



# $\beta$ -Duality of the Sequence Space $\ell_p^\lambda$ and a Geometric Realization via Wulff Shapes

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## Abstract

In the present paper, we investigate the structural and geometric properties of the non-absolute type sequence space  $\ell_p^\lambda$ . Our primary novel contribution lies in demonstrating that the natural norm on its  $\beta$ -dual coincides with the dual norm of  $\ell_p^\lambda$ . We then investigate geometric properties of the space and prove that  $\ell_p^\lambda$  inherits uniform convexity and uniform smoothness from the classical space  $\ell_p$  for  $1 < p < \infty$ , while these properties fail in the case  $p = 1$ . Finally, by considering finite-dimensional truncations, we interpret the dual norm geometrically through the Wulff construction. This yields a visualization of the norm unit sphere as a Wulff shape and shows that geometric features of the resulting shape reflect the convexity and smoothness properties of the underlying sequence space.

**Keywords:**  $\beta$ -dual, Dual norm, Wulff shape, Sequence space, Geometric properties.

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## 1. Introduction

The theory of sequence spaces has developed into an important branch of functional analysis due to its close connection with summability theory, operator theory, and matrix transformations. In particular, the construction of new sequence spaces via generalized difference operators or matrix domains has attracted considerable attention in recent years. Such constructions often extend classical spaces while preserving essential functional-analytic structures, allowing deeper investigations into duality, compactness, and geometric properties. Several studies have focused on sequence spaces generated by weighted means or triangular matrices, leading to various generalizations of classical spaces such as  $\ell_p$ ,  $c$ , and  $c_0$ . See [1–7] for more information.

Among these constructions, Mursaleen and Noman introduced the non-absolute type sequence space  $\ell_p^\lambda$  generated by a triangle matrix determined by a strictly increasing sequence  $\lambda = (\lambda_k)$  of positive real numbers [8, 9]. Their work established several fundamental structural results, including the characterization of its  $\beta$ -dual and basic topological properties. More recently, further developments have connected such sequence spaces with geometric constructions arising from convex geometry and crystallography. In particular, Malkowsky et al. [10, 11] demonstrated that dual norms arising from sequence spaces can be interpreted geometrically through the Wulff's construction, which describes equilibrium crystal shapes determined by surface energy function.

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Motivated by these developments, the present paper aims to analyze the dual norm of the space  $\ell_p^\lambda$  and to investigate additional geometric aspects of the space  $\ell_p^\lambda$ . Our study can be viewed as a natural continuation of the work of Mursaleen and Noman, where we analyze the geometric structure of the space and how it connects to the geometric structure of the resulting Wulff shapes. In particular, we first examine several geometric properties of  $\ell_p^\lambda$ , including uniform convexity and uniform smoothness, and show that these properties are inherited from the classical space  $\ell_p$ . Furthermore, by employing the  $\beta$ -dual norm obtained for  $\ell_p^\lambda$ , we demonstrate that this dual norm naturally induces a Wulff shape in finite-dimensional truncations of the space. This observation allows us to interpret the  $\beta$ -dual norm as a geometric object whose boundary describes the equilibrium crystal corresponding to the norm generated by  $\ell_p^\lambda$ .

## 2. Preliminaries

Let  $\omega$  denote the vector space of all real sequences. If  $x \in \omega$ , then we simply write  $x = (x_k)$  instead of  $x = (x_k)_{k=0}^\infty$ . We also let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ . Any vector subspace of  $\omega$  is called sequence space. The sequence spaces of all bounded, convergent, and null sequences will be denoted by  $\ell_\infty$ ,  $c$ , and  $c_0$ , respectively. Further, the space of absolutely  $p$ -summable sequences for  $1 \leq p < \infty$  is denoted by  $\ell_p$ , that is

$$\ell_p := \left\{ x = (x_k) \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty \right\}.$$

Moreover, we write  $cs$  for the sequence space of all convergent series, that is

$$cs := \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{R} \text{ such that } \lim_{n \rightarrow \infty} \left| \sum_{k=0}^n x_k - l \right| = 0 \right\}.$$

We denote the closed unit ball of a space  $X$  by  $B(X)$  and its corresponding unit sphere by  $S(X)$ .

A sequence space  $X$  is said to be a  $BK$ -space, if it is a Banach space, that is a complete normed space, with continuous coordinates projection  $P_k : X \rightarrow \mathbb{R}$  for  $k \in \mathbb{N}_0$ , where  $\mathbb{R}$  denotes the real field and  $P_k(x) = x_k$  for all  $x = (x_k) \in X$  and every  $k \in \mathbb{N}_0$ . A  $BK$ -space  $X \supset \phi$  of the set of all finitely nonzero sequences is said to have the  $AB$  property if every sequence  $(x^{[n]})$ , where  $x^{[n]}$  denote the  $n$ -th section of a sequence  $x = (x_k)$ , is bounded set in  $X$  for every  $x \in X$ . Closely related to this is the concept of a monotone norm. A sequence space has a monotone norm if  $\|x^{[r]}\| \geq \|x^{[s]}\|$  holds for all  $r > s$  and  $\|x\| = \sup \|x^{[r]}\|$ . It is a standard result that any sequence space equipped with a monotone norm automatically possesses the  $AB$  property.

For a sequence space  $X$ , the  $\beta$ -dual of  $X$  is defined by

$$X^\beta := \left\{ a = (a_k) \in \omega : \sum_{k=0}^\infty a_k x_k \text{ converges for all } x \in X \right\}.$$

Further, let  $A = (a_{nk})_{n,k=0}^\infty$  be an infinite matrix of real numbers. Also, we write  $A_n$  for the sequence in the  $n$ -th row of  $A$ , that is  $A_n = (a_{nk})_{k=0}^\infty$  for every  $n \in \mathbb{N}_0$ . Moreover, if  $x = (x_k) \in \omega$  then we define the  $A$ -transform of  $x$  as the sequence  $Ax = ((Ax)_n)_{n=0}^\infty$ , where

$$(Ax)_n = \sum_{k=0}^\infty a_{nk} x_k \tag{1}$$

provided the series on the right hand side of Eq. (1) converges for each  $n \in \mathbb{N}_0$ . Furthermore, the sequence  $x$  is said to be  $A$ -summable to  $a \in \mathbb{R}$  if  $Ax$  converges to  $a$  which is called the  $A$ -limit of  $x$ . An infinite matrix  $A$  is classified as a triangle matrix if  $a_{nk} = 0$  for all  $k, n \in \mathbb{N}_0$  with  $k > n$  and  $a_{nn} \neq 0$  for all  $n \in \mathbb{N}_0$ .

For an arbitrary sequence space  $X$ , the matrix domain of an infinite matrix  $A$  in  $X$  is defined as the set of all sequences  $x = (x_k)$  such that the transform  $Ax = y = (y_n)$ , given by

$$y_n = \sum_{k=0}^{\infty} a_{nk}x_k,$$

is in  $X$ . Formally, the matrix domain of  $A$  in  $X$ , denoted by  $X_A$ , is

$$X_A = \left\{ x \in \omega : Ax = \left( \sum_{k=0}^{\infty} a_{nk}x_k \right) \in X \right\}. \tag{2}$$

### 3. Sequence Space $\ell_p^\lambda$ and Its $\beta$ -Dual

Following Mursaleen and Noman [8, 9], we let  $\lambda = (\lambda_k)_{k=0}^\infty$  be a strictly increasing sequence of positive reals and  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ . Then, the matrix  $\Lambda = (\lambda_{nk})$  is defined by

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

for all  $n, k \in \mathbb{N}_0$ , where we take  $\lambda_{-1} = 0$  for convenience. A significant characteristic of the matrix  $\Lambda$  is that it is a triangle matrix, that is  $\lambda_{nn} \neq 0$  and all entries above the main diagonal are zero, which also guarantees that it defines a regular limitation method.

To investigate the characteristics of sequence spaces defined through infinite matrix transformations, a clear understanding of the transformations induced by the matrix  $\Lambda$  is required. In particular, the analysis of several points in this discussion depends on the ability to associate the original sequence with the transformed sequence in both directions. Therefore, the initial step required is to examine the fundamental properties of the matrix  $\Lambda$ , particularly the existence of its inverse, which will be used repeatedly in subsequent discussions.

It is well-known that the inverse of a triangle matrix must exist, and is also a triangle [12]. Applying this fact to  $\Lambda$ , we obtain the following result.

**Proposition 1.** *The inverse of the matrix  $\Lambda$ , denoted by  $\Lambda^{-1} = (\lambda_{nk})^{-1}$ , is given by*

$$(\lambda_{nk})^{-1} = \begin{cases} -\frac{\lambda_k}{\lambda_{k+1} - \lambda_k}, & \text{for } k = n - 1, \\ \frac{\lambda_n}{\lambda_n - \lambda_{n-1}}, & \text{for } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* We seek a matrix  $M = (m_{nk})$  such that  $\Lambda M = M \Lambda = I$ . To do this, consider the equation

$$\sum_{j=0}^{\infty} \lambda_{nj}m_{jk} = \delta_{nk}, \tag{3}$$

where  $\delta_{nk}$  is the Kronecker delta, which equals 1 if  $n = k$  and 0 if  $n \neq k$ . Since  $\Lambda$  is a triangle, its inverse is also a unique triangle matrix. Thus,  $m_{jk} = 0$  for  $k > j$ . Combining this with the fact that  $\lambda_{nj} = 0$  for  $j > n$ , Eq. (3) can be rewritten as

$$\sum_{j=k}^n \frac{\lambda_j - \lambda_{j-1}}{\lambda_n} m_{jk} = \delta_{nk}.$$

Multiplying both sides by  $\lambda_n$ , we obtain

$$\sum_{j=k}^n (\lambda_j - \lambda_{j-1}) m_{jk} = \lambda_n \delta_{nk}.$$

Setting  $n = k$ , we obtain  $m_{kk} = \frac{\lambda_k}{\lambda_k - \lambda_{k-1}}$ . Next, setting  $n = k + 1$  yields

$$\begin{aligned} (\lambda_k - \lambda_{k-1}) m_{kk} + (\lambda_{k+1} - \lambda_k) m_{k+1,k} &= 0 \\ (\lambda_k - \lambda_{k-1}) \left( \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \right) + (\lambda_{k+1} - \lambda_k) m_{k+1,k} &= 0 \\ \lambda_k + (\lambda_{k+1} - \lambda_k) m_{k+1,k} &= 0 \\ m_{k+1,k} &= -\frac{\lambda_k}{\lambda_{k+1} - \lambda_k}. \end{aligned}$$

Subsequently, setting  $n = k + 2$ , we have

$$\begin{aligned} (\lambda_k - \lambda_{k-1}) m_{kk} + (\lambda_{k+1} - \lambda_k) m_{k+1,k} + (\lambda_{k+2} - \lambda_{k+1}) m_{k+2,k} &= 0 \\ \lambda_k - \lambda_k + (\lambda_{k+2} - \lambda_{k+1}) m_{k+2,k} &= 0 \\ (\lambda_{k+2} - \lambda_{k+1}) m_{k+2,k} &= 0. \end{aligned}$$

Since the sequence  $\lambda = (\lambda_n)$  is strictly increasing, then  $\lambda_{k+2} - \lambda_{k+1} > 0$ , so it must be that  $m_{k+2,k} = 0$ . By induction, it is easy to show that  $\sum_{j=k}^{n-1} (\lambda_j - \lambda_{j-1}) m_{jk} = 0$  for  $n \geq k + 2$ , so that  $m_{nk} = 0$  for  $n \geq k + 2$ . Therefore, the matrix  $M = \Lambda^{-1}$  satisfies the equation  $\Lambda M = M \Lambda = I$ .  $\square$

Recently, the sequence space  $\ell_p^\lambda$  of non-absolute type has been introduced in [8] and [9] as the space of all sequences whose  $\Lambda$ -transforms are in the space  $\ell_p$ , where  $1 \leq p < \infty$ . More concretely,

$$\ell_p^\lambda := \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p < \infty \right\}.$$

With the notation of Eq. (2), we can redefine the space  $\ell_p^\lambda$  as the matrix domain of triangle  $\Lambda$  in the space  $\ell_p$ , that is  $\ell_p^\lambda = (\ell_p)_\Lambda$ , where  $1 \leq p < \infty$ . It is well-known that if  $A$  is a triangle and  $X$  is a  $BK$ -space, then  $X_A$  is also a  $BK$ -space with the norm given by  $\|x\|_{X_A} = \|Ax\|_X$  for all  $x \in X_A$  [13]. Since  $\ell_p$  is a  $BK$ -space, then as a corollary  $(\ell_p)_\Lambda = \ell_p^\lambda$  is a  $BK$ -space.

**Theorem 1** ([8]). *Let  $1 \leq p < \infty$ . The sequence space  $\ell_p^\lambda$  is a  $BK$ -space with the norm defined by  $\|x\|_{\ell_p^\lambda} = \|\Lambda x\|_{\ell_p}$ .*

**Theorem 2** ([8]). *Let  $1 \leq p < \infty$ . The spaces  $\ell_p^\lambda$  and  $\ell_p$  are isometrically isomorphic, that is  $\ell_p^\lambda \cong \ell_p$ .*

*Proof.* The mapping  $T : \ell_p^\lambda \rightarrow \ell_p$  defined by  $Tx = y = \Lambda x$  is an isometric isomorphism. A more detailed proof can be found in [8] and is therefore omitted.  $\square$

**Theorem 3.** *The space  $\ell_p^\lambda$  has monotone norm, where  $1 \leq p < \infty$ .*

*Proof.* Let  $x = (x_k) \in \ell_p^\lambda$  and  $1 \leq p < \infty$ . We write  $x^{[r]} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$ . Define the norms

$$\|x\| = \left( \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p \right)^{\frac{1}{p}} \quad \text{and} \quad \|x^{[r]}\| = \left( \sum_{n=0}^r \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p \right)^{\frac{1}{p}}$$

for all  $x \in \ell_p^\lambda$ . Then, for  $s > r$ , we have that

$$\|x^{[r]}\| = \left( \sum_{n=0}^r \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p \right)^{\frac{1}{p}} \leq \left( \sum_{n=0}^s \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p \right)^{\frac{1}{p}} = \|x^{[s]}\|,$$

that is  $\|x^{[s]}\| \geq \|x^{[r]}\|$ . Since the sequence  $\|x^{[r]}\|$  is monotonically increasing and bounded above, it converges to its supremum. It follows that

$$\begin{aligned} \sup \|x^{[r]}\| &= \sup \left( \left( \sum_{n=0}^r \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p \right)^{\frac{1}{p}} \right) \\ &= \left( \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^p \right)^{\frac{1}{p}} \\ &= \|x\|. \end{aligned}$$

Since  $\|x^{[s]}\| \geq \|x^{[r]}\|$  for  $s > r$  and  $\|x\| = \sup \|x^{[s]}\|$ , we conclude that the space  $\ell_p^\lambda$  has monotone norm. This completes the proof.  $\square$

Since any space with a monotone norm has  $AB$ , we obtain the following corollary:

**Corollary 1.** *The space  $\ell_p^\lambda$  has  $AB$ , where  $1 \leq p < \infty$ .*

We recall the result concerning the  $\beta$ -dual of the space  $\ell_p^\lambda$  given by Mursaleen and Noman in [9]. This dual is very useful in the study of sequence spaces and matrix transformations. In particular, the  $\beta$ -dual norm is used to represent the corresponding Wulff shapes.

**Theorem 4 ([9]).** *Define the sets  $d_\infty^\lambda$  and  $e_q^\lambda$  as follows:*

$$d_\infty^\lambda = \left\{ a = (a_k) \in \omega : \sup_{k \in \mathbb{N}_0} \left| \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k \right| < \infty \right\},$$

and

$$e_q^\lambda = \left\{ a = (a_k) \in \omega : \sum_{k=0}^{\infty} \left| \bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k \right|^q < \infty \right\}$$

where

$$\bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) = \frac{a_k}{\lambda_k - \lambda_{k-1}} - \frac{a_{k+1}}{\lambda_{k+1} - \lambda_k} \text{ for all } k \in \mathbb{N}_0.$$

Then  $(\ell_1^\lambda)^\beta = d_\infty^\lambda$  and  $(\ell_p^\lambda)^\beta = d_\infty^\lambda \cap e_q^\lambda$ , where  $1 < p < \infty$ .

Now, we are ready to examine the dual norm which is the main focus of the paper.

**Theorem 5.** Let  $1 < p < \infty$ . Define the natural norm on the  $\beta$ -dual space of  $\ell_p^\lambda$  as follows:

$$\|a\|_{(\ell_p^\lambda)^\beta} = \left( \sum_{k=0}^{\infty} \left| \bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k \right|^q \right)^{\frac{1}{q}} \quad \text{and} \quad \|a\|_{(\ell_1^\lambda)^\beta} = \sup_{k \in \mathbb{N}_0} \left| \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k \right|. \quad (4)$$

Then,  $\|a\|_{\ell_p^\lambda}^* = \|a\|_{(\ell_p^\lambda)^\beta}$  for every  $a \in (\ell_p^\lambda)^\beta$  where  $1 < p < \infty$  and  $\|a\|_{\ell_1^\lambda}^* = \|a\|_{(\ell_1^\lambda)^\beta}$  for every  $a \in (\ell_1^\lambda)^\beta$ .

*Proof.* Let  $1 \leq p < \infty$  and  $a \in (\ell_p^\lambda)^\beta$  be given. Consider the equation

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^{n-1} \bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k y_k + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n y_n, \quad n \in \mathbb{N}_0. \quad (5)$$

Let us begin with the case  $1 < p < \infty$  first. By Theorem 4, the necessary condition for the sequence  $a \in (\ell_p^\lambda)^\beta$  is  $a \in d_\infty^\lambda$ . Therefore, the following applies:

$$\sup_{n \in \mathbb{N}_0} \left| \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n \right| < \infty.$$

Since  $x \in \ell_p^\lambda$  if and only if  $y = \Lambda x \in \ell_p$  and the inclusion  $\ell_p \subset c_0$  holds for  $1 \leq p < \infty$ , then  $y \in c_0$ . Consequently, by taking the limit as  $n \rightarrow \infty$  in Eq. (5) and using the fact that  $\bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k \in \ell_q$  because  $a \in (\ell_p^\lambda)^\beta$ , we obtain

$$\sum_k a_k x_k = \sum_k \bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k y_k. \quad (6)$$

Consider the mapping  $T$  defined in Theorem 2. As a consequence of the isometry, the mapping  $T$  bijectively maps the unit sphere in the space  $\ell_p^\lambda$  onto the unit sphere in the space  $\ell_p$ . Now, define the linear functional  $f_a$  by  $f_a(x) = \sum_k a_k x_k$ . From Eq. (6),

$$f_a(x) = \sum_k a_k x_k = \sum_k \bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k y_k$$

where  $y = Tx = \Lambda x$ . Write  $b_k := \bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k$ . Consequently, for every  $x \in \ell_p^\lambda$ , we have  $f_a(x) = g_b(Tx) = g_b(y)$ , where  $g_b(y) = \sum_k b_k y_k$ . Thus,  $f_a = g_b \circ T$ . Now, note that

$$\|f_a\|_{(\ell_p^\lambda)^*} = \sup_{x \in S(\ell_p^\lambda)} |f_a(x)| = \sup_{x \in S(\ell_p^\lambda)} |g_b(Tx)| = \sup_{y \in S(\ell_p)} |g_b(y)| = \|g_b\|_{\ell_p^*}.$$

Using the fact that  $\ell_p^*$  and  $\ell_q$  are isometrically isomorphic, we obtain that

$$\|a\|_{\ell_p^\lambda}^* = \|f_a\|_{(\ell_p^\lambda)^*} = \|g_b\|_{\ell_p^*} = \|b\|_{\ell_q} = \left( \sum_k \left| \bar{\Delta} \left( \frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k \right|^q \right)^{1/q}.$$

It can be proved similarly for the case  $p = 1$  by using the  $\beta$ -dual characterization of  $\ell_1^\lambda$ .  $\square$

## 4. Some Geometric Properties of the Space $\ell_p^\lambda$

In this section, we investigate some geometric properties, namely convexity and smoothness, of the space  $\ell_p^\lambda$  for  $1 \leq p < \infty$ .

We may begin by stating some definitions and quoting the following theorems that are necessary for the subsequent proofs.

**Definition 1** ([14]). A normed space  $X$  is said to be uniformly convex if for every  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon)$  such that for every  $x, y \in S_X$  satisfying  $\|x - y\| \geq \varepsilon$ , the following holds:

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

**Definition 2** ([14]). A normed space  $X$  is said to be uniformly smooth if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for every  $x \in S(X)$  and every  $y \in X$  with  $\|y\| \leq \delta$ , the following holds:

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|.$$

**Theorem 6** ([15]). *The space  $\ell_p$  is uniformly convex and uniformly smooth for  $1 < p < \infty$ .*

Now, we are ready to prove the convexity and smoothness properties of the space  $\ell_p^\lambda$  for  $1 \leq p < \infty$ .

**Theorem 7.** *The space  $\ell_p^\lambda$  is uniformly convex for  $1 < p < \infty$ .*

*Proof.* Let  $1 < p < \infty$ . Since  $\ell_p$  is uniformly convex, then for every  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that for every  $u, v \in \ell_p$  with  $\|u\|_{\ell_p} = \|v\|_{\ell_p} = 1$  and  $\|u - v\|_{\ell_p} \geq \varepsilon$ , the following inequality holds:

$$\left\| \frac{u + v}{2} \right\|_{\ell_p} \leq 1 - \delta.$$

Now, let  $x, y \in \ell_p^\lambda$  be arbitrary such that  $\|x\|_{\ell_p^\lambda} = \|y\|_{\ell_p^\lambda} = 1$  and  $\|x - y\|_{\ell_p^\lambda} \geq \varepsilon$ . Apply the mapping  $T$  defined in Theorem 2. Using the fact that  $T$  is an isometric isomorphism, we obtain  $\|Tx\|_{\ell_p} = \|x\|_{\ell_p^\lambda} = 1$ ,  $\|Ty\|_{\ell_p} = \|y\|_{\ell_p^\lambda} = 1$ , and  $\|Tx - Ty\|_{\ell_p} = \|T(x - y)\|_{\ell_p} = \|x - y\|_{\ell_p^\lambda} \geq \varepsilon$ , where  $Tx, Ty \in \ell_p$ . By linearity of  $T$ ,

$$\left\| \frac{x + y}{2} \right\|_{\ell_p^\lambda} = \left\| T \left( \frac{x + y}{2} \right) \right\|_{\ell_p} = \left\| \frac{Tx + Ty}{2} \right\|_{\ell_p} \leq 1 - \delta.$$

Hence, the space  $\ell_p^\lambda$  is uniformly convex for  $1 < p < \infty$ . □

**Theorem 8.** *The space  $\ell_1^\lambda$  is not uniformly convex.*

*Proof.* It will be shown that  $\ell_1^\lambda$  is not strictly convex. Then, the contrapositive of the fact that uniform convexity implies strict convexity is used to demonstrate that  $\ell_1^\lambda$  is not uniformly convex.

Let us choose  $x, y \in \ell_1^\lambda$  given by

$$x = \left( 1, -\frac{\lambda_0}{\lambda_1 - \lambda_0}, 0, 0, 0, \dots \right) \quad \text{and} \quad y = \left( 0, \frac{\lambda_1}{\lambda_1 - \lambda_0}, -\frac{\lambda_1}{\lambda_2 - \lambda_1}, 0, 0, \dots \right).$$

The  $\Lambda$ -transforms of the sequences  $x$  and  $y$  are given by  $\Lambda x = (1, 0, 0, 0, \dots)$  and  $\Lambda y = (0, 1, 0, 0, \dots)$ . Consequently,  $\|x\|_{\ell_1^\lambda} = \|y\|_{\ell_1^\lambda} = 1$ , which means that  $x, y \in S(\ell_1^\lambda)$  with  $x \neq y$ .

Now, note that

$$\frac{\Lambda x + \Lambda y}{2} = \left( \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \dots \right).$$

Using the isometry between  $\ell_1^\lambda$  and  $\ell_1$ , we obtain

$$\left\| \frac{x+y}{2} \right\|_{\ell_1^\lambda} = \left\| \Lambda \left( \frac{x+y}{2} \right) \right\|_{\ell_1} = \left\| \frac{\Lambda x + \Lambda y}{2} \right\|_{\ell_1} = 1.$$

Since  $\left\| \frac{x+y}{2} \right\|_{\ell_1^\lambda} \not\leq 1$ , it can be concluded that  $\ell_1^\lambda$  is not strictly convex. Consequently,  $\ell_1^\lambda$  is also not uniformly convex.  $\square$

**Theorem 9.** *The space  $\ell_p^\lambda$  is uniformly smooth for  $1 < p < \infty$ .*

*Proof.* Let  $1 < p < \infty$ . Since  $\ell_p$  is uniformly smooth, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $u \in S(\ell_p)$  and  $\|v\|_{\ell_p} < \delta$ , then

$$\|u+v\|_{\ell_p} + \|u-v\|_{\ell_p} < 2 + \varepsilon\|v\|_{\ell_p}.$$

Now, take any  $x \in S(\ell_p^\lambda)$  and  $y \in \ell_p^\lambda$  with  $\|y\|_{\ell_p^\lambda} < \delta$ . Define  $u = Tx$  and  $v = Ty$  using the mapping  $T$  defined in Theorem 2. Since  $T$  is an isometry, we obtain  $\|u\|_{\ell_p} = \|Tx\|_{\ell_p} = \|x\|_{\ell_p^\lambda} = 1$  and  $\|v\|_{\ell_p} = \|Ty\|_{\ell_p} = \|y\|_{\ell_p^\lambda} < \delta$ . By again using the isometry and the linearity of  $T$ , it follows that

$$\|x+y\|_{\ell_p^\lambda} + \|x-y\|_{\ell_p^\lambda} < 2 + \varepsilon\|y\|_{\ell_p^\lambda},$$

which means that the space  $\ell_p^\lambda$  is uniformly smooth for  $1 < p < \infty$ .  $\square$

**Theorem 10.** *The space  $\ell_1^\lambda$  is not uniformly smooth.*

*Proof.* Let an arbitrary  $\delta > 0$  be given. Choose  $u = (1, 0, 0, \dots) \in \ell_1$ . Define

$$v = \left( 0, \frac{t}{N}, \frac{t}{N}, \dots, \frac{t}{N}, 0, 0, \dots \right),$$

where the nonzero terms occur at the indices  $k = 1, 2, \dots, N$  and  $0 < t < \delta$ . Then

$$\|v\|_{\ell_1} = \sum_k |v_k| = \sum_{k=1}^N |v_k| = N \cdot \frac{t}{N} = t < \delta.$$

Moreover,

$$u+v = \left( 1, \frac{t}{N}, \frac{t}{N}, \dots, \frac{t}{N}, 0, 0, \dots \right) \quad \text{and} \quad u-v = \left( 1, -\frac{t}{N}, -\frac{t}{N}, \dots, -\frac{t}{N}, 0, 0, \dots \right).$$

Hence,  $\|u+v\|_{\ell_1} = \|u-v\|_{\ell_1} = 1+t$ . Consequently,  $\|u+v\|_{\ell_1} + \|u-v\|_{\ell_1} = 2+2t = 2+2\|v\|_{\ell_1}$ . Now consider again the mapping  $T$  defined in Theorem 2. Define  $x = T^{-1}u$  and  $y = T^{-1}v$  in  $\ell_1^\lambda$ . Since  $T$  is linear and isometric, we have

$$\|x\|_{\ell_1^\lambda} = \|Tx\|_{\ell_1} = \|u\|_{\ell_1} = 1$$

and

$$\|y\|_{\ell_1^\lambda} = \|Ty\|_{\ell_1} = \|v\|_{\ell_1} = t < \delta.$$

Furthermore,  $\|x \pm y\|_{\ell_1^\lambda} = \|T(x \pm y)\|_{\ell_1} = \|Tx \pm Ty\|_{\ell_1} = \|u \pm v\|_{\ell_1}$ . Therefore, we obtain

$$\|x + y\|_{\ell_1^\lambda} + \|x - y\|_{\ell_1^\lambda} = \|u + v\|_{\ell_1} + \|u - v\|_{\ell_1} = 2 + 2\|v\|_{\ell_1} = 2 + 2\|y\|_{\ell_1^\lambda}.$$

Observe that

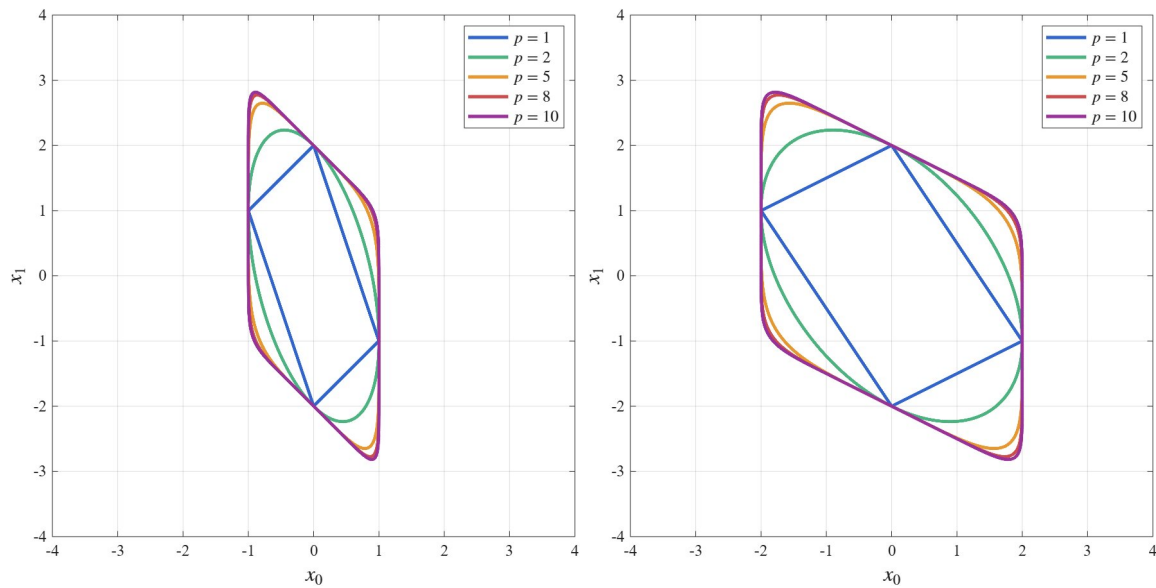
$$\|x + y\|_{\ell_1^\lambda} + \|x - y\|_{\ell_1^\lambda} = 2 + 2\|y\|_{\ell_1^\lambda} > 2 + \|y\|_{\ell_1^\lambda}.$$

Thus, for every  $\delta > 0$  and for  $\varepsilon_0 = 1$ , there exist  $x \in S(\ell_1^\lambda)$  and  $y \in \ell_1^\lambda$  with  $\|y\|_{\ell_1^\lambda} < \delta$  such that

$$\|x + y\|_{\ell_1^\lambda} + \|x - y\|_{\ell_1^\lambda} \geq 2 + \varepsilon_0\|y\|_{\ell_1^\lambda}.$$

We conclude that the sequence space  $\ell_1^\lambda$  is not uniformly smooth. □

The following is a visualization of the unit sphere of the  $\ell_p^\lambda$  sequence space for several sequences  $\lambda_k$  in  $\mathbb{R}^2$ .



**Fig. 1:** (Left) Unit sphere of  $\ell_p^\lambda$  for  $p = 1, 2, 5, 8, 10$  and  $\lambda_k = k + 1$ . (Right) Unit sphere of  $\ell_p^\lambda$  for  $p = 1, 2, 5, 8, 10$  and  $\lambda_k = 2^{k+1}$ .

It can be seen from Fig. 1 that the convexity and smoothness of the classical sequence space  $\ell_p$  are carried over to the space  $\ell_p^\lambda$ . The difference is that the non-absolute property of the space  $\ell_p^\lambda$  causes its unit ball to be non-symmetric with respect to each coordinate axis.

### 5. Wulff Shapes Induced by Finite Truncation of $\ell_p^\lambda$ -Norm

In this section, we study Wulff shapes, which are the sets obtained by Wulff’s geometric construction for the shape of a crystal at equilibrium. Many studies have been conducted on Wulff shapes. For instance, Han and Nishimura investigated the convexity properties of Wulff shapes in [16] and [17]. Furthermore, Wu and Ou-Yang [18] recently reexamined the variational principle of the equilibrium shape problem of crystals with respect to Legendre transformations.

Let  $n$  be a positive integer and  $\partial B^n$  denotes the unit sphere in  $\mathbb{R}^{n+1}$ . Given a continuous function  $F : \partial B^n \rightarrow \mathbb{R}_+$ , where  $\mathbb{R}_+$  is the set consisting of positive real numbers, the Wulff shape associated with  $F$  is the following set:

$$C_F = \bigcap_{\vec{e} \in \partial B^n} H_{\vec{e}} = \bigcap_{\vec{e} \in \partial B^n} \{\vec{x} \in \mathbb{R}^{n+1} : \vec{x} \cdot \vec{e} \leq F(\vec{e})\}. \tag{7}$$

Here,  $H_{\vec{e}}$  is called the half space and the dot in the center stands for the standard dot product of  $\vec{x}, \vec{e} \in \mathbb{R}^{n+1}$ . By definition, a Wulff shape  $C_F$  is a convex body in  $\mathbb{R}^{n+1}$  that contains the origin as an interior point. For a more detailed explanation of convex geometry, see [19].

The Wulff construction given in Eq. (7) can be equivalently restated using Legendre transformation.

**Definition 3.** Let  $\gamma : \partial B^n \rightarrow \mathbb{R}_+$  be a continuous function. The Legendre transform of  $\gamma$  is given by

$$\gamma_*(\vec{e}) = \inf \left\{ \frac{\gamma(\vec{u})}{\vec{e} \cdot \vec{u}} : \vec{e} \in \partial B^n \text{ and } \vec{e} \cdot \vec{u} > 0 \right\}. \quad (8)$$

Using the Legendre transformation, Faling [20] provides more applicable results in crystal formation as follows.

**Theorem 11 ([20]).** Let  $F : \partial B^n \rightarrow \mathbb{R}_+$  be a continuous function. Then a point  $\vec{x}$  is on the boundary  $\partial C_F$  of Wulff shape  $C_F$  corresponding to  $F$  if and only if

$$F(\vec{e}) \geq \vec{x} \cdot \vec{e} \text{ for all } \vec{e} \in \partial B^n \text{ and } F(\vec{e}_0) = \vec{x} \cdot \vec{e}_0 \text{ for some } \vec{e}_0 \in \partial B^n.$$

*Proof.* Let  $\vec{x} \in \partial C_F$ . By definition,  $\partial C_F \subset \overline{C_F}$ . Since  $C_F$  is closed, then  $\overline{C_F} = C_F$  so that  $\vec{x} \in C_F$ . Consequently,  $\vec{x} \cdot \vec{e} \leq F(\vec{e})$  for every  $\vec{e} \in \partial B^n$ . Now, suppose that  $\vec{x} \cdot \vec{e} < F(\vec{e})$  for every  $\vec{e} \in \partial B^n$ . Define the continuous function  $g(\vec{e}) := F(\vec{e}) - \vec{x} \cdot \vec{e}$  with  $\vec{e} \in \partial B^n$ . Since  $\partial B^n$  is compact,  $g$  is continuous, and  $g(\vec{e}) > 0$  for all  $\vec{e}$ , then by the Maximum-Minimum Theorem,  $g$  attains a positive minimum on  $\partial B^n$ . More precisely, there exists  $\delta := \min_{\vec{e} \in \partial B^n} g(\vec{e}) > 0$ , so that for every  $\vec{e} \in \partial B^n$  we have  $\vec{x} \cdot \vec{e} \leq F(\vec{e}) - \delta$ . Choose  $r := \delta/2$  and take any  $\vec{y}$  with  $\|\vec{y} - \vec{x}\| < r$ . Then, for every  $\vec{e} \in \partial B^n$ , we obtain

$$\vec{y} \cdot \vec{e} = \vec{x} \cdot \vec{e} + (\vec{y} - \vec{x}) \cdot \vec{e} \leq \vec{x} \cdot \vec{e} + \|\vec{y} - \vec{x}\| \|\vec{e}\| < (F(\vec{e}) - \delta) + r = F(\vec{e}) - \delta/2 < F(\vec{e}).$$

Thus,  $\vec{y} \cdot \vec{e} \leq F(\vec{e})$  for every  $\vec{e} \in \partial B^n$ , so that  $\vec{y} \in C_F$ . Since  $\vec{y}$  is chosen arbitrarily with  $\|\vec{y} - \vec{x}\| < r$ , it follows that  $B(\vec{x}, r) \subset C_F$ , so  $\vec{x}$  is an interior point of  $C_F$ . This contradicts the assumption that  $\vec{x} \in \partial C_F$ . Therefore, there must exist some  $\vec{e}_0 \in \partial B^n$  such that  $\vec{x} \cdot \vec{e}_0 = F(\vec{e}_0)$ .

Conversely, assume that  $F(\vec{e}) \geq \vec{x} \cdot \vec{e}$  for every  $\vec{e} \in \partial B^n$  and that there exists  $\vec{e}_0 \in \partial B^n$  such that  $F(\vec{e}_0) = \vec{x} \cdot \vec{e}_0$ . By definition,  $\vec{x} \in C_F$ . For any  $r > 0$ , choose  $0 < \varepsilon < r$  and consider the point  $\vec{x} + \varepsilon \vec{e}_0$ . We have  $(\vec{x} + \varepsilon \vec{e}_0) \cdot \vec{e}_0 = \vec{x} \cdot \vec{e}_0 + \varepsilon = F(\vec{e}_0) + \varepsilon > F(\vec{e}_0)$  so that  $\vec{x} + \varepsilon \vec{e}_0 \notin C_F$ . By construction,  $\|(\vec{x} + \varepsilon \vec{e}_0) - \vec{x}\| = \|\varepsilon \vec{e}_0\| = |\varepsilon| \cdot \|\vec{e}_0\| = \varepsilon < r$ . This means that  $\vec{x} + \varepsilon \vec{e}_0 \in B(\vec{x}, r)$ . Thus,  $B(\vec{x}, r) \cap C_F \neq \emptyset$  and  $B(\vec{x}, r) \cap (C_F)^c \neq \emptyset$ , which implies  $\vec{x} \in \partial C_F$ .  $\square$

**Theorem 12 ([20]).** Let  $F : \partial B^n \rightarrow \mathbb{R}_+$  be a continuous function and  $CF : \partial B^n \rightarrow \mathbb{R}_+$  be defined by

$$CF(\vec{e}) = \inf \{ F(\vec{u})(\vec{e} \cdot \vec{u})^{-1} : \vec{u} \in \partial B^n \text{ and } \vec{e} \cdot \vec{u} > 0 \}. \quad (9)$$

Then, the boundary  $\partial C_F$  of Wulff shape corresponding to  $F$  is given by

$$\partial C_F = \{ \vec{x} = CF(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \}. \quad (10)$$

In particular, if  $n = 2$  then a parametric representation for the boundary  $\partial C_F$  of Wulff's shape corresponding to  $F$  is

$$\vec{x}(u_1, u_2) = CF(\vec{e}(u_1, u_2))\vec{e}(u_1, u_2) \text{ for } (u_1, u_2) \in D = (-\pi/2, \pi/2) \times (0, 2\pi). \quad (11)$$

**Corollary 2.** Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^{n+1}$ . Define the dual norm, induced by the dot product, by

$$\|\vec{a}\|^* = \sup \{ |\vec{a} \cdot \vec{x}| : \|\vec{x}\| = 1 \} = \sup_{\vec{x} \neq 0} \frac{|\vec{a} \cdot \vec{x}|}{\|\vec{x}\|}. \quad (12)$$

Then,  $C_{\|\cdot\|} = \{ \vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\|^* \leq 1 \}$ .

*Proof.* Let  $\vec{x} \in C_{\|\cdot\|}$ . Then, by Eq. (7), for every  $\vec{e} \in \partial B^n$  we have  $\vec{x} \cdot \vec{e} \leq \|\vec{e}\|$ . Since  $-\vec{e} \in \partial B^n$  as well, the inequality  $-\vec{x} \cdot \vec{e} \leq \|\vec{e}\|$  also holds. Combining with the previous inequality, we obtain  $|\vec{x} \cdot \vec{e}| \leq \|\vec{e}\|$  for every  $\vec{e} \in \partial B^n$ . Now, take any  $\vec{y} \neq 0$ . Write  $\vec{y} = t\vec{e}$  with  $t = \|\vec{y}\|_2 > 0$ . By homogeneity of norm,

$$\frac{|\vec{x} \cdot \vec{y}|}{\|\vec{y}\|} = \frac{|\vec{x} \cdot (t\vec{e})|}{\|t\vec{e}\|} = \frac{|\vec{x} \cdot \vec{e}|}{\|\vec{e}\|} \leq 1.$$

By taking the supremum over  $\vec{y} \neq 0$ , we obtain  $\|\vec{x}\|^* \leq 1$ . Thus, we obtain  $x \in \{ \vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\|^* \leq 1 \}$  or equivalently  $C_{\|\cdot\|} \subseteq \{ \vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\|^* \leq 1 \}$ .

Conversely, let  $\vec{x} \in \mathbb{R}^{n+1}$  satisfy  $\|\vec{x}\|^* \leq 1$ . Then, by Eq. (12), we have for every  $\vec{z} \neq 0$ ,

$$\|\vec{x} \cdot \vec{z}\| \leq \|\vec{x}\|^* \|\vec{z}\| \leq \|\vec{z}\|. \quad (13)$$

In particular, applying Eq. (13) to  $\vec{e} \in \partial B^n$ , we obtain  $\vec{x} \cdot \vec{e} \leq |\vec{x} \cdot \vec{e}| \leq \|\vec{e}\|$ . Thus,  $\vec{x} \in C_{\|\cdot\|}$  or equivalently  $\{ \vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\|^* \leq 1 \} \subseteq C_{\|\cdot\|}$ .  $\square$

**Corollary 3.** Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^{n+1}$  and define  $CF : \partial B^n \rightarrow \mathbb{R}_+$  as in Eq. (9). Then for every  $\vec{e} \in \partial B^n$ ,

$$CF(\vec{e}) = \frac{1}{\|\vec{e}\|^*}. \quad (14)$$

Consequently, the boundary  $\partial C_{\|\cdot\|}$  of Wulff shape corresponding to  $\|\cdot\|$  is given by

$$\partial C_{\|\cdot\|} = \left\{ \vec{x} = \frac{1}{\|\vec{e}\|^*} \vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \right\}. \quad (15)$$

*Proof.* From Eq. (9),

$$CF(\vec{e}) = \inf \{ \|\vec{u}\| (\vec{e} \cdot \vec{u})^{-1} : \vec{u} \in \partial B^n, \vec{e} \cdot \vec{u} > 0 \} = \left( \sup \left\{ \frac{\vec{e} \cdot \vec{u}}{\|\vec{u}\|} : \vec{u} \in \partial B^n, \vec{e} \cdot \vec{u} > 0 \right\} \right)^{-1}.$$

It remains to show that

$$\sup \left\{ \frac{\vec{e} \cdot \vec{u}}{\|\vec{u}\|} : \vec{u} \in \partial B^n \text{ and } \vec{e} \cdot \vec{u} > 0 \right\} = \|\vec{e}\|^*.$$

For any  $\vec{v} \neq 0$ , put  $\vec{v} = \vec{u}/\|\vec{u}\| \in \partial B^n$ . Then,

$$\frac{|\vec{e} \cdot \vec{u}|}{\|\vec{u}\|} = \frac{|\vec{e} \cdot (\vec{v}\|\vec{u}\|)|}{\|\vec{v}\|\|\vec{u}\|} = \frac{|\vec{e} \cdot \vec{v}|}{\|\vec{v}\|}.$$

Hence,

$$\sup_{\vec{v} \neq 0} \frac{|\vec{e} \cdot \vec{v}|}{\|\vec{v}\|} = \sup_{\vec{u} \in \partial B^n} \frac{|\vec{e} \cdot \vec{u}|}{\|\vec{u}\|}.$$

Now, note that for any  $\vec{u} \in \partial B^n$ ,  $-\vec{u} \in \partial B^n$  as well. Therefore, the supremum can be restricted to the region  $\vec{e} \cdot \vec{u} > 0$  without changing its value. Thus, we obtain,

$$\sup_{\vec{u} \in \partial B^n} \frac{|\vec{e} \cdot \vec{u}|}{\|\vec{u}\|} = \sup_{\substack{\vec{u} \in \partial B^n \\ \vec{e} \cdot \vec{u} > 0}} \frac{\vec{e} \cdot \vec{u}}{\|\vec{u}\|}.$$

We conclude that  $CF(\vec{e}) = (\|\vec{e}\|^*)^{-1}$ . The identity in Eq. (15) follows from Eq. (14). This completes the proof.  $\square$

**Remark 1** ([10]). In the special case of  $n = 2$ , we obtain from Eq. (11) and Eq. (15) the following parametric representation for the boundary of Wulff shapes corresponding to a norm  $\|\cdot\|$  in  $\mathbb{R}^{n+1}$

$$\vec{x}(u_1, u_2) = (\|\vec{e}(u_1, u_2)\|^*)^{-1} \cdot \vec{e}(u_1, u_2) \text{ for } (u_1, u_2) \in D$$

and the potential surface has a parametric representation

$$\vec{y}(u_1, u_2) = (\|\vec{e}(u_1, u_2)\|) \cdot \vec{e}(u_1, u_2) \text{ for } (u_1, u_2) \in D,$$

where  $\vec{e}(u_1, u_2) = (\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1)$  for  $(u_1, u_2) \in D = (\pi/2, \pi/2) \times (0, 2\pi)$ .

To apply the construction Eq. (7) to our setting, we choose the function from the norm induced by the space  $\ell_p^\lambda$ . Since Wulff's construction produces a convex body in a finite-dimensional Euclidean space, we first introduce a finite-dimensional model that is consistent with and suitable for geometric representation and visualization.

We consider the finite-dimensional normed space

$$\ell_{p,\lambda}^{n+1} := \left( \mathbb{R}^{n+1}, \|\cdot\|_{p,n+1} \right),$$

where  $\|\cdot\|_{p,n+1}$  denotes the  $\ell_p^\lambda$ -norm restricted to vectors  $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ . Equivalently, for  $1 \leq p < \infty$ ,

$$\|x\|_{p,n+1} = \left( \sum_{m=0}^n \left| \frac{1}{\lambda_m} \sum_{k=0}^m (\lambda_k - \lambda_{k-1}) x_k \right|^p \right)^{1/p}.$$

Note that since  $\|\cdot\|_{p,n+1}$  is a norm, it is continuous. Thus, we obtain the Wulff shape associated with  $\|\cdot\|_{p,n+1}$  as

$$C_{\|\cdot\|_{p,n+1}} = \bigcap_{\vec{e} \in \partial B^n} \{ \vec{x} \in \mathbb{R}^{n+1} : \vec{x} \cdot \vec{e} \leq \|\vec{e}\|_{p,n+1} \} = \{ \vec{x} \in \mathbb{R}^{n+1} : \|x\|_{p,n+1}^* \leq 1 \}. \quad (16)$$

We will show that the normed spaces of finite dimension that we have defined possess the same geometric properties as the original space.

**Theorem 13.** *Let*

$$X_n := \left\{ x = (x_k) \in \ell_p^\lambda : x_k = 0 \text{ for all } k > n \right\},$$

where  $1 \leq p < \infty$ . Define the map  $T_n : X_n \rightarrow \mathbb{R}^{n+1}$  by  $T_n(x) = (x_0, x_1, \dots, x_n)$ . Assume that the norm  $\|\cdot\|_{p,n+1}$  on  $\mathbb{R}^{n+1}$  is the norm induced from  $\ell_p^\lambda$  by zero extension, that is,

$$\|(a_0, \dots, a_n)\|_{p,n+1} = \|(a_0, \dots, a_n, 0, 0, \dots)\|_{\ell_p^\lambda}.$$

Then:

- (i)  $X_n$  is a closed vector subspace of  $\ell_p^\lambda$ .
- (ii)  $X_n$  and  $\ell_{p,\lambda}^{n+1}$  are isometrically isomorphic, that is  $X_n \cong \ell_{p,\lambda}^{n+1}$ .

*Proof.* (i) It is straightforward to prove that  $X_n$  is a subspace of  $\ell_p^\lambda$ . To prove its closedness, recall that  $\ell_p^\lambda$  is a BK-space. This means that the projection map  $P_k : \ell_p^\lambda \rightarrow \mathbb{R}$  given by  $P_k(x) = x_k$  is continuous for every  $x \in \ell_p^\lambda$  and  $k \in \mathbb{N}_0$ . Since the kernel of  $P_k$  is  $\ker(P_k) = \{x \in \ell_p^\lambda : x_k = 0\}$ , we can write

$$X_n = \bigcap_{k=n+1}^{\infty} \ker(P_k).$$

Since each  $P_k$  is continuous,  $\ker(P_k)$  is a closed subspace of  $\ell_p^\lambda$ . By using the fact that the intersection of any collection of closed sets is closed, it follows that  $X_n$  is a closed subspace of  $\ell_p^\lambda$ .

(ii) The linearity and injectivity of  $T_n$  are straightforward to establish. To demonstrate surjectivity, let  $a = (a_0, \dots, a_n) \in \mathbb{R}^{n+1}$ . We can define its zero extension, namely  $S_n(a) = (a_0, \dots, a_n, 0, 0, \dots)$ . It is clear that  $S_n(a) \in \ell_p^\lambda$ , and consequently  $S_n(a) \in X_n$ . Furthermore,  $T_n(S_n(a)) = a$ , which shows that  $T_n$  is surjective. It is also evident that  $T_n$  is an isometry. Thus,  $X_n \cong \ell_{p,\lambda}^{n+1}$ .  $\square$

**Corollary 4.** *Let  $1 < p < \infty$ . The space  $\ell_{p,\lambda}^{n+1}$  is uniformly convex and uniformly smooth for every  $n \in \mathbb{N}$ .*

*Proof.* Since uniform convexity and uniform smoothness pass to closed subspaces, we have by Theorem 13(i) that  $X_n$  is uniformly convex and uniformly smooth. By Theorem 13(ii), we conclude that  $\ell_{p,\lambda}^{n+1}$  possesses the same geometric properties.  $\square$

Before proceeding to the final part to visualize the Wulff shape induced by the  $\ell_p^\lambda$  norm, we first present the following result concerning the convexity and smoothness of the resulting Wulff shape.

**Theorem 14.** *Let  $C_n$  be the Wulff shape associated with the norm  $\|\cdot\|_{p,n+1}$ . Then,  $C_n$  is strictly convex and its boundary  $\partial C_n$  is smooth.*

*Proof.* By Corollary 4,  $\ell_{p,\lambda}^{n+1}$  is uniformly convex and uniformly smooth. Duality principles implies that its dual is also uniformly convex and uniformly smooth. Consequently, the corresponding unit sphere is strictly convex and smooth.  $\square$

Finally, we present some visualizations and representations of the potential surfaces and the corresponding Wulff shapes of the  $\ell_p^\lambda$  spaces. We adopt the convention that  $x_k = 0$  for every integer  $k > 2$ , so that the only terms affecting the value of the norm are the first three terms of the sequence  $x \in \ell_p^\lambda$ .

We introduce the norm of the  $\ell_p^\lambda$  spaces on  $\mathbb{R}^3$  in a natural way as

$$\|(x_0, x_1, x_2)\|_{p,3} = \left( |x_0|^p + \left| \frac{1}{\lambda_1}(\lambda_0 x_0 + (\lambda_1 - \lambda_0)x_1) \right|^p + \left| \frac{1}{\lambda_2}(\lambda_0 x_0 + (\lambda_1 - \lambda_0)x_1 + (\lambda_2 - \lambda_1)x_2) \right|^p \right)^{1/p}$$

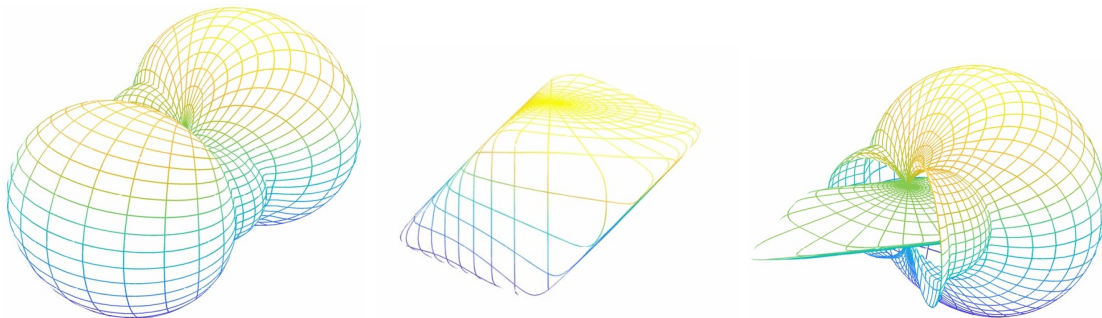
for all  $(x_0, x_1, x_2) \in \mathbb{R}^3$  and  $1 \leq p < \infty$ .

The dual norm, which is the same as its  $\beta$ -dual norm, is given by

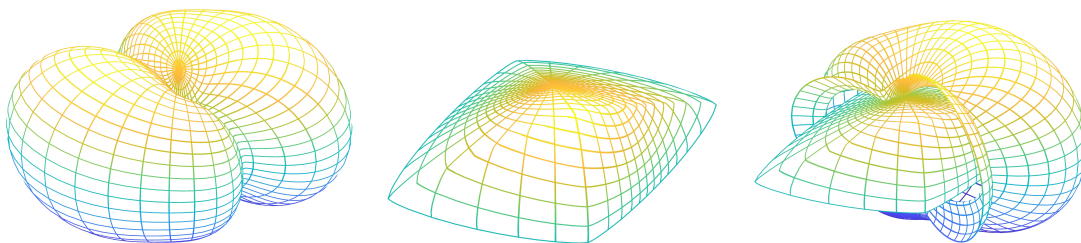
$$\|(x_0, x_1, x_2)\|_{p,3}^* = \left( \left| \lambda_0 \left( \frac{x_0}{\lambda_0} - \frac{x_1}{\lambda_1 - \lambda_0} \right) \right|^q + \left| \lambda_1 \left( \frac{x_1}{\lambda_1 - \lambda_0} - \frac{x_2}{\lambda_2 - \lambda_1} \right) \right|^q + \left| \lambda_2 \left( \frac{x_2}{\lambda_2 - \lambda_1} \right) \right|^q \right)^{1/q}$$

for all  $(x_0, x_1, x_2) \in \mathbb{R}^3$  and  $1 < p < \infty$  where  $q = p/(p - 1)$ .

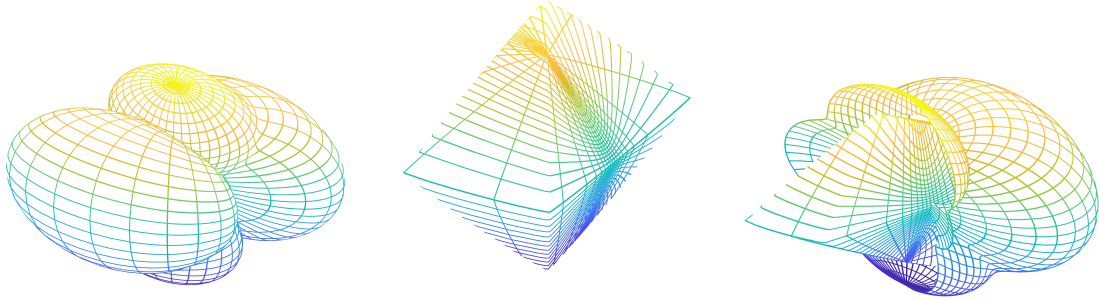
**Example 1.** We give some examples of the visualization of our space for several values of  $p$  and  $\Lambda = (\lambda_0, \lambda_1, \lambda_2)$ . We choose  $p = 1.1$ ,  $\Lambda = (5, 10, 13)$ ;  $p = 5.5$ ,  $\Lambda = (3, 7, 11)$ ; and  $p = 20$ ,  $\Lambda = (3, 4, 11)$  in Fig. 2, Fig. 3, and Fig. 4, respectively.



**Fig. 2:** Potential surface given by the norm of  $\ell_p^\lambda$  for  $p = 1.1$  and  $\Lambda = (5, 10, 13)$ , and its corresponding Wulff shape



**Fig. 3:** Potential surface given by the norm of  $\ell_p^\lambda$  for  $p = 5.5$  and  $\Lambda = (3, 7, 11)$ , and its corresponding Wulff shape



**Fig. 4:** Potential surface given by the norm of  $\ell_p^\lambda$  for  $p = 20$  and  $\Lambda = (3, 4, 11)$ , and its corresponding Wulff shape

## 6. Conclusion

In this paper, we investigated several structural and geometric aspects of the non-absolute type sequence space  $\ell_p^\lambda$ . In particular, we examined the  $\beta$ -dual of the space and showed that the natural norm on  $(\ell_p^\lambda)^\beta$  coincides with the dual norm induced by the Banach space structure of  $\ell_p^\lambda$ . We further analyzed several geometric properties of the space  $\ell_p^\lambda$ . By exploiting the isometric isomorphism between  $\ell_p^\lambda$  and the classical space  $\ell_p$ , we showed that for  $1 < p < \infty$  the space  $\ell_p^\lambda$  inherits important geometric characteristics such as uniform convexity and uniform smoothness, while the space  $\ell_1^\lambda$  fails to possess these properties. These observations clarify how the geometric behavior of  $\ell_p^\lambda$  reflects that of the underlying classical sequence space.

Finally, we considered finite-dimensional truncations of the  $\ell_p^\lambda$  norm in order to obtain geometric realizations through Wulff construction. In this setting, the dual norm determines a corresponding Wulff shape whose boundary can be interpreted as the equilibrium crystal associated with the norm-induced potential surface. Although this visualization is necessarily restricted to finite dimensions, it provides a useful geometric perspective on the analytic structure of the space.

Future work may consider extending this approach to other classes of non-absolute sequence spaces generated by matrix domains, as well as investigating further geometric features of the associated dual norms and their potential applications in convex geometry.

## CRedit Authorship Contribution Statement

**Adrian Taruna Barus:** Visualization, Proofs & Methodology, Writing–Original Draft.

**Meisya Shekina:** Conceptualization, Formal Analysis, Writing–Review & Editing.

## Declaration of Generative AI and AI-assisted technologies

Generative AI-assisted technology was used in a limited manner during the preparation of this manuscript. Specifically, ChatGPT (OpenAI, GPT-5.2) was employed solely for language proofreading and minor stylistic refinement, including grammar, clarity, and readability. The AI tool did not contribute to the development of mathematical concepts, theoretical results, proofs, data analysis, visualizations, or scientific interpretations. All scientific content, results, and conclusions remain the sole responsibility of the authors.

## Declaration of Competing Interest

The authors declare that there are no potential or actual conflicts of interest that could influence the outcomes or interpretation of this research. There are no financial or non-financial relationships affecting this research.

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## Data and Code Availability

The codes supporting the findings of this study are available from the corresponding author upon reasonable request and subject to confidentiality agreements.

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