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On A-Distance (Anti)magic Labeling of Prime Graph over The Ring $\mathbb{Z}_n$

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Abstract

Let $G = (V, E)$ be a simple graph with n vertices, and let A be an abelian group of order n. Let $f : V(G) \to A$ be a bijection, and define the weight of a vertex x by $w(x) = \sum_{y \in N(x)} f(y)$, where $N(x)$ denotes the open neighborhood of x. We call f an A-distance magic labeling if all vertex weights are equal, and an A-distance antimagic labeling if the weights of distinct vertices are pairwise different. The prime graph of a commutative ring R is the simple graph with vertex set R in which two vertices x and y are adjacent if and only if $xRy = \{0_R\}$. In this paper, we investigate the existence of A-distance magic and A-distance antimagic labelings of the prime graph over the ring $\mathbb{Z}_n$ for several values of n.

Keywords: Abelian group; commutative ring; A-distance antimagic; A-distance magic; prime graph.

1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices. A *distance magic* (DM) labeling of G is a bijection from $V(G)$ to $\{1, 2, \ldots, n\}$ such that the weights of all vertices are equal, where the *weight* of a vertex is the sum of the labels of its neighbors [1–3]. A *distance antimagic* (DA) labeling of G is a bijection from $V(G)$ to $\{1, 2, \ldots, n\}$ such that distinct vertices have distinct weights [4, 5]. A graph that admits a DM (resp. DA) labeling is called a DM (resp. DA) graph. For further results on DM and DA labelings, we refer the reader to [6].

Fronček (2013) introduced the concept of DM labelings using elements of Abelian groups [7], and Cichacz *et al.* (2016) introduced the group-based notion of DA labelings [8]. Let $(A, +)$ be an Abelian group of order n, and let $f : V(G) \to A$ be a bijection. For a vertex $x \in V(G)$, the *weight* of x under f is defined by

$$w(x) = \sum_{y \in N(x)} f(y),$$

where $N(x) = \{y \in V(G) : xy \in E(G)\}$. We say that f is an A-*distance magic* (A-DM) labeling if $w(x)$ is constant for all $x \in V(G)$, and an A-*distance antimagic* (A-DA) labeling if $|\{w(x) : x \in V(G)\}| = n$. A graph that admits an A-DM (resp. A-DA) labeling is called an A-DM (resp. A-DA) graph. Clearly, any DA labeling is also a $\mathbb{Z}_n$-DA labeling, whereas the

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converse need not hold. Moreover, under an A-DA labeling, each element of A occurs as the weight of exactly one vertex of G.

Several results on A-DM labelings of regular graphs can be found in [9–12]. For product graphs, a number of results on A-DM (and A-DA) labelings can be found in [13–17]. Further results for other graph families can be found in [18–20]. An open problem in this area is the following: under what conditions does a graph G admit an A-DM (or A-DA) labeling?

Research on DM and DA labelings of graphs constructed from rings was initiated by Sivakumaran *et al.* in 2024 [21]. For other types of graph labelings, graphs arising from rings have also been studied by many researchers [22–25]. In [23, 24], Khuluq *et al.* investigated $\mathbb{Z}_k$-vertex-magic labelings of the prime graph of the ring $\mathbb{Z}_n$. The *prime graph* of a commutative ring R with zero element 0_R, denoted by $PG(R)$, is the simple graph with vertex set $V(PG(R)) = R$ in which two distinct vertices x and y are adjacent if and only if $xRy = \{0_R\}$ [26, 27]. Motivated by [21, 23, 24], in this paper we investigate the existence of DM and DA labelings of $PG(\mathbb{Z}_n)$ for all integers $n > 1$. We also investigate the existence of A-DM labeling of $PG(\mathbb{Z}_n)$ for $n = p^k, pq, pqr$ for distinct primes p, q, and r, and A-DA labeling of $PG(\mathbb{Z}_n)$ for all integers $n > 1$.

2. Methods

This study is conducted through a systematic literature review followed by rigorous theoretical analysis. The research methodology is given as follows:

1. Investigate the properties of the graphs $PG(\mathbb{Z}_n)$ for $n = p^k$, $n = pq$, and $n = pqr$, where p, q, r are distinct primes.
2. Examine the existence or nonexistence of DM and DA labelings of $PG(\mathbb{Z}_n)$.
3. Examine the existence or nonexistence of A-DM and A-DA labelings of $PG(\mathbb{Z}_n)$.
4. Determine explicit A-DM labeling of $PG(\mathbb{Z}_n)$ for $n = p^k$, $n = pq$, and $n = pqr$, where p, q, r are distinct primes.

For basic concepts and terminology in graph theory, we refer the reader to [28]. Throughout this paper, all graphs $G = (V, E)$ are finite simple graphs on n vertices, and every vertex has positive degree. The prime graph considered in this study is constructed from the ring $\mathbb{Z}_n$. The following result summarizes several fundamental properties of $PG(R)$, which will be used to determine the structure of $PG(\mathbb{Z}_n)$ and to obtain the graph constructions. Additional properties of $PG(R)$ can be found in [26].

Theorem 1. [26] *Let $(R, +, \cdot)$ be a ring with zero element 0_R and $PG(R)$ be its prime graph.*

1. *Every non-zero vertex $x \in R$ is adjacent to 0_R.*
2. *$d(x, y) = 2$ if and only if the vertices x and y are not adjacent.*
3. *If R is commutative ring unity 1_R, then the vertices x and y are adjacent if and only if $xy = 0_R$.*
4. *If $1_R \neq x \in R$ is a unit, then x is only adjacent to 0_R.*
5. *If $R = \mathbb{Z}_p$ for p primes or $p = 4$, then $PG(\mathbb{Z}_p) \cong K_{1,p-1}$.*

All groups considered in this paper are finite Abelian groups. For group-theoretic terminology, we refer the reader to [29]. The operation in an Abelian group A is written additively, its identity element is denoted by 0_A, and the inverse of $x \in A$ is denoted by $-x$. For $a \in A$, $\langle a \rangle$ denotes the additive subgroup of A generated by a.

Theorem 2. [29] *Every finite Abelian group A of order n is isomorphic to a group of the form*

$$\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}},$$

where $p_1, p_2, \ldots, p_m$ are primes, not necessarily distinct, and $n_1, n_2, \ldots, n_m$ are positive integers such that

$$n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}.$$

For example, any Abelian group of order 4 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, and any Abelian group of order 6 is isomorphic to $\mathbb{Z}_6$ since $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. To prove our results, we also use the following lemma.

Lemma 1. [30] *Let A be an Abelian group of order n.*

1. *If n is odd, then $\sum_{x \in A} x = 0_A$.*
2. *If n is even, then $\sum_{x \in A} x = \sigma(A)$, where $\sigma(A)$ denotes the sum of all elements of order 2.*

3. Results and Discussion

We first show that the graph $PG(\mathbb{Z}_n)$ admits neither DM nor DA labelings, except for some small values of n. We begin with the DM labeling of $PG(\mathbb{Z}_n)$, followed by the DA labeling of $PG(\mathbb{Z}_n)$.

Theorem 3. *For every integer $n > 1$, the graph $PG(\mathbb{Z}_n)$ is DM if and only if $n = 3$.*

Proof.

($\Rightarrow$) Suppose that $PG(\mathbb{Z}_n)$ is DM graph with DM labeling $f : V(PG(\mathbb{Z}_n)) \to \{1, 2, \ldots, n\}$ and $n \neq 3$. For $n = 2$, we have $PG(\mathbb{Z}_2) \cong P_2$, and hence it is not a DM graph. For $n \geq 4$, let x be a unit in $\mathbb{Z}_n$. By Theorem 1, we obtain

$$f(0) = w(x) = w(0) = \sum_{y \in N(0)} f(y).$$

Let $f(0) = m$. Then

$$m = \frac{n(n+1)}{2} - m \quad \Longrightarrow \quad m = \frac{n(n+1)}{4}.$$

Since $m \leq n$, it follows that $n(n+1) \leq 4n$, which implies $n \leq 3$. Contradiction with $n \geq 4$.

($\Leftarrow$) For $n = 3$, the graph $PG(\mathbb{Z}_3) \cong P_3$, and hence it is a DM graph. □

Theorem 4. *For every integer $n > 1$, the graph $PG(\mathbb{Z}_n)$ is DA if and only if $n = 2$.*

Proof.

($\Rightarrow$) Suppose that $PG(\mathbb{Z}_n)$ is DA graph with DA labeling $f : V(PG(\mathbb{Z}_n)) \to \{1, 2, \ldots, n\}$ and $n \neq 2$. Since $n \neq 2$, then $1 \neq n - 1$, 1 and $n - 1$ are units in $\mathbb{Z}_n$. Since every unit is adjacent only to 0, we obtain $w(1) = f(0) = w(n-1)$, which is a contradiction.

($\Leftarrow$) For $n = 2$, the graph $PG(\mathbb{Z}_2) \cong P_2$, and hence it is a DA graph. □

Next, we investigate the existence of A-DM labeling of the graph $PG(\mathbb{Z}_n)$ for $n = p^k$, $n = pq$, and $n = pqr$, where p, q, and r are distinct primes. We begin with the case $n = p$ and then generalize to $n = p^k$.

Theorem 5. *Let p be a prime and A be any Abelian group of same order. The graph $PG(\mathbb{Z}_p)$ is A-DM if and only if p is odd.*

Proof.

($\Rightarrow$) By contraposition, for $p = 2$, we have $PG(\mathbb{Z}_2) \cong P_2$. Therefore, $PG(\mathbb{Z}_2)$ is not an A-DM graph.

($\Leftarrow$) For any odd prime p, the graph $PG(\mathbb{Z}_p) \cong K_{1,p-1}$. Let $V(PG(\mathbb{Z}_p)) = \{0\} \cup \{x_1, x_2, \ldots, x_{p-1}\}$ and $A = \{0_A, a_1, \ldots, a_{p-1}\}$. Define a bijection $f : V(PG(\mathbb{Z}_p)) \to A$ by $f(0) = 0_A$ and $f(x_i) = a_i$ for $i = 1, 2, \ldots, p-1$. For each vertex x_i, we have $N(x_i) = \{0\}$, so

$$w(x_i) = \sum_{y \in N(x_i)} f(y) = f(0) = 0_A.$$

For the vertex 0, by Lemma 1 we obtain

$$w(0) = \sum_{i=1}^{p-1} f(x_i) = \sum_{i=1}^{p-1} a_i = 0_A,$$

Thus, f is an A-DM labeling of $PG(\mathbb{Z}_p)$. □

Theorem 6. *Let p be a prime and $k > 1$ be an integer. The graph $PG(\mathbb{Z}_{p^k})$ is A-DM for any Abelian group A of order p^k if and only if $(p, k) = (2, 2)$ or $(p, k) = (2, 3)$.*

Proof. The vertex set of $PG(\mathbb{Z}_{p^k})$ can be partitioned as

$$V(PG(\mathbb{Z}_{p^k})) = \{0\} \cup U \cup \bigcup_{i=1}^{k-1} V_{p^i},$$

where U is the set of units in $\mathbb{Z}_{p^k}$ and

$$V_{p^i} = \{mp^i : 1 \le m \le p^{k-i} - 1,\ p \nmid m\}, \qquad i = 1, 2, \ldots, k-1.$$

By Theorem 1, a unit is adjacent only to 0, and 0 is adjacent to every nonzero vertex. For $m_1 p^i \in V_{p^i}$ and $m_2 p^j \in V_{p^j}$, the vertices are adjacent if and only if $(m_1 p^i)(m_2 p^j) \equiv 0 \pmod{p^k}$, which is equivalent to $i + j \ge k$ since $p \nmid m_1 m_2$. For two distinct vertices $m_1 p^i$ and $m_2 p^i$, the vertices are adjacent if and only if $m_1 m_2 p^{2i} \equiv 0 \pmod{p^k}$, which is equivalent to $2i \ge k$ or $i \ge \lceil \frac{k}{2} \rceil$.

($\Rightarrow$) Suppose that $PG(\mathbb{Z}_{p^k})$ is an A-DM graph with A-DM labeling $f : V(PG(\mathbb{Z}_{p^k})) \to A$ and $(p, k) \ne (2, 2), (2, 3)$. We consider to following cases:

- **Case 1 :** $p = 2$ **and** $k \ge 4$
  Consider two distinct vertices $x = 2^{k-2}$ and $y = 3 \cdot 2^{k-2}$ in $V_{2^{k-2}}$. Each of x and

y is adjacent to 0 and every vertex in $\bigcup_{i=2}^{k-1} V_{2^i}$ except itself. Hence,

$$N(x) = \left(\{0\} \bigcup_{i=2}^{k-1} V_{2^i} \right) \setminus \{x\}, \text{ and } N(y) = \left(\{0\} \cup \bigcup_{i=2}^{k-1} V_{2^i} \right) \setminus \{y\}.$$

Since f is an A-DM labeling,

$$w(x) = w(y) \Rightarrow f(0) + \left(\sum_{i=2}^{k-1} \sum_{z \in V_{2^i}} f(z) \right) - f(x) = f(0) + \left(\sum_{i=2}^{k-1} \sum_{z \in V_{2^i}} f(z) \right) - f(y),$$

thus $f(x) = f(y)$. This contradicts the fact that f is a bijection.

- **Case 2 :** $p \geq 3$
  Consider two distinct vertices $x = p^{k-1}$ and $y = (p-1)p^{k-1}$ in $V_{p^{k-1}}$. Each of x and y is adjacent to every vertex in $\bigcup_{i=1}^{k-1} V_{p^i}$ except itself. Hence,

$$N(x) = \left(\{0\} \bigcup_{i=1}^{k-1} V_{p^i} \right) \setminus \{x\}, \text{ and } N(y) = \left(\{0\} \bigcup_{i=1}^{k-1} V_{p^i} \right) \setminus \{y\}.$$

Since f is an A-DM labeling,

$$w(x) = w(y) \Rightarrow f(0) + \left(\sum_{i=1}^{k-1} \sum_{z \in V_{p^i}} f(z) \right) - f(x) = f(0) + \left(\sum_{i=1}^{k-1} \sum_{z \in V_{p^i}} f(z) \right) - f(y),$$

thus $f(x) = f(y)$. This contradicts the fact that f is a bijection.

($\Leftarrow$) Suppose that $(p, k) = (2, 2)$ or $(p, k) = (2, 3)$. First, consider the case $(p, k) = (2, 2)$. By Theorem 2, every Abelian group A of order 4 is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Fig. 1 gives explicit $\mathbb{Z}_4$-DM and $\mathbb{Z}_2 \times \mathbb{Z}_2$-DM labelings of $PG(\mathbb{Z}_4)$.

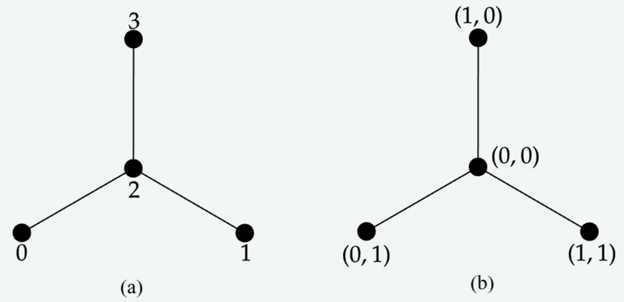

Fig. 1: (a). A $\mathbb{Z}_4$-DM labeling of $PG(\mathbb{Z}_4)$ and (b). A $\mathbb{Z}_2 \times \mathbb{Z}_2$-DM labeling of $PG(\mathbb{Z}_4)$

Hence, it suffices to show that the A-DM property is preserved under group isomorphism.
Let B be either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, and let $f : V(PG(\mathbb{Z}_4)) \to B$ be a B-DM labeling as shown in Fig. 1. If $A \cong B$, let $\varphi : B \to A$ be a group isomorphism, and define $g : V(PG(\mathbb{Z}_4)) \to A$ by $g(x) = \varphi(f(x))$. Since both f and φ are bijections, g is also a bijection. Let w_f and w_g denote the weight under f and g, respectively. For all vertices x, we have

$$w_g(x) = \sum_{y \in N(x)} g(y) = \sum_{y \in N(x)} \varphi(f(y)) = \varphi \left(\sum_{y \in N(x)} f(y) \right) = \varphi(w_f(x)).$$

Because f is a B-DM labeling, w_f is constant for all vertices x. Therefore, w_g is also constant for all vertices x, and hence g is an A-DM labeling of $PG(\mathbb{Z}_4)$. Thus, $PG(\mathbb{Z}_4)$ is an A-DM graph for every Abelian group A of order 4.
Next, consider the case $(p,k) = (2,3)$. By Theorem 2, every Abelian group A of order 8 is isomorphic to one of $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Fig. 2 gives explicit $\mathbb{Z}_8$-DM, $\mathbb{Z}_4 \times \mathbb{Z}_2$-DM, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-DM labelings of $PG(\mathbb{Z}_8)$.

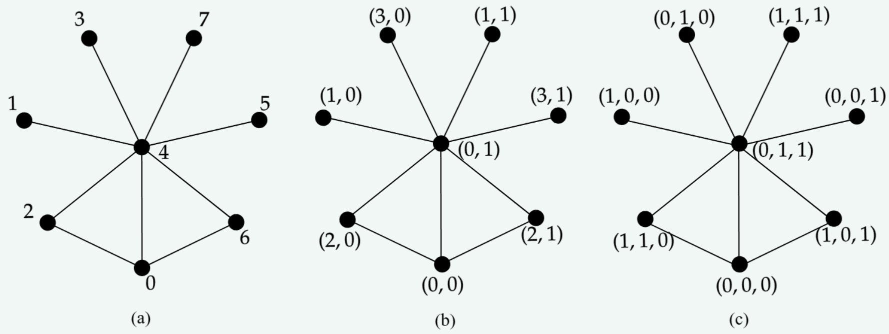

Fig. 2: (a). A $\mathbb{Z}_8$-DM labeling of $PG(\mathbb{Z}_8)$, (b). A $\mathbb{Z}_4 \times \mathbb{Z}_2$-DM labeling of $PG(\mathbb{Z}_8)$, and (c). A $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-DM labeling of $PG(\mathbb{Z}_8)$

By similar argument, it follows that $PG(\mathbb{Z}_8)$ is an A-DM graph for every Abelian group A of order 8. □

Next, we consider the case $n = pq$ for any prime p and q, where $p \neq q$. The following lemma gives a necessary condition for the graph $PG(\mathbb{Z}_{pq})$ to admit an A-DM labeling.

Lemma 2. *Let p and q be distinct primes and let A be an Abelian group of order pq. If $PG(\mathbb{Z}_{pq})$ admits an A-DM labeling f, then*

$$\sum_{y \in V_p} f(y) = \sum_{y \in V_q} f(y) = \sum_{y \in U} f(y) = 0_A,$$

where $V_p = \{mp : 1 \leq m \leq q-1\}$, $V_q = \{mq : 1 \leq m \leq p-1\}$, and U is the set of units in $\mathbb{Z}_{pq}$.

Proof. The vertex set of $PG(\mathbb{Z}_{pq})$ can be written as

$$V(PG(\mathbb{Z}_{pq})) = \{0\} \cup U \cup V_p \cup V_q,$$

where

$$V_p = \{mp : 1 \leq m \leq q-1\}, \quad V_q = \{mq : 1 \leq m \leq p-1\},$$

and U is the set of units of $\mathbb{Z}_{pq}$. Clearly, $|U| = (p-1)(q-1)$, $|V_p| = q-1$, and $|V_q| = p-1$, so $|V(PG(\mathbb{Z}_{pq}))| = pq$. By Theorem 1, every unit is adjacent only to 0, the vertex 0 is adjacent to all nonzero vertices, and each vertex in V_p is adjacent to every vertex in V_q.
Suppose that $PG(\mathbb{Z}_{pq})$ is an A-DM graph with A-DM labeling $f : V(PG(\mathbb{Z}_{pq})) \to A$. For $x \in U$, we have $N(x) = \{0\}$, hence

$$w(x) = f(0). \tag{1}$$

For $x \in V_p$, we obtain

$$w(x) = f(0) + \sum_{y \in V_q} f(y), \tag{2}$$

and for $x \in V_q$,

$$w(x) = f(0) + \sum_{y \in V_p} f(y). \tag{3}$$

Finally, for the vertex 0,

$$w(0) = \sum_{y \in U} f(y) + \sum_{y \in V_p} f(y) + \sum_{y \in V_q} f(y). \tag{4}$$

Since all weights are equal, comparing Eq. (1) with Eqs. (2) and (3) yields

$$\sum_{y \in V_p} f(y) = 0_A \qquad \text{and} \qquad \sum_{y \in V_q} f(y) = 0_A.$$

Substituting these into Eq. (4) and using $w(0) = f(0)$ gives

$$\sum_{y \in U} f(y) = 0_A.$$

□

Now, we can construct an A-DM labeling for the graph $PG(\mathbb{Z}_{pq})$. We begin with the necessary and sufficient conditions for $PG(\mathbb{Z}_{pq})$ to admit a $\mathbb{Z}_{pq}$-DM labeling.

Theorem 7. *For all distinct primes p and q, the graph $PG(\mathbb{Z}_{pq})$ is a $\mathbb{Z}_{pq}$-DM graph if and only if p and q are odd.*

Proof.

($\Rightarrow$) Suppose that $PG(\mathbb{Z}_{pq})$ is a $\mathbb{Z}_{pq}$-DM graph and that p and q are not both odd. WLOG, let $p = 2$ and $q \geq 3$. Let $f : V(PG(\mathbb{Z}_{2q})) \to \mathbb{Z}_{2q}$ be a $\mathbb{Z}_{2q}$-DM labeling of $PG(\mathbb{Z}_{2q})$. Since there is only one element in V_q, and by Lemma 2, for $x \in U$ we have

$$f(0) = w(x) = w(0) = f(q) + \sum_{y \in U} f(y) + \sum_{y \in V_p} f(y) = f(q).$$

This is a contradiction since f is a bijection.

($\Leftarrow$) Let p and q be distinct odd primes. Define a bijection $f : V(PG(\mathbb{Z}_{pq})) \to \mathbb{Z}_{pq}$ by $f(x) = x$ for all $x \in V(PG(\mathbb{Z}_{pq}))$. For $x \in U$, we have $N(x) = \{0\}$, hence

$$w(x) = f(0) = 0.$$

For $x \in V_p$, we have $N(x) = \{0\} \cup V_q = \langle q \rangle$. Since $\langle q \rangle$ is an additive subgroup of $\mathbb{Z}_{pq}$ of odd order p, Lemma 1 implies that the sum of all its elements is 0. Therefore,

$$w(x) = f(0) + \sum_{y \in V_q} f(y) = 0 + (q + 2q + \cdots + (p-1)q) = \sum_{z \in \langle q \rangle} z = 0.$$

Similarly, for $x \in V_q$, we have $N(x) = \{0\} \cup V_p = \langle p \rangle$. Since $\langle p \rangle$ is an additive subgroup of $\mathbb{Z}_{pq}$ of odd order q, Lemma 1 yields

$$w(x) = f(0) + \sum_{y \in V_p} f(y) = 0 + (p + 2p + \cdots + (q-1)p) = \sum_{z \in \langle p \rangle} z = 0.$$

Finally, for the vertex 0, by Lemma 1 we have

$$w(0) = \sum_{y \in U} f(y) + \sum_{y \in V_p} f(y) + \sum_{y \in V_q} f(y) = \left(\sum_{z \in \mathbb{Z}_{pq}} z \right) - 0 = 0.$$

Thus $w(x) = 0$ for every $x \in V(PG(\mathbb{Z}_{pq}))$, and so f is a $\mathbb{Z}_{pq}$-DM labeling of $PG(\mathbb{Z}_{pq})$. □

Corollary 1. *For all distinct primes p and q, and any Abelian group A of order pq, the graph $PG(\mathbb{Z}_{pq})$ is an A-DM graph if and only if p and q are odd.*

Proof. Let A be an Abelian group of order pq. Since p and q are distinct primes, by Theorem 2 we have $A \cong \mathbb{Z}_{pq}$.

($\Rightarrow$) Suppose that $PG(\mathbb{Z}_{pq})$ is an A-DM graph. Then there exists an A-DM labeling $f : V(PG(\mathbb{Z}_{pq})) \to A$. Let $\varphi : A \to \mathbb{Z}_{pq}$ be a group isomorphism, and define $g : V(PG(\mathbb{Z}_{pq})) \to \mathbb{Z}_{pq}$ by $g(x) = \varphi(f(x))$. Since both f and φ are bijections, g is also a bijection. Let w_f and w_g denote the weight under f and g, respectively. For all vertices x, we have

$$w_g(x) = \sum_{y \in N(x)} g(y) = \sum_{y \in N(x)} \varphi(f(y)) = \varphi\left(\sum_{y \in N(x)} f(y)\right) = \varphi(w_f(x)).$$

Thus, g is a $\mathbb{Z}_{pq}$-DM labeling of $PG(\mathbb{Z}_{pq})$. By Theorem 7, it follows that p and q are odd.

($\Leftarrow$) Suppose that p and q are odd. By Theorem 7, the graph $PG(\mathbb{Z}_{pq})$ is a $\mathbb{Z}_{pq}$-DM graph. Hence, there exists a $\mathbb{Z}_{pq}$-DM labeling $f : V(PG(\mathbb{Z}_{pq})) \to \mathbb{Z}_{pq}$. Let $\varphi : \mathbb{Z}_{pq} \to A$ be a group isomorphism, and define $g : V(PG(\mathbb{Z}_{pq})) \to A$ by $g(x) = \varphi(f(x))$. Since both f and φ are bijections, g is also a bijection. It can be shown that g is an A-DM labeling of $PG(\mathbb{Z}_{pq})$. Thus, $PG(\mathbb{Z}_{pq})$ is an A-DM graph. □

Fig. 3 is an example of $\mathbb{Z}_{15}$-DM and $\mathbb{Z}_3 \times \mathbb{Z}_5$-DM labelings of $PG(\mathbb{Z}_{15})$. The isomorphism $\varphi : \mathbb{Z}_{15} \to \mathbb{Z}_3 \times \mathbb{Z}_5$ is defined by $\varphi(x) = (x \mod 3, x \mod 5)$ for all $x \in \mathbb{Z}_{15}$.

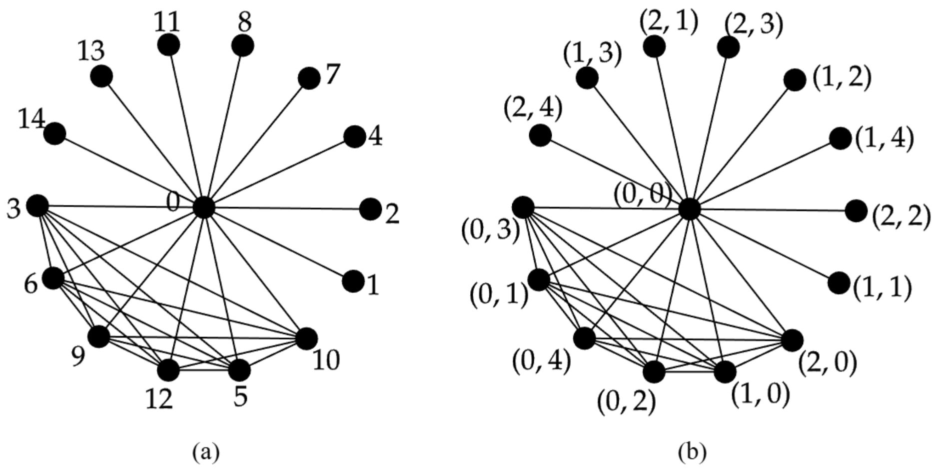

Fig. 3: (a). $\mathbb{Z}_{15}$-DM labeling of $PG(\mathbb{Z}_{15})$ and (b). $\mathbb{Z}_3 \times \mathbb{Z}_5$-DM labeling of $PG(\mathbb{Z}_{15})$

Next, we consider the case $n = pqr$ where p, q, and r are distinct primes. The following lemma gives a necessary condition for the graph $PG(\mathbb{Z}_{pqr})$ to admit an A-DM labeling.

Lemma 3. *Let p, q, r be distinct primes and let A be an Abelian group of order pqr. If $PG(\mathbb{Z}_{pqr})$ admits an A-DM labeling f, then*

$$\sum_{y\in V_p} f(y) = \sum_{y\in V_q} f(y) = \sum_{y\in V_r} f(y) = \sum_{y\in V_{pq}} f(y) = \sum_{y\in V_{pr}} f(y) = \sum_{y\in V_{qr}} f(y) = \sum_{y\in U} f(y) = 0_A,$$

where U is the set of units in $\mathbb{Z}_{pqr}$ and

$$\begin{aligned} V_p &= \{mp : 1 \le m \le qr-1,\ q\nmid m,\ r\nmid m\},\\ V_q &= \{mq : 1 \le m \le pr-1,\ p\nmid m,\ r\nmid m\},\\ V_r &= \{mr : 1 \le m \le pq-1,\ p\nmid m,\ q\nmid m\},\\ V_{pq} &= \{mpq : 1 \le m \le r-1\},\\ V_{pr} &= \{mpr : 1 \le m \le q-1\},\\ V_{qr} &= \{mqr : 1 \le m \le p-1\}. \end{aligned}$$

Proof. The vertex set of $PG(\mathbb{Z}_{pqr})$ can be partitioned as

$$V(PG(\mathbb{Z}_{pqr})) = \{0\} \cup U \cup V_p \cup V_q \cup V_r \cup V_{pq} \cup V_{pr} \cup V_{qr}.$$

By Theorem 1, every unit is adjacent only to 0 and 0 is adjacent to all nonzero vertices. Moreover, for nonzero $x, y \in \mathbb{Z}_{pqr}$, the vertices x and y are adjacent if and only if $xy \equiv 0 \pmod{pqr}$. From this, the following adjacencies are immediate:

- V_p is adjacent exactly to V_{qr} (and to 0),
- V_q is adjacent exactly to V_{pr} (and to 0),
- V_r is adjacent exactly to V_{pq} (and to 0),
- each of V_{pq}, V_{pr}, V_{qr} is adjacent to the other two (and to 0).

Suppose that $f : V(PG(\mathbb{Z}_{pqr})) \to A$ is an A-DM labeling. Then for each $x \in U$, we have $N(x) = \{0\}$ and hence

$$w(x) = f(0). \tag{5}$$

For $x \in V_p$, we have $N(x) = \{0\} \cup V_{qr}$, so

$$w(x) = f(0) + \sum_{y\in V_{qr}} f(y). \tag{6}$$

Similarly, for $x \in V_q$ and $x \in V_r$,

$$w(x) = f(0) + \sum_{y\in V_{pr}} f(y), \tag{7}$$

$$w(x) = f(0) + \sum_{y\in V_{pq}} f(y), \tag{8}$$

respectively. For $x \in V_{pq}$, we have $N(x) = \{0\} \cup V_r \cup V_{pr} \cup V_{qr}$, and thus

$$w(x) = f(0) + \sum_{y\in V_r} f(y) + \sum_{y\in V_{pr}} f(y) + \sum_{y\in V_{qr}} f(y). \tag{9}$$

Likewise,

$$w(x) = f(0) + \sum_{y\in V_q} f(y) + \sum_{y\in V_{pq}} f(y) + \sum_{y\in V_{qr}} f(y) \quad \text{for } x \in V_{pr}, \tag{10}$$

and

$$w(x) = f(0) + \sum_{y\in V_p} f(y) + \sum_{y\in V_{pq}} f(y) + \sum_{y\in V_{pr}} f(y) \quad \text{for } x \in V_{qr}. \tag{11}$$

Finally, for the vertex 0,

$$w(0) = \sum_{y\in U} f(y)+\sum_{y\in V_p} f(y)+\sum_{y\in V_q} f(y)+\sum_{y\in V_r} f(y)+\sum_{y\in V_{pq}} f(y)+\sum_{y\in V_{pr}} f(y)+\sum_{y\in V_{qr}} f(y). \tag{12}$$

Since f is an A-DM labeling, all weights are equal. Comparing Eq. (5) with Eqs. (6), (7), and (8) yields

$$\sum_{y\in V_{qr}} f(y) = \sum_{y\in V_{pr}} f(y) = \sum_{y\in V_{pq}} f(y) = 0_A.$$

Comparing Eq. (5) with Eqs. (9), (10), and (11) then gives

$$\sum_{y\in V_p} f(y) = \sum_{y\in V_q} f(y) = \sum_{y\in V_r} f(y) = 0_A.$$

Substituting these equalities into Eq. (12) and using $w(0) = f(0)$ yields

$$\sum_{y\in U} f(y) = 0_A.$$

□

Now, we construct an A-DM labeling of $PG(\mathbb{Z}_{pqr})$ for distinct primes p, q, and r. We begin with the necessary and sufficient conditions for $PG(\mathbb{Z}_{pqr})$ to admit a $\mathbb{Z}_{pqr}$-DM labeling.

Theorem 8. *For all distinct primes p, q, and r, the graph $PG(\mathbb{Z}_{pqr})$ is a $\mathbb{Z}_{pqr}$-DM graph if and only if p, q, and r are odd.*

Proof.

(⇒) Suppose that $PG(\mathbb{Z}_{pqr})$ is a $\mathbb{Z}_{pqr}$-DM graph and that p, q, and r are not all odd. WLOG, let $p = 2$ and $q, r \geq 3, q \neq r$. Let $f : V(PG(\mathbb{Z}_{2qr})) \to \mathbb{Z}_{2qr}$ be a $\mathbb{Z}_{2qr}$-DM labeling of $PG(\mathbb{Z}_{2qr})$. Since there is only one element in V_{qr}, and by Lemma 3 we have

$$w(0) = \sum_{y\in U} f(y)+\sum_{y\in V_2} f(y)+\sum_{y\in V_q} f(y)+\sum_{y\in V_r} f(y)+\sum_{y\in V_{2q}} f(y)+\sum_{y\in V_{2r}} f(y)+f(qr) = f(qr).$$

Since all vertex weights are equal, for $x \in U$ we have

$$f(0) = w(x) = w(0) = f(qr).$$

This contradicts the fact that f is a bijection.

(⇐) Let p, q, and r be distinct odd primes. Define a bijection $f : V(PG(\mathbb{Z}_{pqr})) \to \mathbb{Z}_{pqr}$ by $f(x) = x$ for all $x \in V(PG(\mathbb{Z}_{pqr}))$. Using the partition of $V(PG(\mathbb{Z}_{pqr}))$ defined above, we consider the sets U, V_p, V_q, V_r, V_{pq}, V_{pr}, and V_{qr}. For each $x \in U$, we have $N(x) = \{0\}$ and hence $w(x) = f(0) = 0$. For $x \in V_p$, we have $N(x) = \{0\} \cup V_{qr} = \langle qr\rangle$. Since $\langle qr\rangle$ is an additive subgroup of $\mathbb{Z}_n$ of odd order p, Lemma 1 implies that

$$w(x) = f(0) + \sum_{y\in V_{qr}} f(y) = \sum_{z\in\langle qr\rangle} z = 0.$$

Similarly, for $x \in V_q$ and $x \in V_r$,

$$w(x) = f(0) + \sum_{y\in V_{pr}} f(y) = \sum_{z\in\langle pr\rangle} z = 0,$$

$$w(x) = f(0) + \sum_{y \in V_{pq}} f(y) = \sum_{z \in \langle pq \rangle} z = 0,$$

respectively. For $x \in V_{pq}$, we have $N(x) = \{0\} \cup V_r \cup V_{pr} \cup V_{qr} = \langle r \rangle$. Since $\langle r \rangle$ is an additive subgroup of $\mathbb{Z}_{pqr}$ of odd order pq, by Lemma 1 we obtain

$$w(x) = f(0) + \sum_{y \in V_r} f(y) + \sum_{y \in V_{pr}} f(y) + \sum_{y \in V_{qr}} f(y) = \sum_{z \in \langle r \rangle} z = 0.$$

Likewise, for $x \in V_{pr}$, we have

$$w(x) = f(0) + \sum_{y \in V_q} f(y) + \sum_{y \in V_{pq}} f(y) + \sum_{y \in V_{qr}} f(y) = \sum_{z \in \langle q \rangle} z = 0,$$

and for $x \in V_{qr}$, we have

$$w(x) = f(0) + \sum_{y \in V_p} f(y) + \sum_{y \in V_{pq}} f(y) + \sum_{y \in V_{pr}} f(y) = \sum_{z \in \langle p \rangle} z = 0.$$

Finally, for the vertex 0, by Lemma 1 we have

$$\begin{aligned} w(0) &= \sum_{y \in U} f(y) + \sum_{y \in V_p} f(y) + \sum_{y \in V_q} f(y) + \sum_{y \in V_r} f(y) + \sum_{y \in V_{pq}} f(y) + \sum_{y \in V_{pr}} f(y) + \sum_{y \in V_{qr}} f(y) \\ &= \left(\sum_{z \in \mathbb{Z}_{pqr}} z \right) - 0 = 0. \end{aligned}$$

Thus $w(x) = 0$ for every $x \in V(PG(\mathbb{Z}_{pqr}))$, and so f is a $\mathbb{Z}_{pqr}$-DM labeling of $PG(\mathbb{Z}_{pqr})$. □

Corollary 2. *For all distinct primes p, q, and r and any Abelian group A of order pqr, the graph $PG(\mathbb{Z}_{pqr})$ is an A-DM graph if and only if p, q, and r are odd.*

Proof. Let A be an Abelian group of order pqr. Since p, q, and r are distinct primes, Theorem 2 implies that $A \cong \mathbb{Z}_{pqr}$.

($\Rightarrow$) Suppose that $PG(\mathbb{Z}_{pqr})$ is an A-DM graph. Then there exists an A-DM labeling $f : V(PG(\mathbb{Z}_{pqr})) \to A$. Let $\varphi : A \to \mathbb{Z}_{pqr}$ be a group isomorphism, and define $g : V(PG(\mathbb{Z}_{pqr})) \to \mathbb{Z}_{pqr}$ by $g(x) = \varphi(f(x))$. Since both f and φ are bijections, g is also a bijection. Let w_f and w_g denote the weight under f and g, respectively. For all vertices x, we have

$$w_g(x) = \sum_{y \in N(x)} g(y) = \sum_{y \in N(x)} \varphi(f(y)) = \varphi \left(\sum_{y \in N(x)} f(y) \right) = \varphi(w_f(x)).$$

Thus, g is a $\mathbb{Z}_{pqr}$-DM labeling of $PG(\mathbb{Z}_{pqr})$. By Theorem 8, it follows that p, q, and r are odd.

($\Leftarrow$) Suppose that p, q, and r are odd. By Theorem 8, the graph $PG(\mathbb{Z}_{pqr})$ is a $\mathbb{Z}_{pqr}$-DM graph. Hence, there exists a $\mathbb{Z}_{pqr}$-DM labeling $f : V(PG(\mathbb{Z}_{pqr})) \to \mathbb{Z}_{pqr}$. Let $\varphi : \mathbb{Z}_{pqr} \to A$ be a group isomorphism, and define $g : V(PG(\mathbb{Z}_{pqr})) \to A$ by $g(x) = \varphi(f(x))$. Since both f and φ are bijections, g is also a bijection. It can be shown that g is an A-DM labeling of $PG(\mathbb{Z}_{pqr})$. Thus, $PG(\mathbb{Z}_{pqr})$ is an A-DM graph. □

Corollary 1 and 2 indicate that $PG(\mathbb{Z}_n)$ is an A-DM graph when n is a product of odd distinct primes. This motivates the following conjecture.

Conjecture 1. *For any distinct primes $p_1, p_2, \ldots, p_k$ and any Abelian group A of order $p_1p_2\ldots p_k$, the graph $PG(\mathbb{Z}_{p_1p_2\ldots p_k})$ is an A-DM graph if and only if $p_1, p_2, \ldots, p_k$ are odd.*

Finally, we investigate A-DA labeling of $PG(\mathbb{Z}_n)$ for all integer $n > 1$. The next result gives a complete characterization of A-DA labelings for $PG(\mathbb{Z}_n)$.

Theorem 9. *Let $n > 1$ be an integer and let A be an Abelian group of order n. The graph $PG(\mathbb{Z}_n)$ is A-DA if and only if $n = 2$.*

Proof. The proof follows the same argument as Theorem 4. □

4. Conclusion

In this paper, we prove that $PG(\mathbb{Z}_n)$ is a DM graph if and only if $n = 3$, and a DA graph if and only if $n = 2$. We also show that $PG(\mathbb{Z}_n)$ is an A-DA graph for any Abelian group A of order n if and only if $n = 2$. For prime powers, we show that $PG(\mathbb{Z}_{p^k})$ is an A-DM graph for any Abelian group A of order p^k if and only if either p is an odd prime and $k = 1$, or $p = 2$ and $k \in \{2, 3\}$. We further show that $PG(\mathbb{Z}_{pq})$ is an A-DM graph for any Abelian group A of order pq if and only if p and q are distinct odd primes, and that $PG(\mathbb{Z}_{pqr})$ is an A-DM graph for any Abelian group A of order pqr if and only if p, q, and r are distinct odd primes. Motivated by these results, we conjecture that the graph $PG(\mathbb{Z}_{p_1p_2\cdots p_k})$ is an A-DM graph for any Abelian group A of order $p_1p_2\cdots p_k$ if and only if $p_1, p_2, \ldots, p_k$ are distinct odd primes.

Declaration of Generative AI and AI-assisted technologies

No generative AI or AI-assisted technologies were used during the preparation of this manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

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