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Compare Bede–Gal between d_H-Differentiability of Set-Valued Functions

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Abstract

This paper presents a comparative study between the Bede–Gal differentiability and the d_H-differentiability for set-valued functions whose values belong to the family of nonempty compact convex subsets of $\mathbb{R}$. The Bede–Gal derivative, originally introduced for fuzzy and interval-valued functions, is redefined for the set-valued framework and analyzed through its metric properties. Meanwhile, the d_H-derivative is formulated in terms of the Pompeiu–Hausdorff metric, allowing differentiability without requiring the existence of the Hukuhara or generalized Hukuhara difference. We establish several results clarifying the relationship between these two concepts, including sufficient conditions under which a function that is Bede–Gal differentiable is also d_H-differentiable, and conversely. Illustrative examples are provided to demonstrate cases where one type of differentiability exists while the other fails. The comparison emphasizes that d_H-differentiability provides a broader and more flexible framework, extending the applicability of the differential calculus in set-valued analysis.

Keywords: Bede–Gal differentiability; d_H-differentiability; Pompeiu–Hausdorff metric; Set-valued function; Generalized Hukuhara difference

1. Introduction

The analysis of differentiability for interval-valued and set-valued functions has become an essential topic in modern mathematical analysis, particularly in areas such as optimization, control theory, and systems with uncertainty. These functions extend classical real-valued mappings by allowing their outputs to be sets, enabling the modeling of imprecise or uncertain data [1, 2]. Understanding differentiability in this context is fundamental for the development of set differential equations and viability theory, which describe dynamic processes involving evolving sets [3]. The theoretical foundations of convexity and correspondences, as developed in [4, 5], also play a crucial role in supporting the mathematical structure behind set-valued mappings and their analytical properties.

Historically, the calculus of interval-valued functions was initiated by Markov [1], who introduced basic interval operations and an early differential framework for interval-valued mappings. Subsequently, Banks and Jacobs [2] developed a general differential calculus for multifunctions, establishing a foundational structure for the modern theory of set-valued analysis. Aubin [3] further extended these ideas through mutational and morphological analysis, providing geometric

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tools for describing the evolution of sets without requiring algebraic inverses. Collectively, these early developments laid the groundwork for subsequent generalizations of differentiability in set-valued contexts.

The Hukuhara difference provided the first systematic attempt to define a derivative for interval-valued functions. However, its applicability was restricted to functions whose intervals evolve monotonically. To overcome this limitation, Stefanini and Bede [6] introduced the generalized Hukuhara (gH) difference, which allows both expanding and contracting intervals. Additional refinements, such as the generalized and π-derivatives, were introduced by Chalco-Cano et al. [7, 8] to extend the analytical framework for interval- and fuzzy-valued functions. Despite their importance, both the Hukuhara and gH differences were later shown not to preserve inclusion isotonicity, raising significant concerns in applications where order relations between sets must be maintained [9]. Motivated by these issues, Stefanini and Bede proposed a new multidimensional gH-difference with improved structural properties and guaranteed existence [10].

A metric-based alternative was subsequently introduced to address the structural limitations of Hukuhara-type differences. The Bede–Gal derivative, defined using the Pompeiu–Hausdorff distance, provides a consistent generalization of differentiability that avoids direct reliance on set subtraction. By measuring the rate of change between sets through a metric limit, this derivative ensures topological consistency and broader existence conditions [11]. More recently, Khastan, Rodríguez-López, and Shahidi [12] proposed a unified framework of differentiability concepts—including the d_H-derivative—which systematically consolidates and extends previously established notions while demonstrating their applications to set differential equations.

Parallel to these developments, metric approaches have also been explored from a functional-analytic perspective. Muslikh et al. [13] introduced the metric derivative for set-valued functions, extending the classical notion of speed in metric spaces and establishing almost-everywhere differentiability for Lipschitz multifunctions. Complementing this direction, Muslikh and Kilicman [14] studied fixed-point principles for derivatives of interval-valued functions, revealing structural insights relevant to iterative methods and differential inclusions.

The present study revisits differentiability of set-valued functions from a metric perspective, emphasizing the unifying role of the Pompeiu–Hausdorff distance. Specifically, we provide a comparative synthesis of the Hukuhara, gH, Bede–Gal, and d_H-type derivatives, clarifying their geometric and analytical relationships while discussing their implications in contemporary set-valued analysis. This investigation aims to contribute to the theoretical understanding of metric-based differentiability and to highlight potential pathways toward more general differential frameworks for multifunctions.

2. Preliminaries

This section recalls the fundamental notions and notations that will be used throughout the paper. We begin with the basic algebraic operations on compact convex subsets of $\mathbb{R}^n$, which provide the algebraic foundation for the study of differentiability in set-valued analysis. Subsequently, we review several differentiability concepts for set-valued and interval-valued functions. These notions arise either from algebraic difference structures or from metric convergence in the Pompeiu–Hausdorff space.

Throughout this paper, we work with two distinct structures on the space of compact convex sets: the algebraic structure induced by Minkowski addition, and the metric structure induced by the Pompeiu–Hausdorff distance. The distinction between these two frameworks plays a crucial role in the comparison of differentiability concepts.

2.1. Algebraic Structure on $\mathcal{K}^n_C$

Let $\mathcal{K}^n$ denote the family of all nonempty compact subsets of $\mathbb{R}^n$, and let $\mathcal{K}^n_C$ be its subfamily of convex compact subsets. For $n = 1$, we simply write $\mathcal{K}$ and $\mathcal{K}_C$. For any $A, B \in \mathcal{K}$ and $c \in \mathbb{R}$, the Minkowski addition and scalar multiplication are defined by

$$A + B := \{a + b \mid a \in A,\, b \in B\}, \qquad cA := \{ca \mid a \in A\}.$$

The Minkowski difference is given by $A - B := A + (-1)B$. It is well known that $\mathcal{K}_C^n$ is closed under Minkowski addition and scalar multiplication, but not under Minkowski subtraction. In general, $A - A = \{0\}$ if and only if A is a singleton. Since the inverse element under Minkowski addition does not exist in $\mathcal{K}_C^n$, an alternative subtraction was introduced so that $A \ominus A = \{0\}$ always holds, known as the Hukuhara difference.

Definition 1. For $A, B \in \mathcal{K}^n$, the Hukuhara difference (H-difference) is defined by [6]:

$$A \ominus B = C \iff A = B + C.$$

The H-difference, whenever it exists, is unique. In particular, for every $A \in \mathcal{K}^n$, the difference $A \ominus A$ exists and equals $\{0\}$. However, the existence of the H-difference is not guaranteed for arbitrary sets: a necessary condition for $A \ominus B$ to exist is that some translation of B is contained in A (see [1, 6]).

Example 2. Let $A = [1, 3]$ and $B = [0, 1]$. We claim that the Hukuhara difference $A \ominus B$ exists and equals $[1, 2]$. To verify this, we show that $[1, 3] = [0, 1] + [1, 2]$. First, let $x \in [0, 1]$ and $y \in [1, 2]$. Then $1 \le x + y \le 3$, so $x + y \in [1, 3]$. Hence, $[0, 1] + [1, 2] \subseteq [1, 3]$. Conversely, let $z \in [1, 3]$. Define

$$z_1 = \min\{1, z - 1\} \in [0, 1], \qquad z_2 = z - z_1.$$

Then $z_2 \in [1, 2]$ and $z = z_1 + z_2$, which implies $z \in [0, 1] + [1, 2]$. Therefore, $[1, 3] \subseteq [0, 1] + [1, 2]$. Thus $[1, 3] = [0, 1] + [1, 2]$, and consequently $A \ominus B = [1, 2]$.

Example 3. Let $A = [1, 3]$ and $B = [0, 4]$. Suppose $A \ominus B = C$ for some $C \in \mathcal{K}$. Then $A = B + C$, implying that every $z \in B + C$ satisfies $x \le z \le 4 + y$, where $x = \min C$ and $y = \max C$. Hence $x = 1$ and $y = -1$, which is impossible. Therefore, the Hukuhara difference $A \ominus B$ does not exist.

To address this limitation, Stefanini [6] proposed the generalized Hukuhara difference (gH-difference), which extends the definition to cover both expanding and contracting cases of sets.

Definition 4. Let $A, B \in \mathcal{K}^n$. The generalized Hukuhara difference (gH-difference) of A and B is defined as [6]:

$$A \ominus_{gH} B = C \iff \begin{cases} A = B + C, \\ \qquad \textit{or} \\ B = A + (-C). \end{cases}$$

It may occur that both relations hold simultaneously for certain sets $A, B, C \in \mathcal{K}^n$. Although this generalization makes the subtraction more flexible than the classical H-difference, the gH-difference still does not always exist, even for compact convex sets. If it exists, the generalized Hukuhara difference is unique.

Example 5. Let $A = [1, 3]$ and $B = [0, 4]$. We show that the generalized Hukuhara difference $A \ominus_{gH} B$ exists and equals $[-1, 1]$. From Example 3, the classical Hukuhara difference $A \ominus B$ does not exist. Therefore, we consider the second case in the definition of the generalized Hukuhara difference, namely $B = A + (-1)C$. We claim that $C = [-1, 1]$. It suffices to prove that $[0, 4] = [1, 3] + [-1, 1]$. First, let $x \in [1, 3]$ and $y \in [-1, 1]$. Then $0 \le x + y \le 4$, so $x + y \in [0, 4]$. Hence $[1, 3] + [-1, 1] \subseteq [0, 4]$. Conversely, let $z \in [0, 4]$. Define

$$z_1 = \min\{3, z + 1\} \in [1, 3], \qquad z_2 = z - z_1.$$

Then $z_2 \in [-1, 1]$ and $z = z_1 + z_2$, which implies $z \in [1, 3] + [-1, 1]$. Thus $[0, 4] \subseteq [1, 3] + [-1, 1]$. Therefore, $[0, 4] = [1, 3] + [-1, 1]$, and consequently $A \ominus_{gH} B = [-1, 1]$.

Example 6. Let $A, B \subset \mathbb{R}^2$ be two compact convex sets defined as follows: A is the triangle with vertices $(2, 0)$, $(4, 0)$, and $(4, 2)$, while B is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$. Assume that there exists $C \in \mathcal{K}^2$ satisfying $A \ominus_{gH} B = C$. If $A = B + C$, then for every $\mathbf{c} \in C$, the translation $\{\mathbf{c}\} + B$ must be contained in A. Geometrically, $\{\mathbf{c}\} + B$ denotes a translation of B by the vector $\mathbf{c} = (x, y)$, representing horizontal and vertical shifts of x and y units, respectively. However, there is no translation vector $\mathbf{c}$ such that $B + \{\mathbf{c}\} \subseteq A$. A similar argument applies to the reverse relation $B = A + (-C)$, so the generalized Hukuhara difference does not exist.

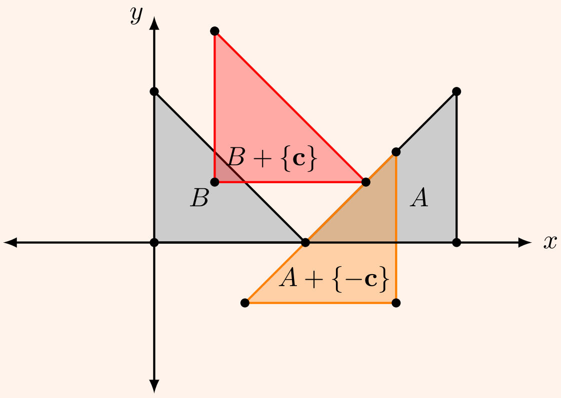

Fig. 1: Translation of B by $\mathbf{c}$ and of A by $-\mathbf{c}$

Example 6 illustrates that even the generalized Hukuhara difference may fail to exist. Although it allows both additive and subtractive behaviors, the existence of $A \ominus_{gH} B$ still requires strong geometric compatibility between A and B.

Proposition 7. *If $A, B \in \mathcal{K}^n_C$ and the Hukuhara difference $A \ominus B$ exists, then the generalized Hukuhara difference $A \ominus_{gH} B$ also exists and*

$$A \ominus_{gH} B = A \ominus B.$$

Proof. If $A \ominus B$ exists, there is a $C \in \mathcal{K}^n_C$ such that $A \ominus B = C \iff A = B + C$. This directly implies $A \ominus_{gH} B = C$, and hence $A \ominus_{gH} B = A \ominus B$. □

The following well-known result, presented by Stefanini [15], summarizes several algebraic properties of the generalized Hukuhara difference in the space of convex compact subsets of $\mathbb{R}^n$.

Lemma 8. *Let $A, B, C, D \in \mathcal{K}_C^n$. Then [11]:*

1. *If $A \ominus B$ exists and $k \in \mathbb{R}$, then $kA \ominus kB$ exists and*

$$kA \ominus kB = k(A \ominus B).$$

2. *If $A \ominus B$ and $C \ominus D$ exist, then $(A + B) \ominus (C + D)$ exists and*

$$(A + B) \ominus (C + D) = (A \ominus C) + (B \ominus D).$$

Corollary 9. *Let $A, B \in \mathcal{K}_C^n$. If $A \ominus B$ exists, then $(A \ominus B) + B = A$.*

Proposition 10. *Let $A, B \in \mathcal{K}_C^n$. Then [15]:*

1. $A \ominus_{gH} A = \{0\}$.
2. $(A + B) \ominus_{gH} B = A,\quad A \ominus_{gH} (A - B) = B,\quad$ *and* $\quad A \ominus_{gH} (A + B) = -B$.
3. *$A \ominus_{gH} B$ exists if and only if both $B \ominus_{gH} A$ and $(-B) \ominus_{gH} (-A)$ exist. In that case,*

$$A \ominus_{gH} B = (-B) \ominus_{gH} (-A) = -(B \ominus_{gH} A).$$

4. *$A \ominus_{gH} B = B \ominus_{gH} A = C$ if and only if $C = -C$. Moreover, $C = \{0\}$ if and only if $A = B$.*

For an interval $[a, b] \subset \mathbb{R}$, we denote its length by $len([a, b]) = b - a$.

Proposition 11. *The gH-difference of two real intervals $[a, b]$ and $[c, d]$ always exists and is given by [9]:*

$$[a, b] \ominus_{gH} [c, d] = [\min\{a - c, b - d\}, \max\{a - c, b - d\}].$$

Moreover,

$$[a, b] \ominus_{gH} [c, d] = \begin{cases} [a - c, b - d], & \text{if } len([a, b]) \geq len([c, d]), \\ [b - d, a - c], & \text{if } len([a, b]) < len([c, d]). \end{cases}$$

Corollary 12. *The Hukuhara difference $[a, b] \ominus [c, d]$ exists if and only if $len([a, b]) \geq len([c, d])$. In that case,*

$$[a, b] \ominus [c, d] = [a - c, b - d].$$

Although the generalized Hukuhara difference always exists for real intervals, this phenomenon is essentially one-dimensional. In higher dimensions, the existence of Hukuhara-type differences depends on nontrivial geometric containment conditions, and may fail even for compact convex sets. Thus, the interval case does not fully reflect the structural limitations present in $\mathbb{R}^n$.

It should be emphasized that both the Hukuhara and generalized Hukuhara differences are defined purely within the algebraic structure induced by Minkowski addition. No metric structure is involved in their formulation.

This limitation motivated the development of alternative differentiability notions. In particular, the Bede–Gal derivative remains within the generalized Hukuhara framework, combining algebraic difference with metric convergence, whereas the d_H-derivative is formulated purely in terms of metric convergence in $(\mathcal{K}_C^n, d_H)$.

2.2. Metric Structure on $\mathcal{K}_C^n$

We now introduce the metric framework governing convergence in the space of compact convex subsets of $\mathbb{R}^n$.

Definition 13. The Pompeiu–Hausdorff distance d_H on $\mathcal{K}^n$ is defined by [12]:

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

It is well known that $(\mathcal{K}^n, d_H)$ is a complete metric space, and $\mathcal{K}_C^n$ is a closed subspace of $\mathcal{K}^n$ [3]. Before introducing continuity and differentiability notions, we recall several properties of the Pompeiu–Hausdorff distance that will be used throughout this work. These properties show that the metric structure is compatible with Minkowski addition and scalar multiplication, allowing the development of a coherent analytic framework for set-valued functions.

Lemma 14. *Let $A, B, C, D \in \mathcal{K}_C^n$. Then [12]:*

1. $d_H(\lambda A, \lambda B) = |\lambda|\, d_H(A, B)$ *for any* $\lambda \in \mathbb{R}$.
2. $d_H(A + C, B + C) = d_H(A, B)$.
3. $d_H(A + B, C + D) \le d_H(A, C) + d_H(B, D)$.
4. $d_H(\lambda A, \mu A) = |\lambda - \mu|\, d_H(A, \{\mathbf{0}\})$ *whenever* $\lambda\mu \ge 0$.

These properties highlight the interplay between the metric and algebraic structures on $\mathcal{K}_C^n$. In particular, they ensure that variations of set-valued functions can be analyzed through metric convergence while remaining consistent with Minkowski operations.

2.3. Continuity and Differentiability

Let $T = (a, b) \subset \mathbb{R}$ be an open interval. All limits involving convergence of sets are understood with respect to the metric d_H.

Definition 15. Let $F : T \to \mathcal{K}_C^n$. The function F is said to be continuous at $t_0 \in T$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that [12]:

$$d_H(F(t), F(t_0)) < \varepsilon \quad \text{whenever} \quad t \in T,\ |t - t_0| < \delta.$$

Having established the notion of continuity in the metric space $(\mathcal{K}_C^n, d_H)$, we now recall differentiability concepts for set-valued functions.

Definition 16. A function $F : T \to \mathcal{K}^n$ is said to be Hukuhara differentiable at $t_0 \in T$ if there exists $F'_H(t_0) \in \mathcal{K}^n$ such that the limits

$$\lim_{h\to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

exist and are equal to $F'_H(t_0)$ [7].

Definition 17. Let $t \in T$ and h be such that $t + h \in T$. A function $F : T \to \mathcal{K}_C^n$ is called generalized Hukuhara differentiable (gH-differentiable) at $t_0 \in T$ if there exists $F'_{gH}(t_0) \in \mathcal{K}_C^n$

such that

$$\lim_{h\to 0} \frac{F(t_0+h)\ominus_{gH} F(t_0)}{h}$$

exists and equals $F'_{gH}(t_0)$ [12].

The previous notions involve limits taken with respect to the metric d_H. However, their formulation still depends on the algebraic existence of Hukuhara-type differences, since the increments are defined through Minkowski subtraction.

Remark on frameworks. The Hukuhara-type derivatives use algebraic set differences to form increments, whereas the d_H-derivative is defined purely through metric approximation in $(\mathcal{K}_C^n, d_H)$.

Definition 18. Let $F : T \to \mathcal{K}_C^n$. We say that F has the Bede–Gal derivative at $t \in T$ if there exists $F'(t) \in \mathcal{K}_C^n$ such that, for sufficiently small $h > 0$, the H-difference exists and one of the following cases [12]:

$$\text{(i)}\quad \lim_{h\to 0^+} d_H\left(\frac{F(t+h)\ominus F(t)}{h}, F'(t)\right) = \lim_{h\to 0^+} d_H\left(\frac{F(t)\ominus F(t-h)}{h}, F'(t)\right) = 0,$$

$$\text{(ii)}\quad \lim_{h\to 0^+} d_H\left(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)\right) = \lim_{h\to 0^+} d_H\left(\frac{F(t-h)\ominus F(t)}{-h}, F'(t)\right) = 0,$$

$$\text{(iii)}\quad \lim_{h\to 0^+} d_H\left(\frac{F(t+h)\ominus F(t)}{h}, F'(t)\right) = \lim_{h\to 0^+} d_H\left(\frac{F(t-h)\ominus F(t)}{-h}, F'(t)\right) = 0,$$

$$\text{(iv)}\quad \lim_{h\to 0^+} d_H\left(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)\right) = \lim_{h\to 0^+} d_H\left(\frac{F(t)\ominus F(t-h)}{h}, F'(t)\right) = 0.$$

If case (i) holds, F is said to be (i)-differentiable at t, and analogous notations are used for the other cases.

Remark 19. In Definition 18, the phrase "for sufficiently small $h > 0$" means that there exists $\delta > 0$ such that the required Hukuhara differences are well-defined for all $h \in (0, \delta)$. The convergence in cases (i)-(iv) is understood in the metric space $(\mathcal{K}_C^n, d_H)$.

Lemma 20. *Let $F : T \to \mathcal{K}_C^n$. If F has the Bede–Gal derivative at $t \in T$ in at least two cases of Definition 18, then $F'(t)$ is a singleton [11].*

It is possible that on different subintervals the function F satisfies the Bede–Gal differentiability of different types.

Example 21. Let $T = (0, 1)$ and define $F : T \to \mathcal{K}_C$ by $F(t) = [t^2, t]$ for each $t \in T$. We first show that $F'(t) = [2t, 1]$ for $t \in \left(0, \frac{1}{2}\right)$ and that F is (i)-differentiable on this interval. Fix $\varepsilon > 0$ and set $\delta := \min\{1 - 2t, \varepsilon\}$. Let h satisfy $0 < h < \delta$. Observe that

$$len(F(t+h)) - len(F(t)) = h - h^2 - 2ht = h(1 - 2t - h) > 0,$$

since $t < \frac{1}{2}$ and $h < 1 - 2t$. Hence the Hukuhara difference $F(t+h) \ominus F(t)$ exists and

$$F(t+h) \ominus F(t) = [2th + h^2,\, h].$$

Similarly,

$$len(F(t)) - len(F(t-h)) = h^2 + h - 2ht = h(h + 1 - 2t) > 0,$$

so $F(t) \ominus F(t-h)$ exists and $F(t) \ominus F(t-h) = [2th - h^2,\, h]$. Therefore,

$$d_H\left(\frac{F(t+h) \ominus F(t)}{h},\, [2t,1]\right) = d_H\left([2t+h,\ 1],\ [2t,\ 1]\right) = \max\{h,0\} = h < \varepsilon,$$

which implies

$$\lim_{h\to 0^+} d_H\left(\frac{F(t+h) \ominus F(t)}{h}, [2t,1]\right) = 0.$$

Moreover,

$$d_H\left(\frac{F(t) \ominus F(t-h)}{h},\, [2t,1]\right) = d_H\left([2t-h,\ 1],\ [2t,\ 1]\right) = \max\{h,0\} = h < \varepsilon,$$

so

$$\lim_{h\to 0^+} d_H\left(\frac{F(t) \ominus F(t-h)}{h}, [2t,1]\right) = 0.$$

Hence F is (i)-differentiable on $\left(0, \frac{1}{2}\right)$ with derivative $F'(t) = [2t, 1]$.

Next, let $t \in \left[\frac{1}{2},\, 1\right)$. We show that $F'(t) = [1, 2t]$ and that F is (ii)-differentiable, but not (i)-differentiable. For $t \geq \frac{1}{2}$,

$$len(F(t+h)) - len(F(t)) = h - h^2 - 2ht < -h^2 < 0,$$

so $F(t+h) \ominus F(t)$ does not exist. Thus F is not (i)-differentiable.

Fix $\varepsilon > 0$ and set $\delta := \min\{2t-1, \varepsilon\}$. Let $0 < h < \delta$. Then $len(F(t)) > len(F(t+h))$ and $len(F(t-h)) > len(F(t))$, so both $F(t) \ominus F(t+h)$ and $F(t-h) \ominus F(t)$ exist. We compute

$$F(t) \ominus F(t+h) = [-2th - h^2, -h], \qquad F(t-h) \ominus F(t) = [h^2 - 2ht, -h].$$

Hence,

$$d_H\left(\frac{F(t) \ominus F(t+h)}{-h},\ [1,2t]\right) = d_H\left([1,\ 2t+h],\ [1,\ 2t]\right) = h < \varepsilon.$$

Likewise,

$$d_H\left(\frac{F(t-h) \ominus F(t)}{-h},\ [1,2t]\right) = d_H\left([1,\ 2t-h],\ [1,\ 2t]\right) = h < \varepsilon.$$

Therefore,

$$\lim_{h\to 0^+} d_H\left(\frac{F(t) \ominus F(t+h)}{-h}, [1,2t]\right) = \lim_{h\to 0^+} d_H\left(\frac{F(t-h) \ominus F(t)}{-h}, [1,2t]\right) = 0$$

and hence F is (ii)-differentiable on $\left[\frac{1}{2}, 1\right)$ with $F'(t) = [1, 2t]$.

The Bede–Gal derivative partially relaxes the strict algebraic requirements of Hukuhara-type differentiability by combining metric convergence with algebraic differences. Nevertheless, it still requires the existence of H-differences for sufficiently small increments.

We now introduce a purely metric notion of differentiability, which is formulated entirely in terms of the Pompeiu–Hausdorff distance and does not depend on the existence of algebraic set differences. Finally, we recall the most general form of metric differentiability, the d_H-derivative.

Definition 22. Let $F : T \to \mathcal{K}_C^n$ [12].

1. F is right d_H-differentiable at $t \in T$ if there exists $F'_+(t) \in \mathcal{K}_C^n$ such that either

$$\lim_{h\to 0^+} \frac{d_H\big(F(t+h),\, F(t) + hF'_+(t)\big)}{h} = 0 \quad \text{or} \quad \lim_{h\to 0^-} \frac{d_H\big(F(t),\, F(t-h) + hF'_+(t)\big)}{h} = 0.$$

2. F is left d_H-differentiable at $t \in T$ if there exists $F'_-(t) \in \mathcal{K}_C^n$ such that either

$$\lim_{h\to 0^-} \frac{d_H\big(F(t+h),\, F(t) + hF'_-(t)\big)}{h} = 0 \quad \text{or} \quad \lim_{h\to 0^+} \frac{d_H\big(F(t),\, F(t-h) + hF'_-(t)\big)}{h} = 0.$$

Definition 23. Let $F : T \to \mathcal{K}_C^n$. We say that F is d_H-differentiable at $t \in T$ if there exists $F'(t) \in \mathcal{K}_C^n$ such that one of the following holds [12]:

$$\text{(i)} \quad \lim_{h\to 0^+} \frac{d_H\big(F(t+h),\, F(t) + hF'(t)\big)}{h} = \lim_{h\to 0^+} \frac{d_H\big(F(t),\, F(t-h) + hF'(t)\big)}{h} = 0,$$

$$\text{(ii)} \quad \lim_{h\to 0^-} \frac{d_H\big(F(t+h),\, F(t) + hF'(t)\big)}{h} = \lim_{h\to 0^-} \frac{d_H\big(F(t),\, F(t-h) + hF'(t)\big)}{h} = 0,$$

$$\text{(iii)} \quad \lim_{h\to 0^+} \frac{d_H\big(F(t+h),\, F(t) + hF'(t)\big)}{h} = \lim_{h\to 0^-} \frac{d_H\big(F(t+h),\, F(t) + hF'(t)\big)}{h} = 0,$$

$$\text{(iv)} \quad \lim_{h\to 0^+} \frac{d_H\big(F(t),\, F(t-h) + hF'(t)\big)}{h} = \lim_{h\to 0^-} \frac{d_H\big(F(t),\, F(t-h) + hF'(t)\big)}{h} = 0.$$

If case (i) holds, F is called (i)-d_H-differentiable at t, and analogous notations are used for the other cases.

The d_H-derivative removes the algebraic requirement that increments be expressed through Hukuhara-type differences. Instead, it characterizes differentiability through metric approximation by affine set-valued mappings.

To illustrate the definition, we consider a homothetic family obtained by scaling a fixed convex set. As expected, its d_H-derivative coincides with the classical derivative of the scaling factor.

Example 24. Let $A \in \mathcal{K}_C^n$ be fixed, let $T \subset \mathbb{R}^+$, and let $m \neq 0$. Define $F : T \to \mathcal{K}_C^n$ by $F(t) = t^m A$. We show that F is d_H-differentiable with $F'(t) = mt^{m-1}A$. If $A = \{\mathbf{0}\}$ the statement is trivial, so assume $A \neq \{\mathbf{0}\}$ and denote $c := d_H(A, \{\mathbf{0}\}) > 0$.
We verify case (i) in Definition 23. Let $\varepsilon > 0$. Since $f(t) = t^m$ is differentiable with $f'(t) = mt^{m-1}$, there exists $\delta' > 0$ such that for all h with $0 < h < \delta'$ we have

$$\left| \frac{(t+h)^m - t^m}{h} - mt^{m-1} \right| < \frac{\varepsilon}{c} \implies \frac{|(t+h)^m - t^m - hmt^{m-1}|}{h} < \frac{\varepsilon}{c}.$$

Now choose $\delta := \min\left\{\delta', \frac{t}{|m|}\right\}$. For $0 < h < \delta$ we have $(t+h)^m > 0$ and $t^m + hmt^{m-1} > 0$, hence by Lemma 14,

$$\frac{d_H((t+h)^m A, t^m A + hmt^{m-1}A)}{h} = \frac{|(t+h)^m - t^m - hmt^{m-1}|}{h} d_H(A, \{\mathbf{0}\}) < \frac{\varepsilon}{c} \cdot c = \varepsilon.$$

Thus

$$\lim_{h\to 0^+} \frac{d_H((t+h)^m A, t^m A + hmt^{m-1}A)}{h} = 0.$$

Similarly, since $f(t) = t^m$ is differentiable, there exists $\delta'' > 0$ such that for all $0 < h < \delta''$

$$\frac{|t^m - (t-h)^m - hmt^{m-1}|}{h} < \frac{\varepsilon}{c}.$$

Now choose $\delta := \min\left\{\delta'', \frac{t}{|m|}, t\right\}$. For $0 < h < \delta$ we have $t - h > 0$, hence $(t-h)^m > 0$. We claim that $(t-h)^m + hmt^{m-1} > 0$ for $0 < h < \delta$. Indeed, if $m > 0$ then $hmt^{m-1} > 0$ and the claim is immediate. If $m < 0$, write $m = -k$ with $k > 0$. Since $h < \frac{t}{k}$, we obtain

$$(t-h)^m + hmt^{m-1} = (t-h)^{-k} - hk\,t^{-k-1} > t^{-k} - hk\,t^{-k-1} = \frac{t - hk}{t^{k+1}} > 0.$$

Therefore, $(t-h)^m$ and $(t-h)^m + hmt^{m-1}$ have the same sign (both positive), and by Lemma 14,

$$\frac{d_H(t^m A, (t-h)^m A + hmt^{m-1}A)}{h} = \frac{|t^m - (t-h)^m - hmt^{m-1}|}{h} d_H(A, \{\mathbf{0}\}) < \frac{\varepsilon}{c} \cdot c = \varepsilon,$$

which proves that

$$\lim_{h \to 0^+} \frac{d_H(t^m A, (t-h)^m A + hmt^{m-1}A)}{h} = 0.$$

Therefore, F is d_H-differentiable with derivative $F'(t) = mt^{m-1}A$.

The next result provides a convenient closed-form for the Pompeiu–Hausdorff distance between two intervals in $\mathbb{R}$, which we will repeatedly use for interval-valued computations.

Lemma 25. *Let $[a,b]$ and $[c,d]$ be two closed intervals in $\mathbb{R}$. Then*

$$d_H([a,b],[c,d]) = \max\{|a-c|,\ |b-d|\}.$$

Proof. Using the fact that the gH-difference of intervals always exists and equals

$$[a,b] \ominus_{gH} [c,d] = [\min\{a-c, b-d\},\ \max\{a-c, b-d\}],$$

Let $m = \min\{a-c, b-d\}$ and $M = \max\{a-c, b-d\}$. Then $[a,b] = [c,d] + [m,M]$ or $[c,d] = [a,b] + (-1)[m,M]$, hence by Lemma 14,

$$d_H([m,M],\{0\}) = d_H([m,M] + [c,d], \{0\} + [c,d]) = d_H([a,b],[c,d]),$$

or

$$d_H([m,M],\{0\}) = d_H((-1)[m,M],(-1)\{0\}) = d_H([c,d]+(-1)[m,M],[c,d]+\{0\}) = d_H([a,b],[c,d]).$$

Since

$$d_H([m,M],\{0\}) = \max\left\{\sup_{x\in[m,M]} |x|,\ \inf_{x\in[m,M]} |x|\right\} = \sup_{x\in[m,M]} |x| = \max\{|m|,|M|\},$$

we obtain $d_H([a,b],[c,d]) = \max\{|a-c|,|b-d|\}$. □

It is possible for a set-valued mapping to be d_H-differentiable with different "types" on different subintervals. The following example exhibits this behavior for an interval-valued map.

Example 26. Let $T = (0,1)$ and define $F : T \to \mathcal{K}_C$ by $F(t) = [t^2, t]$ for each $t \in T$. We claim that F is (i)-d_H-differentiable on $t \in \left(0, \frac{1}{2}\right]$ with $F'(t) = [2t, 1]$. Indeed,

$$\begin{aligned}
\lim_{h \to 0^+} \frac{d_H\big(F(t+h),\ F(t) + h[2t,1]\big)}{h} &= \lim_{h \to 0^+} \frac{d_H\big([(t+h)^2, t+h],\ [t^2, t] + [2th, h]\big)}{h} \\
&= \lim_{h \to 0^+} \frac{\max\{|(t+h)^2 - t^2 - 2ht|,\ |(t+h) - (t+h)|\}}{h} \\
&= \lim_{h \to 0^+} \frac{h^2}{h} = 0,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\lim_{h \to 0^+} \frac{d_H\big(F(t),\ F(t-h) + h[2t,1]\big)}{h} &= \lim_{h \to 0^+} \frac{d_H\big([t^2, t],\ [(t-h)^2, t-h] + [2th, h]\big)}{h} \\
&= \lim_{h \to 0^+} \frac{\max\{|t^2 - (t-h)^2 - 2th|,\ |t - (t-h) - h|\}}{h} \\
&= \lim_{h \to 0^+} \frac{h^2}{h} = 0.
\end{aligned}$$

Thus F is (i)-d_H-differentiable on $\left(0, \frac{1}{2}\right]$ with $F'(t) = [2t, 1]$.
We show that F is not (i)-d_H-differentiable on $\left(\frac{1}{2}, 1\right)$. Suppose there exists $F'(t) = [p, q] \in \mathcal{K}_C$ such that case (i) holds for $t \in \left(\frac{1}{2}, 1\right)$. Then

$$\begin{aligned}
\lim_{h \to 0^+} \frac{d_H\big(F(t+h),\ F(t) + h[p,q]\big)}{h} &= \lim_{h \to 0^+} \frac{\max\{|(t+h)^2 - t^2 - hp|,\ |(t+h) - (t+hq)|\}}{h} \\
&= \max\{|2t - p|,\ |1 - q|\},
\end{aligned}$$

forcing $p = 2t$ and $q = 1$. This contradicts the order requirement $p \le q$ for the interval $[p, q]$ when $t > \frac{1}{2}$. Hence, case (i) cannot hold there.
Finally, we prove that F is (ii)-d_H-differentiable on $t \in \left(\frac{1}{2}, 1\right)$ with $F'(t) = [1, 2t]$. We have

$$\begin{aligned}
\lim_{h \to 0^-} \frac{d_H\big(F(t+h),\ F(t) + h[1,2t]\big)}{h} &= \lim_{h \to 0^-} \frac{d_H\big([(t+h)^2, t+h],\ [t^2, t] + [2ht, h]\big)}{h} \\
&= \lim_{h \to 0^-} \frac{\max\{|(t+h)^2 - t^2 - 2ht|,\ |t + h - t - h|\}}{h} \\
&= \lim_{h \to 0^-} \frac{h^2}{h} = 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{h \to 0^-} \frac{d_H\big(F(t),\ F(t-h) + h[1,2t]\big)}{h} &= \lim_{h \to 0^-} \frac{d_H\big([t^2, t],\ [(t-h)^2, t-h] + [2th, h]\big)}{h} \\
&= \lim_{h \to 0^-} \frac{\max\{|t^2 - (t-h)^2 - 2th|,\ |t - (t-h) - h|\}}{h} \\
&= \lim_{h \to 0^-} \frac{h^2}{h} = 0.
\end{aligned}$$

Therefore,

$$F'(t) = \begin{cases} [2t,\ 1], & 0 < t \le \frac{1}{2}, \\ [1,\ 2t], & \frac{1}{2} < t < 1. \end{cases}$$

3. Results and Discussion

Proposition 11 shows that the generalized Hukuhara difference of two intervals is always well defined and depends on the relative lengths of the intervals. However, Example 6 demonstrates that in higher dimensions the existence of the gH-difference cannot be guaranteed in general.

Motivated by these observations, we extend the one-dimensional result to Cartesian products of intervals and derive a necessary and sufficient condition for the existence of the gH-difference in $\mathbb{R}^n$.

Proposition 27 (Extension of [9]). *Let $X_i = [a_i, b_i] \subset \mathbb{R}$ and $Y_i = [c_i, d_i] \subset \mathbb{R}$ for each $i = 1, 2, \ldots, n$. Define $\mathbf{X} = X_1 \times X_2 \times \cdots \times X_n$ and $\mathbf{Y} = Y_1 \times Y_2 \times \cdots \times Y_n$. Then the gH-difference $\mathbf{X} \ominus_{gH} \mathbf{Y}$ exists if and only if*

$$len(X_i) \geq len(Y_i) \text{ for all } i, \quad \text{or} \quad len(X_i) \leq len(Y_i) \text{ for all } i.$$

Moreover, if $\mathbf{X} \ominus_{gH} \mathbf{Y}$ exists, then

$$\mathbf{X} \ominus_{gH} \mathbf{Y} = [m_1, M_1] \times [m_2, M_2] \times \cdots \times [m_n, M_n],$$

where $m_i = \min\{a_i - c_i, b_i - d_i\}$ and $M_i = \max\{a_i - c_i, b_i - d_i\}$.

Proof. The intuition behind the first statement follows from Example 6. Assume that $\mathbf{X} \ominus_{gH} \mathbf{Y}$ exists, and denote $\mathbf{X} \ominus_{gH} \mathbf{Y} = \mathbf{M}$. This implies that either $\mathbf{X} = \mathbf{Y} + \mathbf{M}$ or $\mathbf{Y} = \mathbf{X} + (-1)\mathbf{M}$. Let $\mathbf{m} \in \mathbf{M}$; then we must have either $\mathbf{Y} + \{\mathbf{m}\} \subset \mathbf{X}$ or $\mathbf{X} + \{-\mathbf{m}\} \subset \mathbf{Y}$. Writing $\mathbf{m} = (z_1, z_2, \ldots, z_n)$, for each $(x_1, \ldots, x_n) \in \mathbf{X}$ and $(y_1, \ldots, y_n) \in \mathbf{Y}$ we obtain

$$a_i \leq y_i + z_i \leq b_i \quad \text{or} \quad c_i \leq x_i - z_i \leq d_i, \qquad \forall i = 1, 2, \ldots, n.$$

Suppose that $a_i \leq y_i + z_i \leq b_i$ for all i. If $(y_1, \ldots, y_n) = (c_1, \ldots, c_n)$, then $a_i - c_i \leq z_i \leq b_i - c_i$, and if $(y_1, \ldots, y_n) = (d_1, \ldots, d_n)$, then $a_i - d_i \leq z_i \leq b_i - d_i$. Combining both inequalities gives $a_i - c_i \leq b_i - d_i$, i.e., $len(X_i) \geq len(Y_i)$. Similarly, assuming $c_i \leq x_i - z_i \leq d_i$ for all i leads to $b_i - d_i \leq a_i - c_i$, hence $len(X_i) \leq len(Y_i)$.
Conversely, let $\mathbf{N} = [m_1, M_1] \times \cdots \times [m_n, M_n]$. We show that $\mathbf{X} \ominus_{gH} \mathbf{Y} = \mathbf{N}$. If $len(X_i) \geq len(Y_i)$ for all i, then

$$b_i - a_i \geq d_i - c_i \implies b_i - d_i \geq a_i - c_i,$$

and hence

$$X_i = [a_i, b_i] = [c_i, d_i] + [a_i - c_i, b_i - d_i] = Y_i + [m_i, M_i].$$

Therefore, $\mathbf{X} = \mathbf{Y} + \mathbf{N}$. Similarly, if $len(X_i) \leq len(Y_i)$ for all i, one can analogously prove that $\mathbf{X} + (-1)\mathbf{N} = \mathbf{Y}$. □

Corollary 28. *Let $X_i = [a_i, b_i] \subset \mathbb{R}$ and $Y_i = [c_i, d_i] \subset \mathbb{R}$ for each $i = 1, 2, \ldots, n$. Define $\mathbf{X} = X_1 \times X_2 \times \cdots \times X_n$ and $\mathbf{Y} = Y_1 \times Y_2 \times \cdots \times Y_n$. Then the H-difference $\mathbf{X} \ominus \mathbf{Y}$ exists if and only if $len(X_i) \geq len(Y_i)$ for all i. Moreover, if $\mathbf{X} \ominus \mathbf{Y}$ exists, then*

$$\mathbf{X} \ominus \mathbf{Y} = [a_1 - c_1, b_1 - d_1] \times [a_2 - c_2, b_2 - d_2] \times \cdots \times [a_n - c_n, b_n - d_n],$$

Proposition 29. *Let $F : T \to \mathcal{K}_C^n$ be a set-valued function. If F is H-differentiable on T, then F is also gH-differentiable on T. Moreover, whenever the Hukuhara derivative $F'_H(t)$*

exists, it coincides with the gH-derivative, that is,

$$F'_H(t) = F'_{gH}(t) \quad \text{for all } t \in T.$$

Proof. Let $t \in T$. By the definition of the Hukuhara derivative,

$$F'_H(t) = \lim_{h\to 0^+} \frac{F(t+h) \ominus F(t)}{h} = \lim_{h\to 0^+} \frac{F(t) \ominus F(t-h)}{h}.$$

Let $\varepsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$F(t+h) \ominus F(t) \text{ exists} \quad \text{and} \quad d_H\left(\frac{F(t+h) \ominus F(t)}{h}, F'_H(t)\right) < \varepsilon, \quad \text{for every } h \text{ with } 0 < h < \delta_1,$$
$$F(t) \ominus F(t-h) \text{ exists} \quad \text{and} \quad d_H\left(\frac{F(t) \ominus F(t-h)}{h}, F'_H(t)\right) < \varepsilon, \quad \text{for every } h \text{ with } 0 < h < \delta_2.$$

Choose $\delta := \min\{\delta_1, \delta_2\}$. According to Proposition 7, the Hukuhara and generalized Hukuhara differences coincide whenever the former exists. Hence,

$$F(t+h) \ominus F(t) = F(t+h) \ominus_{gH} F(t) \quad \text{and} \quad F(t) \ominus F(t-h) = F(t) \ominus_{gH} F(t-h)$$

for every $0 < h < \delta$. Therefore,

$$d_H\left(\frac{F(t+h) \ominus_{gH} F(t)}{h}, F'_H(t)\right) < \varepsilon \quad \text{and} \quad d_H\left(\frac{F(t) \ominus_{gH} F(t-h)}{h}, F'_H(t)\right) < \varepsilon.$$

This shows that $F'_H(t) = F'_{gH}(t)$, as desired. □

The converse of Proposition 29 does not hold in general, as shown by the following counterexample.

Example 30. Let $T = (-1, 0)$ and define $F : T \to \mathcal{K}_C$ by $F(x) = [2x, x]$ for each $x \in T$. We show that F is gH-differentiable but not Hukuhara differentiable. Take h such that $|h| < -x$. For $h > 0$ we have

$$[2x, x] = [2x + 2h, x + h] + (-1)[h, 2h] \iff [2x + 2h, x + h] \ominus_{gH} [2x, x] = [h, 2h].$$

Thus,

$$\frac{F(x+h) \ominus_{gH} F(x)}{h} = \frac{[h, 2h]}{h} = [1, 2].$$

For $h < 0$,

$$[2x + 2h, x + h] = [2x, x] + [2h, h] \iff [2x + 2h, x + h] \ominus_{gH} [2x, x] = [2h, h],$$

which gives

$$\frac{F(x+h) \ominus_{gH} F(x)}{h} = \frac{[2h, h]}{h} = [1, 2].$$

Hence,

$$F'_{gH}(x) = \lim_{h\to 0} \frac{F(x+h) \ominus_{gH} F(x)}{h} = [1, 2] \quad \forall x \in T.$$

Next, we show that F is not Hukuhara differentiable. Suppose that for some $h > 0$ the Hukuhara difference $F(x + h) \ominus F(x)$ exists and equals $A \in \mathcal{K}_C$. Then

$$F(x+h) \ominus F(x) = A \iff [2x + 2h, x + h] = [2x, x] + A.$$

Since A is compact, let its minimal and maximal elements be a and b, respectively. Then the minimal and maximal elements of $[2x, x] + A$ are $2x + a$ and $x + b$. To satisfy the equality above, we must have $2h = a$ and $b = h$, which contradicts $a \leq b$. Therefore, $F(x+h) \ominus F(x)$ does not exist for any $h > 0$, and F is not Hukuhara differentiable.

Theorem 31. *If $F : T \to \mathcal{K}_C^n$ is Bede–Gal differentiable at $t_0 \in T$, then F is continuous at t_0.*

Proof. Let F satisfy the (i)-differentiability condition at $t_0 \in T$; the remaining cases can be proved analogously. Then

$$\lim_{h \to 0^+} d_H\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, F'(t_0)\right) = \lim_{h \to 0^+} d_H\left(\frac{F(t_0) \ominus F(t_0-h)}{h}, F'(t_0)\right) = 0.$$

Let $\varepsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} &F(t_0+h) \ominus F(t_0) \text{ exists} \quad \text{and} \quad d_H\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, F'(t_0)\right) < \varepsilon \quad \text{for all } 0 < h < \delta_1, \\ &F(t_0) \ominus F(t_0-h) \text{ exists} \quad \text{and} \quad d_H\left(\frac{F(t_0) \ominus F(t_0-h)}{h}, F'(t_0)\right) < \varepsilon \quad \text{for all } 0 < h < \delta_2. \end{aligned}$$

Its clear $d_H(F(t_0+h), F(t_0)) = 0 < \varepsilon$ for $h = 0$. Observe that for $h > 0$,

$$\begin{aligned} d_H(F(t_0+h), F(t_0)) &= d_H([F(t_0+h) \ominus F(t_0)] + F(t_0), F(t_0)) \\ &= d_H(F(t_0+h) \ominus F(t_0), \{0\}) \\ &= h\, d_H\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, \{0\}\right) \\ &\leq h\left[d_H\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, F'(t_0)\right) + d_H(F'(t_0), \{0\})\right]. \end{aligned}$$

Let $C_\varepsilon := \varepsilon + d_H(F'(t_0), \{0\})$. Choose $\delta := \min\left\{\delta_1, \delta_2, \frac{\varepsilon}{C_\varepsilon}\right\}$. Then for $0 < h < \delta$ we obtain

$$d_H(F(t_0+h), F(t_0)) < h(\varepsilon + d_H(F'(t_0), \{0\})) \leq \frac{\varepsilon}{C_\varepsilon} \cdot C_\varepsilon = \varepsilon.$$

A similar argument shows that $d_H(F(t_0-h), F(t_0)) < \varepsilon$ for all $0 < h < \delta$. Therefore, $d_H(F(t), F(t_0)) < \varepsilon$ for all t such that $|t - t_0| < \delta$ and hence F is continuous at t_0. □

Remark 32. Theorem 31 was originally established by Amrahov *et al.* [11] using the inclusion-based definition of the Hausdorff distance. Here we provide an equivalent proof formulated under the analytic definition of the Pompeiu–Hausdorff metric, which simplifies the argument and clarifies the role of metric properties.

Theorem 33. *Let $f, g : T \to \mathbb{R}$ be differentiable real-valued functions such that $f(t) \leq g(t)$ for every $t \in T$. Suppose $t_0 \in T$ and there exists a neighborhood $I = (t_0 - \delta, t_0 + \delta) \subset T$ such that*

$$f'(t) \leq g'(t) \quad \text{for all } t \in I, \qquad \text{or} \qquad f'(t) \geq g'(t) \quad \text{for all } t \in I. \tag{1}$$

Then $F(t) = [f(t), g(t)]$ is Bede–Gal differentiable at t_0 and

$$F'(t_0) = [\min\{f'(t_0), g'(t_0)\},\ \max\{f'(t_0), g'(t_0)\}].$$

Proof. Assume first that $f'(t) \le g'(t)$ for all $t \in I$. The other case can be proved analogously. We show that F is (i)-differentiable at t_0. Let $\ell(t) = g(t) - f(t)$. Then

$$\ell'(t) = g'(t) - f'(t) \ge 0 \quad \text{for } t \in I,$$

so ℓ is nondecreasing on I. Hence for $0 < h < \delta$, $\ell(t_0 - h) \le \ell(t_0) \le \ell(t_0 + h)$ which implies

$$f(t_0) - f(t_0 - h) \le g(t_0) - g(t_0 - h), \qquad f(t_0 + h) - f(t_0) \le g(t_0 + h) - g(t_0).$$

Therefore the Hukuhara differences exist and $F(t_0 + h) \ominus F(t_0) = [f(t_0 + h) - f(t_0),\ g(t_0 + h) - g(t_0)]$. Hence

$$d_H\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, [f'(t_0), g'(t_0)]\right) = d_H\left(\left[\frac{f(t_0+h) - f(t_0)}{h}, \frac{g(t_0+h) - g(t_0)}{h}\right], [f'(t_0), g'(t_0)]\right)$$
$$= \max\left\{\left|\frac{f(t_0+h) - f(t_0)}{h} - f'(t_0)\right|, \left|\frac{g(t_0+h) - g(t_0)}{h} - g'(t_0)\right|\right\}.$$

Let $\varepsilon > 0$. Since f and g are differentiable at t_0, there exist $\delta_1, \delta_2 > 0$ such that

$$\left|\frac{f(t_0+h) - f(t_0)}{h} - f'(t_0)\right| < \varepsilon \quad \text{for } 0 < h < \delta_1,$$

and

$$\left|\frac{g(t_0+h) - g(t_0)}{h} - g'(t_0)\right| < \varepsilon \quad \text{for } 0 < h < \delta_2.$$

Let $\delta' = \min\{\delta_1, \delta_2, \delta\}$. Then for $0 < h < \delta'$ we obtain

$$d_H\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, [f'(t_0), g'(t_0)]\right) < \varepsilon.$$

Thus

$$\lim_{h \to 0^+} d_H\left(\frac{F(t_0+h) \ominus F(t_0)}{h}, [f'(t_0), g'(t_0)]\right) = 0.$$

A similar argument yields

$$\lim_{h \to 0^+} d_H\left(\frac{F(t_0) \ominus F(t_0-h)}{h}, [f'(t_0), g'(t_0)]\right) = 0,$$

so F satisfies the (i)-differentiability condition at t_0. □

Remark 34. The condition in Eq. (1) guarantees the existence of the relevant Hukuhara differences. Example ex:bede–gal does not satisfy Eq. (1) at $t_0 = \frac{1}{2}$, since $f'(t) < 1$ for $t < \frac{1}{2}$ and $f'(t) > 1$ for $t > \frac{1}{2}$. Hence there is no $\delta > 0$ such that either $f'(t) \le g'(t)$ or $f'(t) \ge g'(t)$ holds for all $t \in (t_0 - \delta, t_0 + \delta)$.

After establishing the relationships among the Hukuhara, generalized Hukuhara, and Bede–Gal derivatives, we now turn to the most general notion of differentiability considered in [12], namely the d_H-derivative. Unlike the previous concepts, the d_H-derivative is defined purely in terms of the Pompeiu–Hausdorff metric and does not depend on the algebraic structure of set differences. A fundamental property of this derivative is that differentiability implies continuity.

Theorem 35. *If $F : T \to \mathcal{K}_C^n$ is d_H-differentiable at $t \in T$, then F is continuous at t [12].*

The converse of Theorem 35 does not hold in general; that is, continuity with respect to the Pompeiu–Hausdorff metric does not necessarily imply d_H-differentiability. The following example illustrates this phenomenon.

Example 36. Let $T = (-1, 1)$ and define $F : T \to \mathcal{K}_C$ by $F(t) = [t, |2t|]$ for all $t \in T$. We first show that F is continuous at $t = 0$. Let $\varepsilon > 0$. For any $t \in T$ with $|t| < \frac{\varepsilon}{2}$, we have

$$d_H(F(t), F(0)) = d_H([t, |2t|], \{0\}) = \max\{|t|, |2t|\} = 2|t| < \varepsilon,$$

which proves that F is continuous at $t = 0$.
Next, we show that F is not d_H-differentiable at $t = 0$. It suffices to consider case (i) in Definition 23. Assume, for contradiction, that $F'(0)$ exists. Since $F'(0) \in \mathcal{K}_C$, there exist real numbers $p \leq q$ such that $F'(0) = [p, q]$.
From the first limit in case (i), we obtain

$$\begin{aligned}
0 &= \lim_{h \to 0^+} \frac{d_H(F(0),\, F(h) + hF'(0))}{h} \\
&= \lim_{h \to 0^+} \frac{d_H(\{0\},\, [h, 2h] + h[p, q])}{h} \\
&= \lim_{h \to 0^+} \frac{d_H(\{0\},\, [h + hp,\, 2h + hq])}{h} \\
&= \lim_{h \to 0^+} \frac{\max\{|h(1 + p)|,\, |h(2 + q)|\}}{h} \\
&= \max\{|1 + p|,\, |2 + q|\}.
\end{aligned}$$

Hence $1 + p = 0$ and $2 + q = 0$, so $p = -1$ and $q = -2$, which contradicts $p \leq q$. Therefore, F is continuous but not d_H-differentiable at $t = 0$.

Motivated by Example 26, we obtain the following result.

Theorem 37. *Let $f, g : T \to \mathbb{R}$ be real-valued functions such that $f(t) \leq g(t)$ for all $t \in T$, and define the interval-valued function $F : T \to \mathcal{K}_C$ by $F(t) = [f(t), g(t)]$. If both f and g are differentiable on T, then F is d_H-differentiable and*

$$F'(t) = [\min\{f'(t), g'(t)\},\, \max\{f'(t), g'(t)\}].$$

Proof. We consider two cases. We will show that if $f'(t) \leq g'(t)$, then F is (i)-d_H-differentiable, whereas if $f'(t) \geq g'(t)$, then F is (ii)-d_H-differentiable. We prove case (i); case (ii) can be shown analogously.
Assume $f'(t) \leq g'(t)$ and set $F'(t) = [f'(t), g'(t)]$. Then

$$\begin{aligned}
\frac{d_H(F(t + h),\, F(t) + hF'(t))}{h} &= \frac{d_H([f(t + h), g(t + h)],\, [f(t) + hf'(t),\, g(t) + hg'(t)])}{h} \\
&= \frac{\max\{|f(t + h) - f(t) - hf'(t)|,\, |g(t + h) - g(t) - hg'(t)|\}}{h}.
\end{aligned}$$

Let $\varepsilon > 0$. Since f and g are differentiable at t, there exist $\delta_1, \delta_2 > 0$ such that

$$\frac{|f(t+h) - f(t) - hf'(t)|}{h} < \varepsilon \quad \text{for } 0 < h < \delta_1,$$
$$\frac{|g(t+h) - g(t) - hg'(t)|}{h} < \varepsilon \quad \text{for } 0 < h < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < h < \delta$ we obtain

$$\frac{d_H(F(t+h),\, F(t) + hF'(t))}{h} < \varepsilon.$$

Hence

$$\lim_{h \to 0^+} \frac{d_H(F(t+h),\, F(t) + hF'(t))}{h} = 0.$$

Similarly,

$$\lim_{h \to 0^+} \frac{d_H(F(t),\, F(t-h) + hF'(t))}{h} = 0,$$

which proves (i)-d_H-differentiability. ☐

The assumption $f(t) \leq g(t)$ in Theorem 37 is essential. Even if we redefine $F(t) = [\min\{f(t), g(t)\},\, \max\{f(t), g(t)\}]$, the function F may still fail to be d_H-differentiable, as shown below.

Example 38. Let $T = (1, 3)$ and define $f, g : T \to \mathbb{R}$ by $f(t) = t$ and $g(t) = \frac{t}{2} + 1$. Define $F : T \to \mathcal{K}_C$ by

$$F(t) = [\min\{f(t), g(t)\}, \max\{f(t), g(t)\}] = \begin{cases} \left[t, \frac{t}{2} + 1\right], & 1 \leq t \leq 2, \\ \left[\frac{t}{2} + 1, t\right], & 2 < t < 3. \end{cases}$$

We show that F is not d_H-differentiable at $t = 2$. As an example, we verify that F fails to satisfy case (i) of the definition. Assume that there exists $F'(2) \in \mathcal{K}_C$, say $F'(2) = [p, q]$. Then,

$$\begin{aligned} \lim_{h \to 0^+} \frac{d_H(F(2+h), F(2) + h[p,q])}{h} &= \lim_{h \to 0^+} \frac{d_H\left(\left[2 + h, \frac{h}{2} + 2\right], [2 + hp, 2 + hq]\right)}{h} \\ &= \lim_{h \to 0^+} \frac{\max\{|h - hp|, |\frac{h}{2} - hq|\}}{h} \\ &= \max\{|1 - p|, |\frac{1}{2} - q|\}. \end{aligned}$$

For the above limit to be zero, we must have $p = 1$ and $q = \frac{1}{2}$, which contradicts the requirement $p \leq q$. Hence, F is not d_H-differentiable at $t = 2$.

4. Conclusion

This paper compared the Bede–Gal and dH differentiability concepts for set-valued functions under the Pompeiu–Hausdorff metric. We clarified their relationships with the Hukuhara and generalized Hukuhara derivatives and identified conditions ensuring their existence. The results show that while Bede–Gal differentiability extends classical Hukuhara-type ideas, the dH-derivative offers a broader and more flexible metric-based formulation that remains applicable even when algebraic set differences fail. Illustrative examples highlight the distinct behaviors of the two derivatives. Future research may explore extensions to nonconvex sets and applications to set differential equations.

CRediT Authorship Contribution Statement

Wildan Bagus Wicaksono: Conceptualization, Methodology, Formal Analysis, Investigation, Writing Original Draft, Visualization. **Mohamad Muslikh:** Supervision, Validation, Review & Editing, Project Administration, Corresponding Author.

Declaration of Generative AI and AI-assisted Technologies

The authors acknowledge the use of generative AI-assisted tools in preparing this manuscript. ChatGPT (OpenAI, GPT-5.1) was used solely to assist with English translation and language refinement. All mathematical ideas, results, and analyses were fully developed by the authors, who verified the accuracy and suitability of all AI-assisted outputs.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

This study is theoretical in nature and does not involve any empirical data or computational code. All results, proofs, and analyses were developed analytically by the authors. Therefore, no datasets or code were generated or used in the preparation of this manuscript.

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