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Approximation Properties on a Set Based on Equivalence Relations and Dominance Relations

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Abstract

An equivalence relation on a set forms equivalence classes so that the concept of approximation is formed on that set (rough set). The concept of approximation on a set is developing very rapidly. Some papers replace the equivalence relation with other relations, one of which is the dominance relation. This paper provides the approximation properties that hold to the equivalence and dominance relations using the concept of three types of approximations defined on a set. In addition, it identifies the properties that hold in the equivalence relations but do not necessarily hold in the dominance relations. Furthermore, this paper proves that the three types of approximations on a set $w.r.t$ an equivalence relation are identical, but this result does not necessarily hold for a dominance relation, especially for the third type.

Keywords: Approximation; dominance relation; equivalence relation

1. Introduction

The rough set theory was first introduced by Pawlak [1]. The theory uses equivalence relations. An equivalence relation on a set forms equivalence classes, thus forming the concept of lower approximation and upper approximation on a set (rough set). The concept of rough sets has developed in various fields [2–13].

The concept of rough sets has been generalized to algebraic structures by employing equivalence relations induced by normal subgroups in groups, ideals in rings, and submodules in modules. Since 2021, the authors have investigated rough set models involving several normal subgroups, ideals, and submodules [14–18].

Abu-Donia [19] in 2008 introduced three types of lower and upper approximation concepts for a set to a binary relation. Abu-Donia's concept was developed by Salama et al. [20] in 2022 using a dominance relation. The equivalence relations are reflexive, symmetric and transitive, while the dominance relations are reflexive, antisymmetric and transitive. This paper investigates the approximation properties that hold to the equivalence and dominance relations regarding the three types of approximation concepts for a set. It also determines the properties that hold for the equivalence relations but not necessarily hold for the dominance relations. The relationship between the three types of approximation concepts for a set to the equivalence relation and the dominance relation has not been discussed in the literature [1, 19, 20]. The main contribution of this paper is showing that the relationship between the concept of three types of approximation on a set $w.r.t$ the equivalence relation and the dominance relation.

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2. Equivalence Relations and Dominance Relations in a Set

Throughout this paper, we denote X as a non-empty set, $\mathcal{B}$ as a binary relation $\mathcal{B}$ on X, $\mathcal{E}$ represents an equivalence relation $\mathcal{E}$ on X, while $\mathcal{D}$ denotes a dominance relation $\mathcal{D}$ on X.

Definition 1. [21] $\mathcal{B}$ is a subset of $X \times X = \{(a,b)|a,b \in X\}$. In this case $(a,b) \in \mathcal{B}$ means that a is related to b.

Definition 2. [21] an equivalence relation $\mathcal{E}$ satisfies

(1) reflexive, that is, for every $a \in X$ holds $(a,a) \in \mathcal{E}$,
(2) symmetric, that is, if $(a,b) \in \mathcal{E}$, then $(b,a) \in \mathcal{E}$,
(3) transitive , that is, if $(a,b) \in \mathcal{E}$ and $(b,c) \in \mathcal{E}$, then $(a,c) \in \mathcal{E}$.

Given $\mathcal{E}$. For $x \in X$, the equivalence classes of x are denoted $[x]_{\mathcal{E}} = \{y \in X|(x,y) \in \mathcal{E}\} = \{y \in X|(y,x) \in \mathcal{E}\}$.

Proposition 1. *[21] Given $\mathcal{E}$. If $a,b \in X$, then the relationship between 2 equivalence classes is $[a]_{\mathcal{E}} = [b]_{\mathcal{E}}$ or $[a]_{\mathcal{E}} \cap [b]_{\mathcal{E}} = \emptyset$.*

From Proposition 1, the set X is partitioned into equivalence classes on the set X.

Example 1. Let $X = \{a,b,c,d,e\}$ and $\mathcal{E} = \{(a,a),(b,b),(c,c),(d,d),(e,e),(b,c),(c,b),(d,e),(e,d)\}$ on X, then 3 equivalence classes are formed in X, namely $[a]_{\mathcal{E}} = \{a\}$, $[b]_{\mathcal{E}} = \{b,c\} = [c]_{\mathcal{E}}$, $[d]_{\mathcal{E}} = \{d,e\} = [e]_{\mathcal{E}}$.

$\mathcal{B}$ is antisymmetric if $(a,b) \in \mathcal{B}$ and $(b,a) \in \mathcal{B}$, then $a = b$.

Definition 3. [20] A dominance relation $\mathcal{D}$ satisfies reflexive, antisymmetric and transitive.

Definition 4. [19] Given $\mathcal{B}$. For $a \in X$, the right neighborhood of a is denoted $a\mathcal{B} = \{b \in X|(a,b) \in \mathcal{B}\}$ and the left neighborhood of a is denoted $\mathcal{B}a = \{b \in X|(b,a) \in \mathcal{B}\}$.

Given $\mathcal{E}$. If $a \in X$, then $a\mathcal{E} = \mathcal{E}a = [a]_{\mathcal{E}}$, but this is not necessarily the case for $\mathcal{D}$. In the next section, we understand the three types of lower and upper approximations of a set using the right neighborhood of an element in the set $w.r.t$ $\mathcal{B}$. The following describe the right neighborhood properties of an element in the set $w.r.t$ $\mathcal{D}$.

Proposition 2. *Given $\mathcal{D}$. If $b \in a\mathcal{D}$, then $b\mathcal{D} \subseteq a\mathcal{D}$.*

Proof. For $b \in a\mathcal{D}$ so $(a,b) \in \mathcal{D}$. Let $c \in b\mathcal{D}$ so $(b,c) \in \mathcal{D}$. Since $\mathcal{D}$ is transitive, then $(a,c) \in \mathcal{D}$ so that $c \in a\mathcal{D}$. □

Remark 1. Given $\mathcal{D}$. If $b \notin a\mathcal{D}$, then

(1) it is not necessarily true that $a\mathcal{D} \cap b\mathcal{D} = \emptyset$,
(2) $b \in b\mathcal{D} \nsubseteq a\mathcal{D}$ but allows $a\mathcal{D} \subset b\mathcal{D}$.

This is shown in the following example:

Example 2. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $\mathcal{D} = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_5, x_5), (x_6, x_6), (x_7, x_7), (x_1, x_2), (x_1, x_3), (x_1, x_5), (x_1, x_6), (x_4, x_2), (x_6, x_2), (x_6, x_5), (x_7, x_2)\}$ in X, then the right neighborhood of the element in X is as follows:

$$\begin{aligned} x_1\mathcal{D} &= \{x_1, x_2, x_3, x_5, x_6\}, \\ x_2\mathcal{D} &= \{x_2\}, \\ x_3\mathcal{D} &= \{x_3\}, \\ x_4\mathcal{D} &= \{x_2, x_4\}, \\ x_5\mathcal{D} &= \{x_5\}, \\ x_6\mathcal{D} &= \{x_2, x_5, x_6\}, \\ x_7\mathcal{D} &= \{x_2, x_7\}. \end{aligned}$$

It is seen that

(1) $x_2\mathcal{D} \subset x_1\mathcal{D}$, $x_3\mathcal{D} \subset x_1\mathcal{D}$, $x_5\mathcal{D} \subset x_1\mathcal{D}$, $x_6\mathcal{D} \subset x_1\mathcal{D}$, $x_2\mathcal{D} \subset x_4\mathcal{D}$, $x_2\mathcal{D} \subset x_6\mathcal{D}$, $x_5\mathcal{D} \subset x_6\mathcal{D}$, $x_2\mathcal{D} \subset x_7\mathcal{D}$,

(2) $x_1 \notin x_2\mathcal{D}$, holds $x_1\mathcal{D} \cap x_2\mathcal{D} = \{x_2\}$,

(3) $x_3 \notin x_2\mathcal{D}$, holds $x_3\mathcal{D} \cap x_2\mathcal{D} = \emptyset$,

(4) $x_3 \notin x_2\mathcal{D}$, holds $x_3\mathcal{D} \not\subset x_2\mathcal{D}$,

(5) $x_1 \notin x_6\mathcal{D}$, holds $x_6\mathcal{D} \subset x_1\mathcal{D}$.

Remark 2. The dominance relation D above can be described in a directed graph where $(a, b) \in D$ means there is a directed edge from vertex a to vertex b.

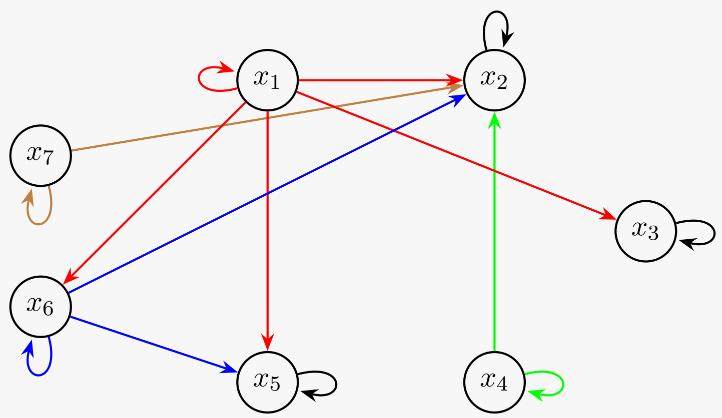

Fig. 1: Dominance Relation D on X

3. Approximation of a Set to the Equivalence Relation and the Dominance Relation

In this section, we provide the definitions of three types of approximations of a set and discuss the properties that hold and those that do not necessarily hold to binary relations, equivalence relations, and dominance relations.

The following is a definition of the lower and upper approximation of a set in Type 1 sense.

Definition 5. [19] Given $\mathcal{B}$ and $S \subseteq X$. The lower and upper approximation of S in Type 1 sense are defined as :

$$\mathcal{L}_{\mathcal{B}}(S) = \{x \in X | x\mathcal{B} \subseteq S\}$$

and

$$U_{\mathcal{B}}(S) = \{x \in X | x\mathcal{B} \cap S \neq \emptyset\}.$$

The above definition can also be stated for an quivalence relations as follows:

Definition 6. Given $\mathcal{E}$ and $S \subseteq X$. The lower and upper approximation of S in Type 1 sense are defined as :

$$\mathcal{L}_{\mathcal{E}}(S) = \{x \in X | [x]_{\mathcal{E}} \subseteq S\}$$

and

$$U_{\mathcal{E}}(S) = \{x \in X | [x]_{\mathcal{E}} \cap S \neq \emptyset\}.$$

The following gives 2 examples of lower and upper approximation of S to $\mathcal{E}$ and $\mathcal{D}$.

Example 3. Given $\mathcal{E}$ and $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$. If $S = \{x_3, x_8, x_9, x_{10}, x_{11}\}$, and equivalence classes $[x_1]_{\mathcal{E}} = \{x_1, x_2, x_4, x_5, x_7\}$, $[x_3]_{\mathcal{E}} = \{x_3, x_6\}$, $[x_8]_{\mathcal{E}} = \{x_8\}$, $[x_9]_{\mathcal{E}} = \{x_9, x_{11}\}$, $[x_{10}]_{\mathcal{E}} = \{x_{10}, x_{12}\}$, then $\mathcal{L}_{\mathcal{E}}(S) = \{x_8, x_9, x_{11}\}$ and $U_{\mathcal{E}}(S) = \{x_3, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}\}$ so that $\mathcal{L}_{\mathcal{E}}(S) \subset S \subset U_{\mathcal{E}}(S)$.

Remark 3. The Equivalence relation $\mathcal{E}$ above can be described in a directed graph where $(a, b), (b, a) \in \mathcal{E}$ means there is a directed edge from vertex a to vertex b and vice versa.

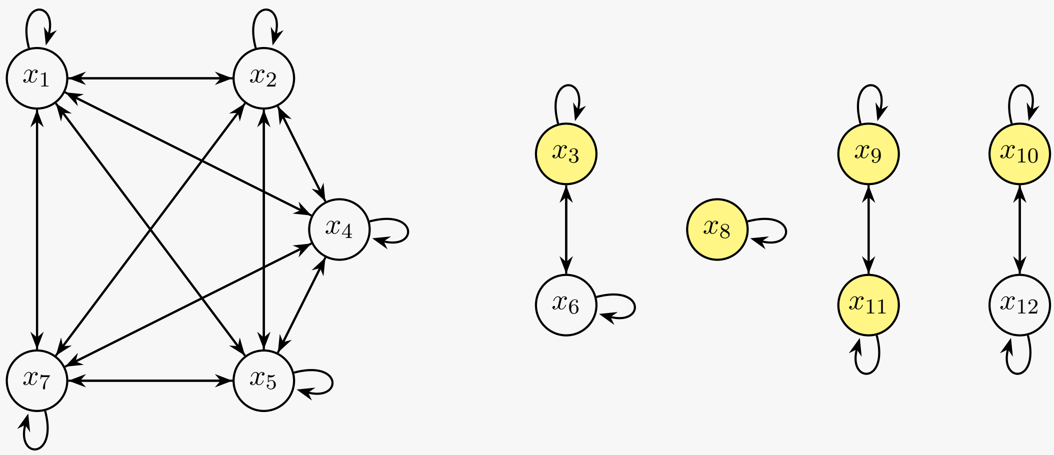

Fig. 2: Equivalence Relation $\mathcal{E}$ on X with $S = \{x_3, x_8, x_9, x_{10}, x_{11}\}$

Example 4. From Example 2, if $S = \{x_1, x_2, x_5, x_6, x_7\}$, then $\mathcal{L}_{\mathcal{D}}(S) = \{x_2, x_5, x_6, x_7\}$ and $U_{\mathcal{D}}(S) = \{x_1, x_2, x_4, x_5, x_6, x_7\} = X - \{x_3\}$ so that $\mathcal{L}_{\mathcal{D}}(S) \subset S \subset U_{\mathcal{D}}(S)$.

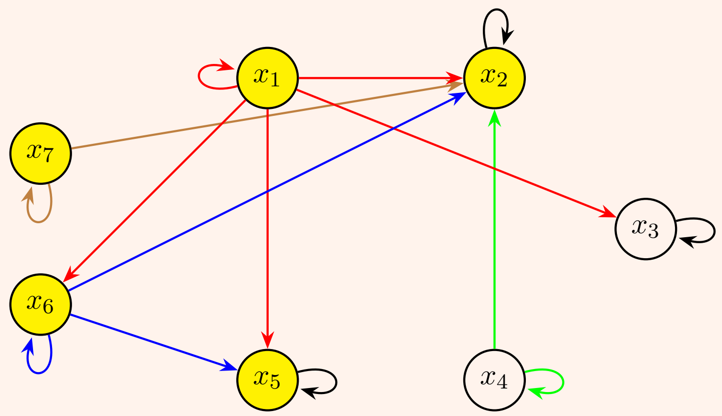

Fig. 3: Dominance Relation D on X with $S = \{x_1, x_2, x_5, x_6, x_7\}$

In Proposition 1 [19], the properties of approximation to any binary relation on a non-empty set apply so that these properties also apply to equivalence relations and domination relations as follows:

Proposition 3. *Given $\mathcal{E}$. If $S, T \subseteq X$, then*

(1) $\mathcal{L}_{\mathcal{E}}(S) = (U_{\mathcal{E}}(S^c))^c$ *and* $U_{\mathcal{E}}(S) = (\mathcal{L}_{\mathcal{E}}(S^c))^c$,
(2) $\mathcal{L}_{\mathcal{E}}(X) = X$,
(3) $U_{\mathcal{E}}(\emptyset) = \emptyset$,
(4) $\mathcal{L}_{\mathcal{E}}(S \cap T) = \mathcal{L}_{\mathcal{E}}(S) \cap \mathcal{L}_{\mathcal{E}}(T)$ *and* $U_{\mathcal{E}}(S \cap T) \subseteq U_{\mathcal{E}}(S) \cap U_{\mathcal{E}}(T)$,
(5) $\mathcal{L}_{\mathcal{E}}(S \cup T) \supseteq \mathcal{L}_{\mathcal{E}}(S) \cup \mathcal{L}_{\mathcal{E}}(T)$ *and* $U_{\mathcal{E}}(S \cup T) = U_{\mathcal{E}}(S) \cup U_{\mathcal{E}}(T)$,
(6) *If* $S \subseteq T$, *then* $\mathcal{L}_{\mathcal{E}}(S) \subseteq \mathcal{L}_{\mathcal{E}}(T)$ *and* $U_{\mathcal{E}}(S) \subseteq U_{\mathcal{E}}(T)$.

Proof. In Proposition 1 [19], binary relations have properties (1) – (6) so that these properties also apply to equivalence relations . ☐

From the paper [1] and Proposition 1 [20] it has been shown that the following properties apply:

Proposition 4. *Given $\mathcal{E}$. If $S \subseteq X$, then*

(1) $\mathcal{L}_{\mathcal{E}}(\emptyset) = \emptyset$,
(2) $\mathcal{L}_{\mathcal{E}}(S) \subseteq S$,
(3) $\mathcal{L}_{\mathcal{E}}(S) = \mathcal{L}_{\mathcal{E}}(\mathcal{L}_{\mathcal{E}}(S))$ *and* $U_{\mathcal{E}}(S) = U_{\mathcal{E}}(U_{\mathcal{E}}(S))$,
(4) $U_{\mathcal{E}}(X) = X$,
(5) $S \subseteq U_{\mathcal{E}}(S)$,
(6) $\mathcal{L}_{\mathcal{E}}(S) \subseteq U_{\mathcal{E}}(S)$.

Remark 4. Proposition 3 and 4 also hold for dominance relation.

Given $\mathcal{E}$. If $S = \bigcup_{x \in X} [x]_{\mathcal{E}}$ for some $x \in X$, then $\mathcal{L}_{\mathcal{E}}(S) = S = U_{\mathcal{E}}(S)$. The following theorem is an approximation property that applies to the equivalence relation.

Proposition 5. *[1] Given $\mathcal{E}$. If $S \subseteq X$, then*

(1) $\mathcal{L}_{\mathcal{E}}\left(U_{\mathcal{E}}\left(S\right)\right) = U_{\mathcal{E}}\left(S\right)$,

(2) $U_{\mathcal{E}}\left(\mathcal{L}_{\mathcal{E}}(S)\right) = \mathcal{L}_{\mathcal{E}}(S)$.

The above characteristics do not necessarily hold to domination relations [20].

Remark 5. Given $\mathcal{D}$. If $S \subseteq X$, then the following property does not necessarily hold

(1) $S \subseteq \mathcal{L}_{\mathcal{D}}\left(U_{\mathcal{D}}\left(S\right)\right)$,

(2) $U_{\mathcal{D}}\left(S\right) = \mathcal{L}_{\mathcal{D}}\left(U_{\mathcal{D}}\left(S\right)\right)$,

(3) $S \supseteq U_{\mathcal{D}}\left(\mathcal{L}_{\mathcal{D}}(S)\right)$,

(4) $U_{\mathcal{D}}\left(\mathcal{L}_{\mathcal{D}}(S)\right) = \mathcal{L}_{\mathcal{D}}(S)$.

This can be demonstrated through the following example:

Example 5. From Example 2, if $S = \{x_2, x_6, x_7\}$, then we have

(1) $\mathcal{L}_{\mathcal{D}}\left(S\right) = \{x_2, x_7\}$,

(2) $U_{\mathcal{D}}\left(S\right) = \{x_1, x_2, x_4, x_6, x_7\}$,

(3) $U_{\mathcal{D}}\left(\mathcal{L}_{\mathcal{D}}(S)\right) = U_{\mathcal{D}}\left(\{x_2, x_7\}\right) = \{x_1, x_2, x_4, x_6, x_7\}$,

(4) $\mathcal{L}_{\mathcal{D}}\left(U_{\mathcal{D}}\left(S\right)\right) = \mathcal{L}_{\mathcal{D}}\left(\{x_1, x_2, x_4, x_6, x_7\}\right) = \{x_2, x_4, x_7\}$.

So that

(1) $S \not\subseteq \mathcal{L}_{\mathcal{D}}\left(U_{\mathcal{D}}\left(S\right)\right)$,

(2) $U_{\mathcal{D}}\left(S\right) \neq \mathcal{L}_{\mathcal{D}}\left(U_{\mathcal{D}}\left(S\right)\right)$,

(3) $S \not\supseteq U_{\mathcal{D}}\left(\mathcal{L}_{\mathcal{D}}(S)\right)$,

(4) $U_{\mathcal{D}}\left(\mathcal{L}_{\mathcal{D}}(S)\right) \neq \mathcal{L}_{\mathcal{D}}(S)$.

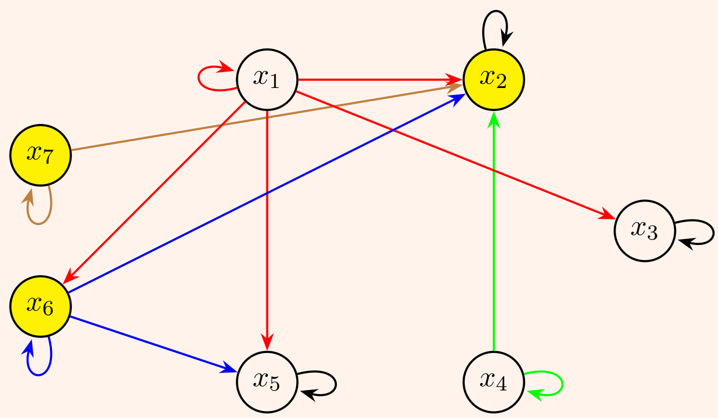

Fig. 4: Dominance Relation D on X with $S = \{x_2, x_6, x_7\}$

The following is a definition of the lower and upper approximation of a set in Type 2 sense.

Definition 7. [19] Given $\mathcal{B}$ and $S \subseteq X$. The lower and upper approximation of S in Type 2 sense are defined as :

$$\mathcal{L}^{*}_{\mathcal{B}}\left(S\right) = \cup\left\{x\mathcal{B} | x\mathcal{B} \subseteq S\right\}$$

and

$$U^{*}_{\mathcal{B}}\left(S\right) = \left(\mathcal{L}^{*}_{\mathcal{B}}\left(S^{c}\right)\right)^{c}$$

The above definition can also be stated for an quivalence relations as follows:

Definition 8. Given $\mathcal{E}$ and $S \subseteq X$. The lower and upper approximation of S in Type 2 sense are defined as :

$$\mathcal{L}_{\mathcal{E}}^{*}\left(S\right) = \cup \left\{ [x]_{\mathcal{E}} | [x]_{\mathcal{E}} \subseteq S \right\}$$

and

$$U_{\mathcal{E}}^{*}\left(S\right) = \left(\mathcal{L}_{\mathcal{E}}^{*}\left(S^{c}\right)\right)^{c}$$

In both dominance and equivalence relations, Definition 5 and Definition 7 are the same. This can be explained as follows:

Proposition 6. *Given $\mathcal{E}$. If $S \subseteq X$, then $\mathcal{L}_{\mathcal{E}}\left(S\right) = \mathcal{L}_{\mathcal{E}}^{*}\left(S\right)$ and $U_{\mathcal{E}}\left(S\right) = U_{\mathcal{E}}^{*}\left(S\right)$.*

Proof. Since for $a, b \in [x]_{\mathcal{E}}$, then we have $[x]_{\mathcal{E}} = [a]_{\mathcal{E}} = [b]_{\mathcal{E}}$ so that

$$\mathcal{L}_{\mathcal{E}}\left(S\right) = \left\{ x \in X | [x]_{\mathcal{E}} \subseteq S \right\} = \cup \left\{ [x]_{\mathcal{E}} | [x]_{\mathcal{E}} \subseteq S \right\} = \mathcal{L}_{\mathcal{E}}^{*}\left(S\right)$$

whereas by using Proposition 3 (1), we obtain

$$U_{\mathcal{E}}^{*}\left(S\right) = \left(\mathcal{L}_{\mathcal{E}}^{*}\left(S^{c}\right)\right)^{c} = \left(\mathcal{L}_{\mathcal{E}}\left(S^{c}\right)\right)^{c} = U_{\mathcal{E}}\left(S\right).$$

□

Proposition 7. *Given $\mathcal{D}$. If $S \subseteq X$, then $\mathcal{L}_{\mathcal{D}}\left(S\right) = \mathcal{L}_{\mathcal{D}}^{*}\left(S\right)$ and $U_{\mathcal{D}}\left(S\right) = U_{\mathcal{D}}^{*}\left(S\right)$.*

Proof. From Proposition 2, it is clear that $\mathcal{L}_{\mathcal{D}}\left(S\right) = \mathcal{L}_{\mathcal{D}}^{*}\left(S\right)$, whereas by using Proposition 3 (1) we obtain

$$U_{\mathcal{D}}^{*}\left(S\right) = \left(\mathcal{L}_{\mathcal{D}}^{*}\left(S^{c}\right)\right)^{c} = \left(\mathcal{L}_{\mathcal{D}}\left(S^{c}\right)\right)^{c} = U_{\mathcal{D}}\left(S\right).$$

□

The following is a definition of the lower and upper approximation of a set in Type 3 sense.

Definition 9. [19] Given $\mathcal{B}$ and $S \subseteq X$. The lower and upper approximation of S in Type 3 sense are defined as :

$$\mathcal{L}_{\mathcal{B}}^{**}\left(S\right) = \left(U_{\mathcal{B}}^{**}\left(S^{c}\right)\right)^{c}$$

and

$$U_{\mathcal{B}}^{**}\left(S\right) = \cup \left\{ x\mathcal{B} | x\mathcal{B} \cap S \neq \emptyset \right\}$$

The above definition can also be stated for an quivalence relations as follows:

Definition 10. Given $\mathcal{E}$ and $S \subseteq X$. The lower and upper approximation of S in Type 3 sense are defined as :

$$\mathcal{L}_{\mathcal{E}}^{**}\left(S\right) = \left(U_{\mathcal{E}}^{**}\left(S^{c}\right)\right)^{c}$$

and

$$U_{\mathcal{E}}^{**}\left(S\right) = \cup \left\{ [x]_{\mathcal{E}} | [x]_{\mathcal{E}} \cap S \neq \emptyset \right\}$$

In the equivalence relation, Definition 5 and Definition 9 are the same definition but not in the dominance relation, Definition 5 and Definition 9 are different definitions, it can be seen that

$U_{\mathcal{D}}(S) \subseteq U_{\mathcal{D}}^{**}(S)$.

Proposition 8. *Given $\mathcal{E}$. If $S \subseteq X$, then $\mathcal{L}_{\mathcal{E}}(S) = \mathcal{L}_{\mathcal{E}}^{**}(S)$ and $U_{\mathcal{E}}(S) = U_{\mathcal{E}}^{**}(S)$.*

Proof. If $a, b \in [x]_{\mathcal{E}}$, then $[x]_{\mathcal{E}} = [a]_{\mathcal{E}} = [b]_{\mathcal{E}}$ so that

$$U_{\mathcal{E}}^{**}(S) = \cup\{[x]_{\mathcal{E}} | [x]_{\mathcal{E}} \cap S \neq \emptyset\} = \{x \in X | [x]_{\mathcal{E}} \cap S \neq \emptyset\} = U_{\mathcal{E}}(S)$$

whereas by using Proposition 3, we obtain

$$\mathcal{L}_{\mathcal{E}}^{**}(S) = (U_{\mathcal{E}}^{**}(S^c))^c = (U_{\mathcal{E}}(S^c))^c = \mathcal{L}_{\mathcal{E}}(S).$$

□

The following example illustrates that dominance relations do not necessarily apply $\mathcal{L}_{\mathcal{D}}(S) = \mathcal{L}_{\mathcal{D}}^{**}(S)$ and $U_{\mathcal{D}}(S) = U_{\mathcal{D}}^{**}(S)$.

Example 6. In Example 2, assume that $S = \{x_2, x_7\}$, then we have

(1) $U_{\mathcal{D}}(S) = X - \{x_3, x_5\}$,
(2) $U_{\mathcal{D}}^{**}(S) = x_1\mathcal{D} \cup x_2\mathcal{D} \cup x_4\mathcal{D} \cup x_6\mathcal{D} \cup x_7\mathcal{D} = X$,
(3) $\mathcal{L}_{\mathcal{D}}(S) = \{x_2, x_7\}$,
(4) $\mathcal{L}_{\mathcal{D}}^{**}(S) = (U_{\mathcal{D}}^{**}(S^c))^c = (U_{\mathcal{D}}^{**}\{x_1, x_3, x_4, x_5, x_6\})^c = (x_1\mathcal{D} \cup x_3\mathcal{D} \cup x_4\mathcal{D} \cup x_5\mathcal{D} \cup x_6\mathcal{D})^c = (X - \{x_7\})^c = \{x_7\}$.

So that

(1) $U_{\mathcal{D}}(S) \neq U_{\mathcal{D}}^{**}(S)$,
(2) $\mathcal{L}_{\mathcal{D}}(S) \neq \mathcal{L}_{\mathcal{D}}^{**}(S)$.

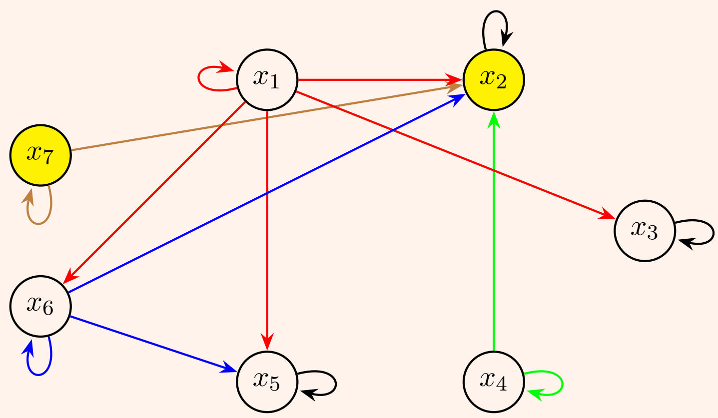

Fig. 5: Dominance Relation D on X with $S = \{x_2, x_7\}$

In Proposition 8, for the equivalence relation, we have $\mathcal{L}_{\mathcal{E}}(S) = \mathcal{L}_{\mathcal{E}}^{**}(S)$ and $U_{\mathcal{E}}(S) = U_{\mathcal{E}}^{**}(S)$ so that the properties that apply to the lower and upper approximation of S of Type 1 apply to Type 3, but they do not necessarily apply to dominance relations.

In Proposition 1 [19], the properties of approximation to any binary relation on a non-empty set apply so that these properties also apply to equivalence relations and domination relations as follows:

Proposition 9. *Given $\mathcal{E}$. If $S, T \subseteq X$, then*

(1) $\mathcal{L}_{\mathcal{E}}^{**}(S) = (U_{\mathcal{E}}^{**}(S^c))^c$ *and* $U_{\mathcal{E}}^{**}(S) = (\mathcal{L}_{\mathcal{E}}^{**}(S^c))^c$,

(2) $\mathcal{L}_{\mathcal{E}}^{**}(X) = X$ *and* $U_{\mathcal{E}}^{**}(\emptyset) = \emptyset$,

(3) $\mathcal{L}_{\mathcal{E}}^{**}(S \cap T) = \mathcal{L}_{\mathcal{E}}^{**}(S) \cap \mathcal{L}_{\mathcal{E}}^{**}(T)$ *and* $U_{\mathcal{E}}^{**}(S \cap T) \subseteq U_{\mathcal{E}}^{**}(S) \cap U_{\mathcal{E}}^{**}(T)$,

(4) $\mathcal{L}_{\mathcal{E}}^{**}(S \cup T) \supseteq \mathcal{L}_{\mathcal{E}}^{**}(S) \cup \mathcal{L}_{\mathcal{E}}^{**}(T)$ *and* $U_{\mathcal{E}}^{**}(S \cup T) = U_{\mathcal{E}}^{**}(S) \cup U_{\mathcal{E}}^{**}(T)$,

(5) *If* $S \subseteq T$, *then* $\mathcal{L}_{\mathcal{E}}^{**}(S) \subseteq \mathcal{L}_{\mathcal{E}}^{**}(T)$ *and* $U_{\mathcal{E}}^{**}(S) \subseteq U_{\mathcal{E}}^{**}(T)$,

(6) $S \subseteq \mathcal{L}_{\mathcal{E}}^{**}(U_{\mathcal{E}}^{**}(S))$,

(7) $U_{\mathcal{E}}^{**}(\mathcal{L}_{\mathcal{E}}^{**}(S)) \subseteq S$.

Proof. In Proposition 1 [19], the binary relation has properties (1) – (7) so that these properties also apply to equivalence relations . □

From the paper [1] and Proposition 9 [20], it has been shown that the following properties apply.

Proposition 10. *Given $\mathcal{E}$. If $S \subseteq X$, then*

(1) $\mathcal{L}_{\mathcal{E}}^{**}(S) \subseteq S$,

(2) $\mathcal{L}_{\mathcal{E}}^{**}(\emptyset) = \emptyset$ *and* $U_{\mathcal{E}}^{**}(X) = X$,

(3) $S \subseteq U_{\mathcal{E}}^{**}(S)$,

(4) $\mathcal{L}_{\mathcal{E}}^{**}(S) \subseteq U_{\mathcal{E}}^{**}(S)$.

Remark 6. Proposition 9 and 10 also hold for dominance relation.

If $S = \bigcup_{x \in X} [x]_{\mathcal{E}}$ for some $x \in X$, then we have $\mathcal{L}_{\mathcal{E}}^{**}(S) = \mathcal{L}_{\mathcal{E}}(S) = S = U_{\mathcal{E}}(S) = U_{\mathcal{E}}^{**}(S)$. The following theorem states the approximation properties that apply to the equivalence relation.

Proposition 11. *Given $\mathcal{E}$. If $S \subseteq X$, then*

(1) $\mathcal{L}_{\mathcal{E}}^{**}(U_{\mathcal{E}}^{**}(S)) = U_{\mathcal{E}}^{**}(S)$,

(2) $U_{\mathcal{E}}^{**}(U_{\mathcal{E}}^{**}(S)) = U_{\mathcal{E}}^{**}(S)$,

(3) $\mathcal{L}_{\mathcal{E}}^{**}(\mathcal{L}_{\mathcal{E}}^{**}(S)) = \mathcal{L}_{\mathcal{E}}^{**}(S)$,

(4) $U_{\mathcal{E}}^{**}(\mathcal{L}_{\mathcal{E}}^{**}(S)) = \mathcal{L}_{\mathcal{E}}^{**}(S)$.

The above characteristics do not necessarily hold to domination relations [20].

Remark 7. Given $\mathcal{D}$. If $S \subseteq X$, then the following properties do not necessarily hold

(1) $\mathcal{L}_{\mathcal{D}}^{**}(U_{\mathcal{D}}^{**}(S)) = U_{\mathcal{D}}^{**}(S)$,

(2) $U_{\mathcal{D}}^{**}(U_{\mathcal{D}}^{**}(S)) = U_{\mathcal{D}}^{**}(S)$,

(3) $\mathcal{L}_{\mathcal{D}}^{**}(\mathcal{L}_{\mathcal{D}}^{**}(S)) = \mathcal{L}_{\mathcal{D}}^{**}(S)$,

(4) $U_{\mathcal{D}}^{**}(\mathcal{L}_{\mathcal{D}}^{**}(S)) = \mathcal{L}_{\mathcal{D}}^{**}(S)$.

Example 7. Let $X^* = \{x_1, x_2, x_3, x_4, x_5\}$ and $\mathcal{D} = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_5, x_5), (x_1, x_2), (x_1, x_3), (x_3, x_2), (x_5, x_2)\}$ on X^*. Then, the right neighborhood of the element in X^* is as follows:

$$x_1\mathcal{D} = \{x_1, x_2, x_3\},$$
$$x_2\mathcal{D} = \{x_2\},$$
$$x_3\mathcal{D} = \{x_2, x_3\},$$
$$x_4\mathcal{D} = \{x_4\},$$
$$x_5\mathcal{D} = \{x_2, x_5\}.$$

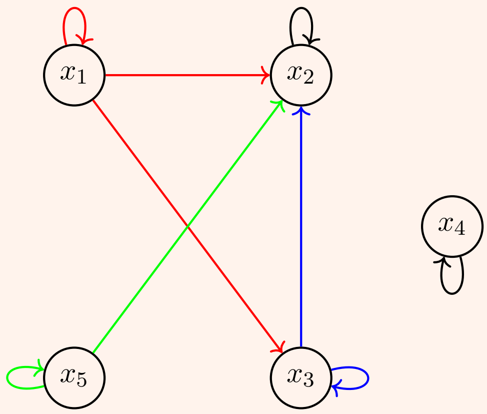

Fig. 6: Dominance Relation D on X^*

If $S = \{x_5\}$, then we have

(1) $U_{\mathcal{D}}^{**}(S) = x_5\mathcal{D} = \{x_2, x_5\}$,

(2) $U_{\mathcal{D}}^{**}(U_{\mathcal{D}}^{**}(S)) = U_{\mathcal{D}}^{**}(\{x_2, x_5\}) = x_1\mathcal{D} \cup x_2\mathcal{D} \cup x_3\mathcal{D} \cup x_5\mathcal{D} = \{x_1, x_2, x_3, x_5\}$,

(3) $\mathcal{L}_{\mathcal{D}}^{**}(U_{\mathcal{D}}^{**}(S)) = \mathcal{L}_{\mathcal{D}}^{**}(\{x_2, x_5\}) = (U_{\mathcal{D}}^{**}(\{x_1, x_3, x_4\}))^c = (x_1\mathcal{D} \cup x_3\mathcal{D} \cup x_4D)^c = (\{x_1, x_2, x_3, x_4\})^c = \{x_5\}$.

So that

(1) $U_{\mathcal{D}}^{**}(U_{\mathcal{D}}^{**}(S)) \neq U_{\mathcal{D}}^{**}(S)$,

(2) $\mathcal{L}_{\mathcal{D}}^{**}(U_{\mathcal{D}}^{**}(S)) \neq U_{\mathcal{D}}^{**}(S)$.

If $T = \{x_2, x_5\}$, then we have

(1) $\mathcal{L}_{\mathcal{D}}^{**}(T) = \mathcal{L}_{\mathcal{D}}^{**}(\{x_2, x_5\}) = (U_{\mathcal{D}}^{**}(\{x_1, x_3, x_4\}))^c = (x_1\mathcal{D} \cup x_3\mathcal{D} \cup x_4\mathcal{D})^c = (\{x_1, x_2, x_3, x_4\})^c = \{x_5\}$,

(2) $\mathcal{L}_{\mathcal{D}}^{**}(\mathcal{L}_{\mathcal{D}}^{**}(T)) = \mathcal{L}_{\mathcal{D}}^{**}(\{x_5\}) = (U_{\mathcal{D}}^{**}(\{x_1, x_2, x_3, x_4\}))^c = (x_1\mathcal{D} \cup x_2\mathcal{D} \cup x_3\mathcal{D} \cup x_4\mathcal{D} \cup x_5\mathcal{D})^c = X^c = \emptyset$,

(3) $U_{\mathcal{D}}^{**}(\mathcal{L}_{\mathcal{D}}^{**}(T)) = U_{\mathcal{D}}^{**}(\{x_5\}) = x_5\mathcal{D} = \{x_2, x_5\}$.

So that

(1) $\mathcal{L}_{\mathcal{D}}^{**}(\mathcal{L}_{\mathcal{D}}^{**}(T)) \neq \mathcal{L}_{\mathcal{D}}^{**}(T)$,

(2) $U_{\mathcal{D}}^{**}(\mathcal{L}_{\mathcal{D}}^{**}(T)) \neq \mathcal{L}_{\mathcal{D}}^{**}(T)$.

4. Conclusions

In an equivalence relation on a set, the definitions of Type 1, 2, and 3 of lower and upper approximations are identical. In a dominance relation on a set, the definitions of Type 1 and Type 2 of lower and upper approximations are identical, whereas the definitions of Type 1 and Type 3 of lower and upper approximations are different.

Based on the discussion in the third section, we found the following results. In the equivalence relations and the dominance relations, there are several properties that are the same for the lower and upper approximations of Type 1 and Type 3 of a set. In Type 1 and Type 3, there are several properties of the lower and upper approximations that hold to the equivalence relation but not necessarily hold to the dominance relation.

CRediT Authorship Contribution Statement

Dian Winda Setyawati: Conceptualization, Methodology, Validation, Formal analysis, Resources, Writing – original draft, Writing – review and editing.
Rinurwati: Conceptualization, Methodology, Formal analysis, Writing – original draft .
Soleha: Conceptualization, Methodology, Formal analysis, Resources,Writing – original draft.

Declaration of Generative AI and AI-assisted technologies

During the preparation of this work, the author used an AI tool (ChatGPT) to create LaTeX commands for graph figure. The authors are fully responsible for the final content of this manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

Not applicable. No datasets were generated or analyzed, and no code was used in this study.

References

[1] Zdzisław Pawlak. "Rough sets". In: *International Journal of Computer and Information Sciences* 11 (1982), pp. 341–356. DOI: 10.1007/BF01001956.

[2] Salvatore Greco, Benedetto Matarazzo, and Roman Slowinski. "Rough approximation by dominance relations". In: *International journal of intelligent systems* 17.2 (2002), pp. 153–171. DOI: 10.1002/int.10014.

[3] Pengfei Zhang et al. "Multi-source information fusion based on rough set theory: A review". In: *Information Fusion* 68 (2021), pp. 85–117. DOI: 10.1016/j.inffus.2020.11.004.

[4] P Mohamed Shakeel, MA Burhanuddin, and Mohammad Ishak Desa. "Automatic lung cancer detection from CT image using improved deep neural network and ensemble classifier". In: *Neural Computing and Applications* 34.12 (2022), pp. 9579–9592. DOI: 10.1007/s00521-020-04842-6.

[5] SHI Zhibo et al. "Evaluation of Airplane AC Power Quality Based on Rough Set Theory." In: *Electric Drive* 52.11 (2022). DOI: 10.19457/j.1001-2095.dqcd22754.

[6] Dragan Pamucar, Ali Ebadi Torkayesh, and Sanjib Biswas. "Supplier selection in healthcare supply chain management during the COVID-19 pandemic: a novel fuzzy rough decision-making approach". In: *Annals of Operations Research* 328.1 (2023), pp. 977–1019. DOI: 10.1007/s10479-022-04529-2.

[7] Ahmet Topal, Nilgun Guler Bayazit, and Yasemen Ucan. “A Method to Handle the Missing Values in Multi-Criteria Sorting Problems Based on Dominance Rough Sets”. In: *Mathematics* 12.18 (2024), p. 2944. DOI: 10.3390/math12182944.

[8] Ying Yu et al. “Feature selection for multi-label learning based on variable-degree multi-granulation decision-theoretic rough sets”. In: *International Journal of Approximate Reasoning* 169 (2024), p. 109181. DOI: 10.1016/j.ijar.2024.109181.

[9] Haoran Fan, Chong Wang, and Shaohua Li. “Novel method for reliability optimization design based on rough set theory and hybrid surrogate model”. In: *Computer Methods in Applied Mechanics and Engineering* 429 (2024), p. 117170. DOI: 10.1016/j.cma.2024.117170.

[10] Yefan Liu et al. “Open-circuit fault diagnosis for the inverter of inductive power transfer systems: a rough-set-theory-based method”. In: *IEEE Transactions on Instrumentation and Measurement* 73 (2024), pp. 1–11. DOI: 10.1109/TIM.2024.3368469.

[11] Sridevi Srinivasan and Shiny Duela Johnson. “Optimizing feature subset for schizophrenia detection using multichannel EEG signals and rough set theory”. In: *Cognitive Neurodynamics* 18.2 (2024), pp. 431–446. DOI: 10.1007/s11571-023-10011-x.

[12] Jie Yang et al. “Constructing three-way decision with fuzzy granular-ball rough sets based on uncertainty invariance”. In: *IEEE Transactions on Fuzzy Systems* (2025). DOI: 10.1109/TFUZZ.2025.3536564.

[13] Abdullah M Elsayed et al. “Allocation and control of multi-devices voltage regulation in distribution systems via rough set theory and grasshopper algorithm: A practical study”. In: *Results in Engineering* 25 (2025), p. 103860. DOI: 10.1016/j.rineng.2024.103860.

[14] Bijan Davvaz et al. “Near approximations in rings”. In: *Applicable Algebra in Engineering, Communication and Computing* 32 (2021), pp. 701–721. DOI: 10.1007/s00200-020-00421-3.

[15] Bijan Davvaz et al. “Near approximations in modules”. In: *Foundations of Computing and Decision Sciences* 46.4 (2021), pp. 319–337. DOI: 10.2478/fcds-2021-0020.

[16] Dian Winda Setyawati and Subiono. “On T-rough groups”. In: *7th International Conference on Mathematics: Pure, Applied and Computation:, ICoMPAC 2021*. American Institute of Physics Inc. 2022, p. 020005. DOI: 10.1063/5.0118156.

[17] Dian Winda Setyawati et al. “Application of near approximations in Cayley graphs”. In: *Discrete Mathematics, Algorithms and Applications* 15.8 (2023), p. 2250180. DOI: 10.1142/S1793830922501804.

[18] Dian Winda Setyawati, Subiono, and Bijan Davvaz. “Near-generalized approximations in groups based on a set-valued mapping”. In: *International Journal of Fuzzy Logic and Intelligent Systems* 24.2 (2024), pp. 125–140. DOI: 10.5391/IJFIS.2024.24.2.125.

[19] H. M. Abu-Donia. “Comparison between different kinds of approximations by using a family of binary relations”. In: *Knowledge-Based Systems* 21.8 (2008), pp. 911–919. DOI: 10.1016/j.knosys.2008.03.046.

[20] A. K. Salah, Essam El-Seidy, and A. S. Salama. “Properties of different types of rough approximations defined by a family of dominance relations”. In: *International Journal of Fuzzy Logic and Intelligent Systems* 22.2 (2022), pp. 193–201. DOI: 10.5391/IJFIS.2022.22.2.193.

[21] Anthony W Knapp. *Basic algebra*. Springer Science & Business Media, 2006. https://www.math.mcgill.ca/darmon/courses/17-18/algebra2/algebra2.html.