = |k - l|H(A, \{0\})$

Proof $(a + b)A = aA + bA, \forall a, b > 0, \forall A \in KC(\mathbb{R}^k)$: Clearly $(a + b)A \subseteq aA + bA$ we shall prove the converse. take any $x \in aA, y \in bA$. Write $x = aw, y = bt, w, t \in A$ Notice that $(a + b)(\frac{x}{a+b} + \frac{y}{a+b}) = (a + b)(\frac{a}{a+b}w + \frac{b}{a+b}t)$ clearly $\frac{a}{a+b}w + \frac{b}{a+b}t \in A$ hence $x + y \in (a + b)A$ \square

3.2. Necessary and Sufficient Condition for Hukuhara and Generalized Hukuhara Difference Existence

Proposition 2. Let $U, V \in \mathcal{K}_c(\mathbb{R}^k)$. The necessary and sufficient condition for the Hukuhara difference $U -_H V$ to exist is that for all $u \in U$ there exists at least a point w such that

$$u \in V + \{w\} \subseteq U.$$

Proof. Necessary condition proof: Since $U -_H V$ then $\exists W$ such that $U = W + V$ hence $\forall u \in U, u = v + w, v \in V, w \in W$. From here we get $u \in V + \{w\} \subseteq W + V = U$
Sufficient condition proof : Pick any $u \in U$. $\exists w_u \in \mathbb{R}^n$ so that

$$u \in V + \{w_u\} \subseteq U.$$

We get

$$U \subseteq \bigcup_{u \in U} (V + \{w_u\}) = V + \bigcup_{u \in U} \{w_u\} = V + W \subseteq U$$

Hence

$$U = V + W.$$

Denote $cl(U)$ as closure of U and $conv(U)$ as the smallest convex sets containing U .

We shall prove

$$U = V + cl(W), U = V + conv(W),$$

$U = V + W \subseteq cl(V) + cl(W) = V + cl(W) \subseteq cl(V + W) = cl(U) = U$ and $U = V + W \subseteq conv(V) + conv(W) = V + conv(W) \subseteq conv(V + W) = conv(U) = U$. Hence $cl(conv(W)) = Q$ so Q is convex and $U = V + Q$. Since U, V are bounded then Q is bounded too. Hence $Q \in KC(\mathbb{R}^n)$. Q unique, if not then $\exists B \in KC(\mathbb{R}^n)$ so that $U = V + B$ hence $0 = H(V + Q, V + B) = H(Q, B) \rightarrow Q = B$. Hence $U -_H V = Q$. \square

Example 10 (Intuition). Consider 2 closed circles on cartesian coordinate centered at its origin with radius respectively 2 and 1. Denote the bigger circle as A and the smaller circle B . Notice that by translating circle B by any points from itself will forms a translated circle B that is inside A . Taking the union of every possible translated circle B will gives us the entire circle A . Hence $A -_H B = B$

Proposition 3. Let $U, V \in \mathcal{K}_c(\mathbb{R}^k)$. The necessary and sufficient condition for the generalized Hukuhara difference $U -_{gH} V$ to exist is that for all $v \in V$ and $u \in U$ there exists at least a point w such that either

$$v \in U - \{w\} \subseteq V$$

or

$$u \in V + \{w\} \subseteq U$$

Proof. By noticing that if $W \in \mathcal{K}_c(\mathbb{R}^k)$ then $-W \in \mathcal{K}_c(\mathbb{R}^k)$ where $-W = (-1)W$ from Minkowski scalar multiplication definition, the proof follows the same logic and argument as in Hukuhara differences case. \square

3.3. Basic Properties in Pompeiu-Hausdorff Differentiation

Proposition 4. The value A^+ is unique.

Proof. Case 1: Let there exist other set B^+ such that both B^+ and A^+ satisfy (i) in Definition 15. By the right-limit definition, $\forall s > 0, \exists c > 0$ so that if $0 < h < c \rightarrow \frac{H(F(t+h), F(t)+hA^+)}{h} < s$ and $\forall y > 0, \exists p > 0$ so that if $0 < h < p \rightarrow \frac{H(F(t+h), F(t)+hB^+)}{h} < y$. Now by taking $b = \min(y, p)$ we have that whenever $0 < h < b$ then

$$\frac{H(F(t+h), F(t)+hA^+)}{h} + \frac{H(F(t+h), F(t)+hB^+)}{h} < y + s$$

Now applying triangle inequality yields

$$\frac{H(F(t+h), F(t)+hA^+)}{h} + \frac{H(F(t+h), F(t)+hB^+)}{h} \geq \frac{H(F(t)+hB^+, F(t)+hA^+)}{h}$$

By Proposition 1 (2) and (1) yields $\frac{H(F(t)+hB^+, F(t)+hA^+)}{h} = \frac{H(hB^+, hA^+)}{h} = H(B^+, A^+)$ hence

$$0 \leq H(B^+, A^+) < y + s, \forall y, s > 0$$

Let $\epsilon = y + s$ since $0 \leq H(B^+, A^+) < \epsilon, \forall \epsilon > 0$ then $H(B^+, A^+) = 0 \rightarrow A^+ = B^+$

Case 2: Let B^+ follow (i) and A^+ follow (ii) in Definition 15. By the right-limit definition, $\forall s > 0, \exists c > 0$ so that if $0 < h < c \rightarrow \frac{H(F(t+h), F(t)+hB^+)}{h} < s$ and by the left limit definition $\forall y > 0, \exists p > 0$ so that if $-p < h < 0 \rightarrow \left| \frac{H(F(t), F(t-h)+hA^+)}{h} \right| = \frac{H(F(t), F(t-h)+hA^+)}{-h} < y$. Substituting $u = -h$ yields $p > u$ hence whenever $p > u$ implies $\frac{H(F(t), F(t+u)-uA^+)}{u} < y$. Consider $\forall h$ satisfying $b = \min(p, c) > h > 0$. By Proposition 1 (1) and (2) respectively we have

$$\begin{aligned} H(\{0\}, -A^+ + B^+) &= \frac{H(\{0\}, -hA^+ + hB^+)}{h} \\ &= \frac{H(F(t+h) + F(t), F(t+h) + F(t) - hA^+ + hB^+)}{h} \end{aligned}$$

Applying Proposition 1 (3) gives us

$$\begin{aligned} 0 \leq H(\{0\}, -A^+ + B^+) &= \frac{H(F(t+h) + F(t), F(t+h) + F(t) - hA^+ + hB^+)}{h} \\ &\leq \frac{H(F(t), F(t+h) - hA^+)}{h} + \frac{H(F(t+h), F(t) + hB^+)}{h} < y + s, \forall y, s > 0 \quad (\text{iii}) \end{aligned}$$

Since $0 \leq H(\{0\}, -A^+ + B^+) < y + s, \forall y, s > 0$ following the same logic as in case (i) we deduce that $H(\{0\}, B^+ - A^+) = 0 \rightarrow \{0\} = B^+ - A^+ \Rightarrow A^+ = B^+$ \square

Proposition 5. $F' = \{0\}$, if $F = B$ (i.e., F is constant).

Proof. If F is constant, then $F(t) = F(t+h) = F(t-h) \forall h, t > 0$ as long as $F(t), F(t-h), F(t+h)$ exist. thus by taking $\delta = \epsilon, \forall \epsilon > 0$ we get for any $h > 0$ satisfying $h < \delta$ we have

$$\frac{1}{h}H(F(t+h), F(t)) = \frac{1}{h}H(\{0\}, \{0\}) = 0 < \epsilon, \frac{1}{h}H(F(t), F(t-h)) = \frac{1}{h}H(\{0\}, \{0\}) = 0 < \epsilon,$$

which implies $F'(t) = \{0\}$ by Definition 16 (1). Since $\{0\} = F'(t)$ then $\{0\}$ is both the right and the left differentiation of F and the uniqueness of the right differentiation of F is guaranteed by proposition 4; Hence, $F'(t)$ is unique and is the singleton $\{0\}$. \square

Proposition 6. If F is differentiable at the point $x = t$, then F is continuous at $x = t$.

Proof. Since the proof for the other cases follows the same logic, we shall prove one case only. Now assume that it satisfies (i) in Definition 15. Since $F'(t)$ follows (i), then by the limit epsilon-delta definition:

$\forall \epsilon > 0 \exists \delta > 0$ so that $\forall h > 0$ satisfying $h < \delta$ we have

$$H(F(t+h), F(t) + hF'(t)) < h\epsilon$$

and

$\forall c > 0 \exists u > 0$ so that $\forall h > 0$ satisfying $h < u$ we have

$$H(F(t), F(t-h) + hF'(t)) < hc.$$

Then pick $u = c$ and $\delta = \epsilon$. So,

$$\forall h > 0 \text{ satisfying } h < \delta, u$$

we get by Proposition 1 (3):

$$\begin{aligned} H(F(t+h), F(t)) &= H(F(t+h) + hF'(t), F(t) + hF'(t)) \\ &\leq H(F(t+h), F(t) + hF'(t)) + H(hF'(t), \{0\}) \\ &< \epsilon h + hH(F'(t), \{0\}) < \epsilon^2 + \epsilon M = \epsilon(\epsilon + M), \end{aligned}$$

where $M = H(F'(t), \{0\})$ is a non-zero constant. And by Proposition 1 (3),

$$H(F(t-h), F(t)) = H(F(t-h) + hF'(t), F(t) + hF'(t))$$

$$\leq H(F(t-h) + hF'(t), F(t)) + H(\{0\}, hF'(t)).$$

Hence,

$$\begin{aligned} & H(F(t-h) + hF'(t), F(t)) + H(\{0\}, hF'(t)) \\ &= H(F(t-h) + hF'(t), F(t)) + hH(\{0\}, F'(t)) < hc + hM < c(c + M). \end{aligned}$$

Hence,

$$\lim_{h \rightarrow 0} H(F(t+h), F(t)) = 0.$$

So F is continuous at $x = t$. □

Proposition 7. *The set of all Hausdorff differentiable set-valued functions that satisfy the following condition among (1), (2), (3), or (4) in definition 16 forms a convex cone.*

Proof. Let the set be denoted by \mathcal{A} . Pick two arbitrary elements of \mathcal{A} , F and G , and without loss of generality assume that F, G satisfies (1).

1. $kF' = (kF)'$. Let $k = 0$. Then, $kF' = 0$ and $(kF)' = 0$. Now, pick any positive k . Let F be Pompeiu-Hausdorff differentiable and satisfy (1). Then by limit epsilon-delta definition $\forall \frac{\epsilon}{k} > 0, \exists \delta > 0$ so that for any $h > 0$ satisfying $h < \delta$ we have

$$\frac{H(F(t+h), F(t) + hF'(t))}{h} < \frac{\epsilon}{k}$$

where ϵ is any positive real number. In this case let δ be the satisfying positive real that satisfy the definition above. Then $\forall h > 0$ satisfying $h < \delta$ we have:

$$\frac{H(F(t+h), F(t) + hF'(t))}{h} < \frac{\epsilon}{k} \implies \frac{H(kF(t+h), kF(t) + hkF'(t))}{h} < \epsilon, \forall \epsilon > 0. \quad (2)$$

This implies kF is differentiable too with $kF'(t)$ as the derivative. By the uniqueness of derivative then $kF' = (kF)'$ where $(kF)'$ denotes the Pompeiu-Hausdorff derivative of kF .

2. $aF' + bG' = (aF + bG)'$, $\forall a, b \geq 0$. Firstly, from the preceding (Proposition 7 (1)) step we have proved that $aF' = (aF)'$ for any Pompeiu-Hausdorff differentiable function F and $\forall a > 0$. By the limit epsilon-delta definition we have, $\forall c > 0, \exists d > 0$ so that for any $h > 0$ satisfying $h < d$ implies

$$H(aF(t+h), aF(t) + h(aF(t))') < hc,$$

and

$$\forall f > 0,$$

$\exists g > 0$ so for every $h > 0$ satisfying $h < g$ implies

$$H(bG(t+h), bG(t) + h(bG(t))') < hf.$$

Hence, by taking $w = \min(d, g)$ we have:

$$\begin{aligned} & \frac{H(bG(t+h), bG(t) + h(bG(t))') + H(aF(t+h), aF(t) + h(aF(t))')}{h} \geq \\ & \frac{H(bG(t+h) + aF(t+h), bG(t) + aF(t) + h((bG(t))' + (aF(t))'))}{h} \end{aligned}$$

But,

$$\frac{H(bG(t+h), bG(t) + h(bG(t))') + H(aF(t+h), aF(t) + h(aF(t))')}{h} \leq c + f$$

So,

$$\frac{H(bG(t+h) + aF(t+h), bG(t) + aF(t) + h((bG(t))' + (aF(t))'))}{h} \leq c + f$$

This implies,

$$\lim_{h \rightarrow 0^+} \frac{1}{h} H(bG(t+h) + aF(t+h), bG(t) + aF(t) + h((bG(t))' + (aF(t))')) = 0$$

Which means that $((aF + bG)'(t))^+$ exist and $((aF + bG)'(t))^+ = aF'(t) + bG'(t)$ where $((aF + bG)'(t))^+$ denotes the right Pompeiu-Hausdorff derivative of $aF(t) + bG(t)$. To prove the left derivative is indeed $aF'(t) + bG'(t)$, we shall use the limit epsilon-delta definition again. By the limit epsilon-delta definition we have, $\forall c > 0, \exists d > 0$ so that for any $h > 0$ satisfying $h < d$ implies $H(aF(t), aF(t-h) + h(aF(t))') < hc$ and $\forall f > 0, \exists g > 0$ so for every $h > 0$ satisfying $h < g$ implies $H(bG(t), bG(t-h) + h(bG(t))') < hf$. Hence, by taking $w = \min(d, g)$ we have:

$$\frac{H(bG(t), bG(t-h) + h(bG(t))') + H(aF(t), aF(t-h) + h(aF(t))')}{h} \geq \frac{H(bG(t) + aF(t), bG(t-h) + aF(t-h) + h((bG(t))' + (aF(t))'))}{h}$$

But,

$$\frac{H(bG(t), bG(t-h) + h(bG(t))') + H(aF(t), aF(t-h) + h(aF(t))')}{h} \leq c + f$$

So,

$$\frac{H(bG(t) + aF(t), bG(t-h) + aF(t-h) + h((bG(t))' + (aF(t))'))}{h} \leq c + f$$

This implies,

$$\lim_{h \rightarrow 0^+} \frac{1}{h} H(bG(t) + aF(t), bG(t-h) + aF(t-h) + h((bG(t))' + (aF(t))')) = 0$$

Which means that $((aF + bG)'(t))^-$ exist and $((aF + bG)'(t))^- = aF'(t) + bG'(t)$ where $((aF + bG)'(t))^-$ denotes the left Pompeiu-Hausdorff derivative of $aF(t) + bG(t)$. Hence we conclude $aF' + bG' = (aF + bG)', \forall a, b \geq 0$.

□

3.4. Generalized Hukuhara Differentiability and Pompeiu-Hausdorff Distance Differentiation Relation

Proposition 8. *F is Pompeiu-Hausdorff differentiable if F is gH-differentiable*

Proof. Let F be gH -differentiable at t . Denote

$$F(t+x) -_{gH} F(t) = Q(x) \quad F'_{gh}(t) = \lim_{x \rightarrow 0^+} \frac{Q(x)}{x}.$$

Therefore, by the ε - δ definition of gH -differentiability, $\forall c > 0, \exists d \in \mathbb{R}^+$ such that if $|x| < d$, $\rightarrow H\left(\frac{F(t+x) -_{gH} F(t)}{x}, F'_{gh}(t)\right) = H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) < c$. We shall prove that $F(t)$ is Pompeiu-Hausdorff differentiable. Note that

$$H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) = \frac{H(Q(x), xF'_{gh}(t))}{x}, \quad \text{by Proposition 1 (1).}$$

Case 1: $F(t+x) = Q(x) + F(t)$

- Take $x > 0$. For any x satisfying $0 < x < d$, we have

$$\begin{aligned} \frac{1}{x}H(F(t+x), F(t) + xF'_{gh}(t)) &= \frac{1}{x}H(Q(x) + F(t), F(t) + xF'_{gh}(t)) \\ &= \frac{1}{x}H(Q(x), xF'_{gh}(t)) \\ &= H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) \end{aligned}$$

But,

$$H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) < c$$

Hence,

$$\frac{1}{x}H(F(t+x), F(t) + xF'_{gh}(t)) < c$$

finally,

$$\lim_{x \rightarrow 0^+} \frac{1}{x}H(F(t+x), F(t) + xF'_{gh}(t)) = 0.$$

- Take $x < 0$. For any x satisfying $x > -d$, we have

$$H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) < c \implies \frac{1}{-x}H(Q(x), xF'_{gh}(t)) = \frac{1}{-x}H(Q(x) + F(t), F(t) + xF'_{gh}(t))$$

Hence,

$$\frac{1}{-x}H(Q(x) + F(t), F(t) + xF'_{gh}(t)) = \frac{1}{-x}H(F(t+x), F(t) + xF'_{gh}(t)) < c.$$

Then,

$$\lim_{x \rightarrow 0^-} \frac{1}{x}H(F(t+x), F(t) + xF'_{gh}(t)) = 0.$$

Thus,

$$\lim_{x \rightarrow 0^-} \frac{1}{x}H(F(t+x), F(t) + xF'_{gh}(t)) = \lim_{x \rightarrow 0^+} \frac{1}{x}H(F(t+x), F(t) + xF'_{gh}(t)) = 0,$$

so $F(t)$ is Pompeiu-Hausdorff differentiable by Definition 16 (3).

Case 2: $F(t+x) - Q(x) = F(t)$

- Take $x < 0$. For any x satisfying $x > -d$, we have

$$\begin{aligned} \frac{1}{-x}H(F(t), F(t+x) - xF'_{gh}(t)) &= \frac{1}{-x}H(-Q(x) + F(t+x), F(t+x) - xF'_{gh}(t)) \\ &= \frac{1}{-x}H(-Q(x), -xF'_{gh}(t)) \end{aligned}$$

Then,

$$\frac{1}{-x}H(-Q(x), -xF'_{gh}(t)) = H\left(\frac{-Q(x)}{-x}, \frac{-xF'_{gh}(t)}{-x}\right) = H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) < c.$$

Hence,

$$\lim_{h \rightarrow 0^+} \frac{1}{h}H(F(t), F(t-h) + hF'(t)) = 0.$$

- Take $x > 0$. For any x satisfying $x < d$, we have

$$H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) < c \implies \frac{1}{x}H(Q(x), xF'_{gh}(t)) = \frac{1}{x}H(Q(x) - F(t+x), -F(t+x) + xF'_{gh}(t))$$

Hence,

$$\frac{1}{x}H(Q(x) - F(t+x), -F(t+x) + xF'_{gh}(t)) = \frac{1}{x}H(-F(t), -F(t+x) + xF'_{gh}(t)).$$

Then,

$$\frac{1}{x}H(-F(t), -F(t+x) + xF'_{gh}(t)) = \frac{1}{x}H(F(t), F(t+x) - xF'_{gh}(t))$$

Hence,

$$\frac{1}{x}H(F(t), F(t+x) - xF'_{gh}(t)) = H\left(\frac{Q(x)}{x}, F'_{gh}(t)\right) < c$$

This implies,

$$\left| \frac{1}{-x}H(F(t), F(t+x) - xF'_{gh}(t)) \right| < c$$

Finally we can conclude that,

$$\lim_{x \rightarrow 0^-} \frac{1}{x}H(F(t), F(t-x) + xF'_{gh}(t)) = 0.$$

Thus,

$$\lim_{x \rightarrow 0^-} \frac{1}{x}H(F(t), F(t-x) + xF'_{gh}(t)) = \frac{1}{x}H(F(t), F(t-x) + xF'_{gh}(t)) = 0,$$

so $F(t)$ is Pompeiu-Hausdorff differentiable by Definition 16 (4). Thus, $F(t)$ is Pompeiu-Hausdorff differentiable by Definition , and by uniqueness of the derivative, we have $F'_{gh}(t)$ as the derivative. □

4. Conclusion

We have reviewed the fundamental properties in Pompeiu-Hausdorff distance differentiability and concepts in generalized Hukuhara differentiability. Any set-valued function is Pompeiu-Hausdorff distance differentiable if it is gH differentiable. The collection of every sets that is

Pompeiu-Hausdorff distance differentiable that satisfy the same definition of differentiable forms convex cone. The Pompeiu-Hausdorff distance behaves the same way as single-valued function like interval calculus in a sense that every set valued function that is Pompeiu-Hausdorff distance differentiable on a particular points implies continuity on that points. The Hukuhara difference is unique and does not always exist, the same goes for generalized version of it in terms of its existence.

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William Surya Putra: Writer of this article .

Mohamad Muslikh: Advisor, and reviewer for this article.

Declaration of Generative AI and AI-assisted technologies

ChatGPT was used in preparing this manuscript to assist in refining the grammar and making the mathematical expressions tidier and more readable. Every mathematical aspect was written and developed by the author.

Declaration of Competing Interest

The author declares no competing interests.

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