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# Spectral Analysis of a Cyclic Tank Mixing System

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**Abstract**

In this paper, we study a cyclic mixing tank system consisting of $n$ identical tanks arranged in a closed-loop configuration, each operating under constant volume with equal inflow and outflow rates. The model is further generalized to an $m$-layer structure, where each layer comprises $n$ circularly interconnected tanks. Based on mass balance principles, the concentration dynamics are formulated as a linear system of first-order ordinary differential equations. By exploiting the structured form of the system matrix, we derive an explicit analytical solution and obtain a closed-form characterization of its spectrum. We show that the eigenvalues of the system are explicitly given by $\lambda_{k,j} = \alpha(-2 + e^{\frac{2\pi i(k-1)}{n}})$, which inherently guarantees the decay of all modes. In particular, the solution exhibits a polynomial-exponential form associated with the multilayer cyclic tank. We also establish that the system is exponentially stable under conditions ensuring decay of all modes. Finally, we also present numerical simulations to validate the analytical results and to illustrate the transient dynamics of the multilayer system.

**Keywords:** cyclic mixing tank; $m$-layer; linear ordinary differential equations; analytical solution; exponential stability.



## 1. Introduction

Interconnected tank models constitute a classical system for describing mass transport and mixing processes in compartmental systems. Such models arise naturally in chemical engineering, environmental systems, pharmacokinetics, and process industries, where material transfer between well-mixed compartments is modeled through balance laws and differential equations [1, 2]. From a mathematical perspective, these systems are formulated as linear ordinary differential equations derived from mass conservation principles [3–5].

Mixing problems involving multiple interconnected tanks have been studied in the literature. A comprehensive discussion of multi-tank configurations and their mathematical structure can be found in [6]. More recent work has examined the stability of cascading tank systems [7] and developed graph-theoretic representations of mixing processes using quasi-digraph structures [8]. In a related direction, a mixture purification model based on a cascading tank configuration was analyzed in [9], emphasizing the role of flow structure in determining long-term behavior.

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Consider a cyclic configuration consisting of $n$ identical tanks arranged in a closed circular structure with pure water inflow and outflow. Each tank has constant volume $V$ and is assumed to be perfectly mixed, so that the concentration of the dissolved substance tends to zero as time approaches infinity. The fluid flows unidirectionally along the circular arrangement: every tank receives inflow from the preceding tank and pure water inflow and transfers its mixture to the subsequent tank at a constant rate $q$ as follows.

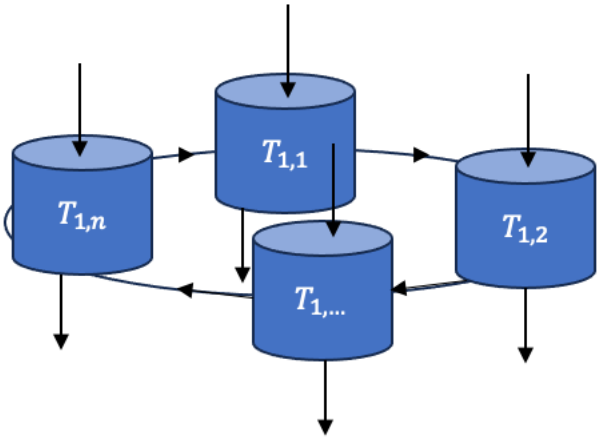


**Fig. 1:** Cyclic Tank Model

Let $x_k(t)$ denote the concentration in tank $k$ at time $t$, for $k = 1, \dots, n$. Applying the principle of mass balance yields

$$\frac{dx_k}{dt} = \alpha \left( x_{k-1} - 2x_k \right), \qquad k = 1, \dots, n,$$

with the cyclic convention $x_0 = x_n$ and $\alpha = \dfrac{q}{V}$, the system can be written compactly as

$$\dot{\mathbf{x}}(t) = \alpha(P - 2I_n)\mathbf{x}(t), \tag{1}$$

where $P$ is the cyclic permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The matrix $P - 2I_n$ is circulant and therefore diagonalizable by the Fourier matrix order $n$ [10, 11].

Many existing studies focus primarily on single-layer configurations or cascading tank structures [6, 7, 9], where the analysis is relatively straightforward due to the absence of vertical interactions between layers. While the multilayer configuration has been introduced in [8], the mathematical analysis of such systems remains limited, particularly in terms of obtaining explicit analytical solutions and understanding the underlying spectral structure.

In particular, previous works have not fully explored how the interplay between horizontal mixing and vertical coupling influences the dynamic behavior of the system. This gap becomes significant when considering more complex transport mechanisms, where interactions across layers may lead to nontrivial transient dynamics and altered stability properties.

In this article, we study a specific multilayer cyclic tank configuration and develop a structured analytical to derive a closed-form solution as the following figure. The proposed approach

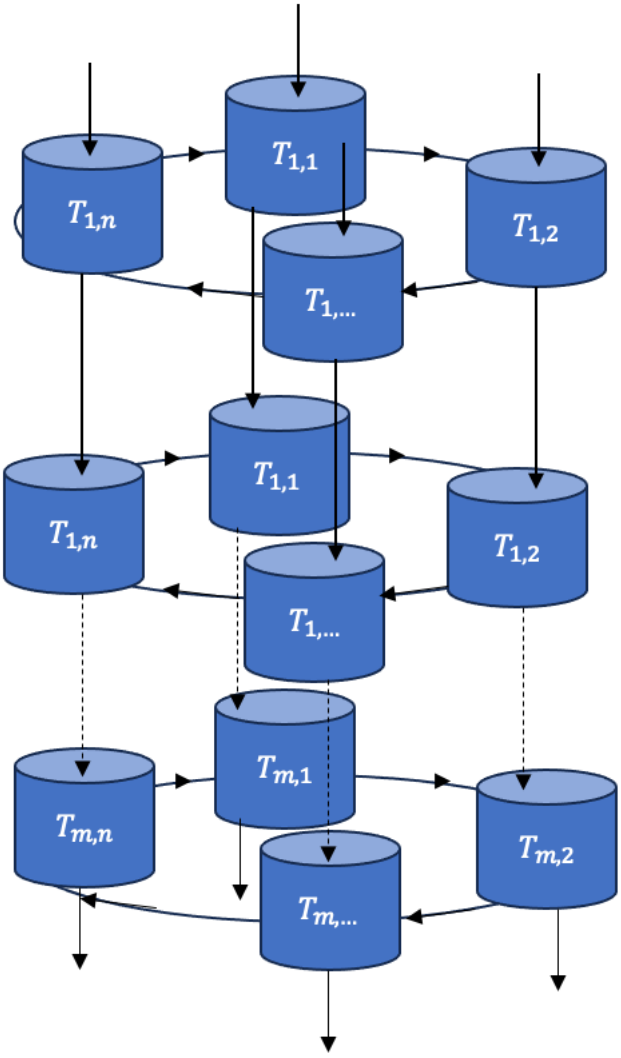


**Fig. 2:** A system comprising $m-$layer, where every layer consist of $n$ tanks

highlights the role of matrix structure, including its spectral properties, in determining the evolution of the system.

The multilayer model is derived based on the principle of mass balance. Let $x_{k,j}(t)$ denote the concentration in tank $k$ of layer $j$. The rate of change of concentration is determined by the balance between horizontal inflow from neighboring tanks within the same layer, vertical inflow from adjacent layers, and total outflow.

Under the assumptions of constant flow rate and identical tank volumes, the governing equations can be expressed in terms of contributions from horizontal and vertical transport. This formulation provides the physical foundation for the block matrix representation introduced in the subsequent section.

In addition to the inter-tank transfer, each tank interacts with the external environment through additional inflow and outflow mechanisms. The external inflow ensures that the liquid volume remains constant, while the additional outflow represents mass loss from the system. Consequently, although the volume of each tank is preserved, the total amount of dissolved substance evolves over time due to transport and removal effects.

The coefficients in the system matrix are constructed to reflect the underlying physical flow mechanisms in the tank system. In particular, the diagonal entries represent the total rate of mass leaving each tank, which consists of horizontal outflow to the next tank in the same layer and vertical transfer to adjacent layers. This total outflow determines the rate at which

concentration decreases within each tank.

The off-diagonal entries correspond to inflow contributions. Horizontal inflow arises from the preceding tank within the same layer, while vertical inflow represents transfer from the corresponding tank in an adjacent layer. These inflow terms increase the concentration in the receiving tank and ensure that mass is redistributed throughout the system.

Under the assumptions of identical flow rates and equal tank volumes, all coefficients are uniform and proportional to the ratio $\alpha = \frac{q}{V}$. This guarantees that the mathematical formulation is fully consistent with the conservation of mass principle, where every decrease in one compartment corresponds to a transfer to another compartment or removal through the outflow.

The objective of this study is to formulate a mathematical model corresponding to the cyclic multi-layer tank configuration depicted in Figure 1 and Figure 2 to derive an explicit solution for system of differential equation.

This paper is organized into four main sections. In the first section, we provide the introduction, outlining the background and motivation of the study. The second section focuses on the methods, describing the approach and techniques used in the research. The third section presents the results and discussion, analyzing the findings in detail and interpreting their significance. Finally, the fourth section concludes the paper, summarizing the key outcomes and highlighting potential implications.

## 2. Methods

This research adopts a mathematical modeling and analytical approach to study a cyclic mixing tank system with constant volume and equal inflow and outflow rates. The methodology consists of four main stages: model formulation, multilayer generalization, analytical solution, and numerical simulation.

### 2.1. Model Formulation

We consider a cyclic system consisting of $n$ identical tanks arranged in a closed-loop configuration. Each tank operates under constant volume $V$ and constant flow rate $q$. The inflow and outflow rates are equal in every tank to ensure volume conservation.

Let $x_k(t)$ denote the solute concentration in the $k$-th tank at time $t$, for $k = 1, 2, \ldots, n$. Under the assumptions of perfect mixing and constant flow rates, the governing equations are derived using the principle of mass balance (rate of accumulation equals inflow minus outflow).

This yields a system of first-order linear ordinary differential equations describing the concentration dynamics.

### 2.2. Multi-Layer Generalization

The model is extended to an $m$-layer configuration, where each layer consists of $n$ cyclically interconnected tanks. The same mass balance principle is applied to every tank in each layer, resulting in a larger coupled linear system.

The entire system is expressed in compact matrix form as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where $\mathbf{x}(t)$ represents the concentration vector of all tanks in the multilayer structure, and $A$ is the system matrix determined by the cyclic interconnection pattern.

### 2.3. Analytical Solution

To obtain the analytical solution, the system is written in matrix form and solved using the matrix exponential:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0).$$

The structure of the system matrix is analyzed to determine its eigenvalues and eigenvectors. By employing spectral decomposition techniques, the solution is expressed as a superposition of exponential modes associated with the eigenvalues of $A$. This approach provides an explicit analytical representation of the concentration dynamics and enables characterization of the system behavior over time.

### 2.4. Numerical Simulation

Finally, numerical simulations are performed to illustrate the analytical results. The simulations demonstrate the temporal evolution of concentrations and confirm the theoretical predictions derived from the analytical solution.

## 3. Results and Discussion

In this section, we analyze the concentration in the cyclic system. We begin by considering the single-layer case. We assume that the tank volumes remain constant by setting the horizontal and vertical flow rates to be equal.

The discussion is then extended to the $m$-layer system. We examine how concentration propagates between tanks both horizontally and vertically. Furthermore, we analyze the concentration peaks in each tank using an approximation approach.

### 3.1. Circular Tank with 1 Layer

An explicit representation of the solution is obtained by analyzing the spectral properties of the system matrix induced by the cyclic tank configuration. Because of the symmetry of the interconnection structure, the matrix has a circulant form and is diagonalizable by the discrete Fourier basis. This property enables explicit computation of the eigenvalues and allows the solution to be expressed through spectral decomposition. The main result is stated in the following theorem.

**Theorem 1.** *Consider the cyclic tank system satisfying Eq. (1). Then, for any initial condition $\mathbf{x}(0) = \mathbf{x}_0$, the solution is given by*

$$\mathbf{x}(t) = \sum_{k=1}^{n} c_k \exp\Big(\alpha(-2+\omega^{k-1})t\Big)\,\mathbf{v}_k,$$

*where $\omega = e^{\frac{2\pi i}{n}}$, $\mathbf{v}_k = (1, \omega^{-(k-1)}, \omega^{-2(k-1)}, \ldots, \omega^{-(n-1)(k-1)})^T$, $k = 1, \ldots, n$ are the discrete Fourier eigenvectors of $A$, and the constants $c_k$ are determined by the initial condition.*

*Proof.* The system is characterized by the coefficient matrix $A$

$$A = \alpha \begin{pmatrix} -2 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix} = \alpha(P - 2I_n),$$

where $P$ is the cyclic permutation matrix.
Let $\omega = e^{\frac{2\pi i}{n}}$ and define $\mathbf{v}_k = (1, \omega^{-(k-1)}, \omega^{-2(k-1)}, \ldots, \omega^{-(n-1)(k-1)})^T$, $k = 1, \ldots, n$.

We obtain that

$$A\mathbf{v}_k = \alpha(-2I_n + P)\mathbf{v}_k = \alpha(-2 + \omega^{k-1})\mathbf{v}_k.$$

Thus the eigenvalues are

$$\lambda_k = \alpha(-2 + \omega^{k-1}) = \alpha(-2 + e^{\frac{2\pi i(k-1)}{n}}), \quad k = 1, \ldots, n.$$

Since $\omega^{k-1}$ lies on the unit circle, we have $\mathrm{Re}(\lambda_k) = \alpha(-2 + \cos\left(\frac{2\pi(k-1)}{n}\right))$. Expanding the initial condition in the Fourier basis $\mathbf{x}_0 = \sum_{k=1}^{n} c_k \mathbf{v}_k$, we obtain the solution

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = \sum_{k=1}^{n} c_k e^{\alpha(-2+\omega^{k-1})t}\mathbf{v}_k.$$

Our proof is completed. □

We provide numerical simulation results to validate and visualize analytical solutions obtained in the previous section. In particular, they demonstrate how the explicit solution structure and spectral properties manifest in the time evolution of the system.

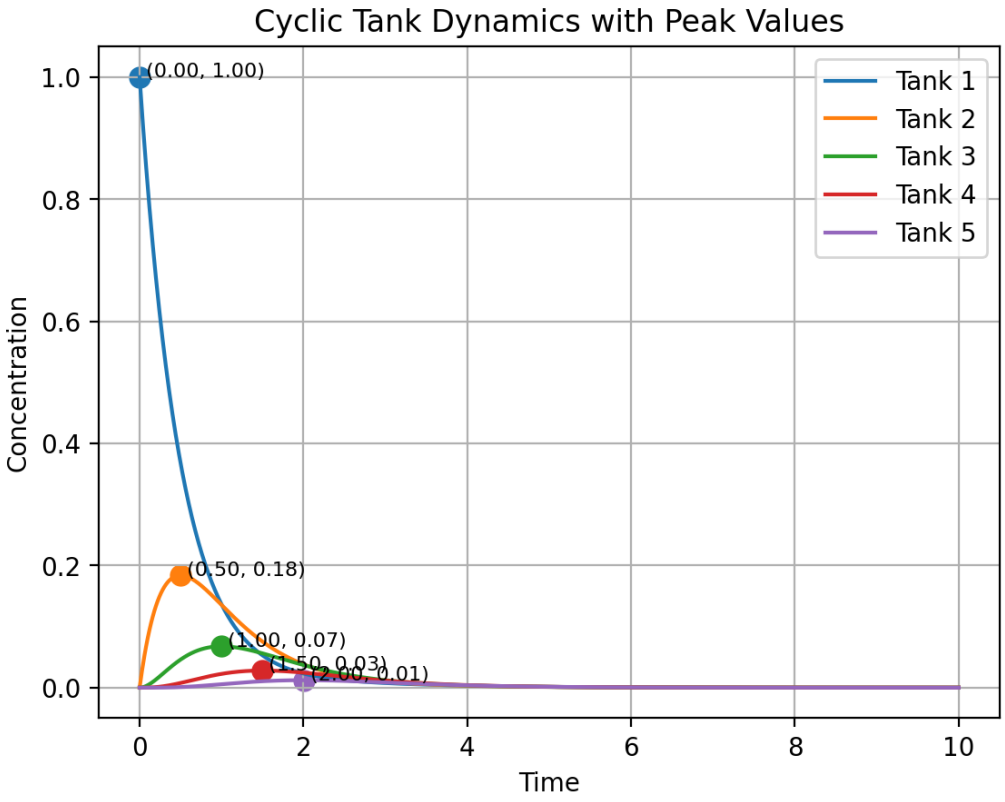


**Fig. 3:** Spectral Solution with initial condition $(1, 0, 0, 0, 0)^T$ and the peak time

Fig. 3 illustrates the concentration dynamics in a single-layer system with $n = 5$ tanks and $\alpha = 1$, starting from the initial condition $(1, 0, 0, 0, 0)$. The concentration in the first tank decreases monotonically, while downstream tanks exhibit a transient peak before decaying to the steady state.

From Theorem 1, we obtain solution of the system is

$$\mathbf{x}(t) = \frac{1}{5}\sum_{k=1}^{5} e^{(-2+\omega^{k-1})t}\begin{pmatrix}1\\ \omega^{-(k-1)}\\ \omega^{-2(k-1)}\\ \omega^{-3(k-1)}\\ \omega^{-4(k-1)}\end{pmatrix}, \text{with } \omega = e^{\frac{2\pi i}{5}}.$$

Since $\omega^{k-1}$ for $k = 1, 2, 3, 4, 5$ the root of unity [12] of $z^5 = 1$, we obtain that $z = \omega^{k-1}$ for $k \neq 1$ are the root of $z^4 + z^3 + z^2 + z + 1$. Hence, we can check that

$$\mathbf{x}(0) = \frac{1}{5}\sum_{k=1}^{5}\begin{pmatrix}1\\ \omega^{-(k-1)}\\ \omega^{-2(k-1)}\\ \omega^{-3(k-1)}\\ \omega^{-4(k-1)}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}.$$

The peak time for $x_1(t)$ satisfies $0 < t_{peak} < 1$. By Taylor series, we obtain that

$$|e^{\omega^{k-1}t}| \approx 1 + t\cos\left(\frac{2\pi(k-1)}{5}\right).$$

It implies peak time satisfy $|e^{\omega^{k-1}t_{peak}}| \approx 1$. Since finding the exact peak time is complex, we consider an approximate model of the cyclic system to simplify the analysis.

$$x_1(t) = \frac{1}{5}\sum_{k=1}^{5} e^{(-2+\omega^{k-1})t} \approx e^{-2t}.$$

Using the integrating factor method, we iteratively compute $x_k$ by

$$x_k(t) = \int_0^t e^{-2(t-s)}x_{k-1}(s)\,ds, \quad j \geq 2.$$

By iterating this relation, one obtains the explicit approximation

$$\begin{aligned}
x_1(t) &\approx e^{-2t}x_1(0) = e^{-2t},\\
x_2(t) &= \int_0^t e^{-2(t-s)}x_1(s)\,ds \approx \int_0^t e^{-2(t-s)}e^{-2s}\,ds = \int_0^t e^{-2t}ds = te^{-2t},\\
x_3(t) &= \int_0^t e^{-2(t-s)}x_2(s)\,ds \approx \int_0^t e^{-2(t-s)}se^{-2s}\,ds = \int_0^t se^{-2t}ds = \frac{t^2}{2}e^{-2t},\\
x_4(t) &= \int_0^t e^{-2(t-s)}x_3(s)\,ds \approx \int_0^t e^{-2(t-s)}\frac{s^2}{2}e^{-2s}\,ds = \frac{1}{2}\int_0^t s^2e^{-2t}ds = \frac{t^3}{3!}e^{-2t},\\
x_5(t) &= \int_0^t e^{-2(t-s)}x_4(s)\,ds \approx \int_0^t e^{-2(t-s)}\frac{s^3}{6}e^{-2s}\,ds = \frac{1}{6}\int_0^t s^3e^{-2t}ds = \frac{t^4}{4!}e^{-2t}.
\end{aligned}$$

So, we conclude that

$$x_k(t) \approx \frac{t^{k-1}}{(k-1)!}e^{-2t}.$$

This approximation allows a direct computation of the peak time by differentiating $x_k(t)$

$$\dot{x}_k(t) \approx \frac{(k-1)t^{k-2} - 2t^{k-1}}{(k-1)!}\,e^{-2t} = 0.$$

Hence, we obtain that

$$t_{\text{peak},k} \approx \frac{k-1}{2}. \tag{2}$$

Similarly, the peak magnitude is approximated by

$$x_k^{\max} \approx \frac{(k-1)^{k-1}}{(k-1)!\,2^{k-1}} e^{-(k-1)}.$$

The accuracy of the analytical approximation was evaluated by comparing it with the numerical solution obtained from `solve_ivp`. Table 1 summarizes the maximum absolute error and the root-mean-square (RMS) error for each tank.

**Table 1:** Comparison of the analytical approximation with the numerical solution: maximum absolute error and RMS error for each tank.

| Tank | Max Absolute Error | RMS Error |
|---|---|---|
| 1 | $5.509 \times 10^{-3}$ | $2.481 \times 10^{-3}$ |
| 2 | $2.516 \times 10^{-3}$ | $1.186 \times 10^{-3}$ |
| 3 | $1.169 \times 10^{-3}$ | $5.730 \times 10^{-4}$ |
| 4 | $5.488 \times 10^{-4}$ | $2.779 \times 10^{-4}$ |
| 5 | $2.593 \times 10^{-4}$ | $1.352 \times 10^{-4}$ |

As shown in the table, the approximation is highly accurate. The maximum error occurs in Tank 1, with a value of $5.509 \times 10^{-3}$, while the error decreases progressively for the downstream tanks. This trend is expected because the approximation becomes increasingly accurate for tanks that are further along the cyclic chain, as the higher-order modes, which are neglected in the approximation, have diminishing contributions. Overall, the small errors indicate that the proposed analytical expression provides a reliable and computationally efficient estimate of the tank concentrations.

We note that the eigenvalues of the system matrix $A$ determine the stability of the cyclic tank system. As each eigenvalue has a strictly negative real part, the solution modes decay over time. This observation immediately leads to the following result concerning the long-term behavior of the system.

**Corollary 1.** *The system (1) is exponentially stable and* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. *Furthermore, because the maximum real part of the eigenvalues is governed by* $\max_k Re(\lambda_k) = -\alpha$, *there exists a constant* $M > 0$ *such that the system state satisfies the explicit decay estimate:*

$$\|\mathbf{x}(t)\| \leq Me^{-\alpha t}$$

*for* $t \geq 0$.

*Proof.* The real part of the eigenvalues is given by $\text{Re}(\lambda_k) = \alpha(-2 + \cos(\frac{2\pi(k-1)}{n}))$ for $k = 1, \cdots, n$. Since the maximum value of the cosine function is 1 (attains at $k = 1$), it follows that the maximal spectral bound is strictly governed by $\alpha$:

$$\text{Re}(\lambda_k) \leq \alpha(-2+1) = -\alpha < 0$$

for all $k$. Since $\alpha = \dfrac{q}{V} > 0$, all eigenvalues have strictly negative real parts. The system $\dot{x} = Ax$ is therefore exponentially stable with a decay rate bounded by $e^{-\alpha t}$. Since $A$ is circulant matrix, then we can write $A = FDF^{-1}$, where $D$ diagonal matrix. We choose

$M = \|F\|\|F^{-1}\|\|\mathbf{x}(0)\|$ and obtain

$$\|\mathbf{x}(t)\| \leq \|F\|\|F^{-1}\|\|\mathbf{x}(0)\|e^{-\alpha t} = Me^{-\alpha t} \qquad \square$$

### 3.2. Circular Tank with $m$ Layer

The $m$-layer tank model extends the single-layer approach by dividing the tank into multiple layers. The system allows for a more detailed representation of mixing and transport processes within the tank. Analysis of the $m$-layer system provides insight into how layer interactions affect concentration distribution and highlight the differences from the single-layer case. Consider a system consisting of $m$ layers with $n$ tanks in each layer. All flow rates are identical and equal to $q$, and each tank has the same volume $V$. Pure water inflow to all tanks in layer 1. The state vector is arranged as

$$\mathbf{x} = \begin{bmatrix} x_{1,1} & \cdots & x_{n,1} & \cdots & x_{1,m} & \cdots & x_{n,m} \end{bmatrix}^T.$$

Define the cyclic permutation matrix

$$P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad L = P - 2I_n.$$

In the first layer, each tank receives pure water inflow from an external source , and subsequently each tank discharges flow to one neighboring tank and to the corresponding tank in the layer below. Therefore, based on the mass balance principle, we obtain

$$\dot{x}_{k,1} = \alpha(x_{k-1,1} - 2x_{k,1}), \quad \text{with} \quad x_{0,1} = x_{n,1}.$$

For layer $j \geq 2$, each tank receives inflow from the corresponding tank in the layer above, namely layer $j-1$. Hence, the incoming concentration is no longer zero. For tank $x_{k,j}$, the inflow contribution from the upper tank is given by $\alpha x_{k,j-1}$. Therefore, we obtain

$$\dot{x}_{k,j} = \alpha(x_{k,j-1} + x_{k-1,j} - 2x_{k,j}), \quad \text{with} \quad x_{0,j} = x_{n,j}.$$

For simplify, the matrix for 2 layer and 3 tanks.

$$A = \alpha \begin{bmatrix} \boxed{-2} & \boxed{0} & \boxed{1} & 0 & 0 & 0 \\ \boxed{1} & \boxed{-2} & \boxed{0} & 0 & 0 & 0 \\ \boxed{0} & \boxed{1} & \boxed{-2} & 0 & 0 & 0 \\ 1 & 0 & 0 & \boxed{-2} & \boxed{0} & \boxed{1} \\ 0 & 1 & 0 & \boxed{1} & \boxed{-2} & \boxed{0} \\ 0 & 0 & 1 & \boxed{0} & \boxed{1} & \boxed{-2} \end{bmatrix}.$$

Entries enclosed in boxes correspond to the matrix $L = P - 2I_n$, which represents the cyclic intra-layer interactions. The blue entries correspond to the identity matrix $I_n$, which arises from inter-layer coupling between adjacent layers

In the general, the system can be written

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \tag{3}$$

where

$$A = \alpha \begin{bmatrix} L & 0 & \cdots & 0 \\ I_n & L & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I_n & L \end{bmatrix},$$

Before we derive the explicit solution the $m$-layer cyclic tank system, it is necessary to introduce several fundamental definitions and theoretical results that will be used throughout the analysis.

**Definition 1.** [13, Definition 2.1] Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$. The **Kronecker product** of $A$ and $B$, denoted by $A \otimes B$, is the block matrix in $\mathbb{C}^{mp\times nq}$ defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

In other words, each entry $a_{ij}$ of $A$ is replaced by the block $a_{ij}B$.

The Kronecker product provides a convenient representation for block-structured matrices arising in the $m$-layer system. We recall several fundamental properties that will be used in the subsequent analysis.

**Lemma 1.** *[14] Let $A, B, C, D$ be matrices of same dimensions and $\alpha$ be scalar. The Kronecker product satisfies the following properties:*

1. $\alpha(A \otimes B) = ((\alpha A) \otimes B) = (A \otimes (\alpha B))$
2. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
3. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.
4. $(A \otimes B)^T = A^T \otimes B^T$.
5. *If $A$ and $B$ are invertible, then* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
6. *If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue of $B$, then $\lambda\mu$ is an eigenvalue of* $A \otimes B$.
7. $e^{A\otimes I + I\otimes B} = e^A \otimes e^B$.

The previous lemma provides the algebraic tools required to manipulate block-structured matrices arising in the system representation. To proceed with the spectral analysis of the cyclic structure, we use the Fourier matrix, which diagonalizes circulant and permutation matrices.

**Definition 2.** [10, 11] Let $n \in \mathbb{N}$ and define $\omega = e^{\frac{2\pi i}{n}}$. The **Fourier matrix** of order $n$, denoted by $F \in \mathbb{C}^{n\times n}$, is defined by

$$F = \left(\omega^{(j-1)(k-1)}\right)_{j,k=1}^{n}.$$

Explicitly,

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}.$$

While the Fourier matrix diagonalizes the cyclic structure of the system, the vertical inter-layer coupling is described by a nilpotent matrix.

**Definition 3.** [15] An operator is called **nilpotent** if some power of it equals 0.

**Definition 4.** Let $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$. A *transpose of Jordan block* of size $k$ associated with the eigenvalue $\lambda$ is the matrix $J_k(\lambda) \in \mathbb{C}^{k\times k}$ defined by

$$J_k(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & \lambda \end{bmatrix}.$$

**Definition 5.** [16] A square matrix $J \in \mathbb{C}^{n\times n}$ is called a **Transpose of Jordan matrix** if it has a block diagonal structure

$$J = \operatorname{diag}\left(J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \ldots, J_{k_r}(\lambda_r)\right),$$

where each $J_{k_i}(\lambda_i)$ is a Jordan block.

The following theorem presents the explicit solution of Differential Equation (3).

**Theorem 2.** *Let $\mathbf{x}(0) = \mathbf{x}_0$. Then the solution of the system (3) is given explicitly by*

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0,$$

*where*

$$e^{At} = \begin{bmatrix} e^{\alpha Lt} & 0 & \cdots & 0 \\ (\alpha t)e^{\alpha Lt} & e^{\alpha Lt} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{(\alpha t)^{m-1}}{(m-1)!}e^{\alpha Lt} & \cdots & (\alpha t)e^{\alpha Lt} & e^{\alpha Lt} \end{bmatrix}.$$

*Moreover, since $L$ is diagonalizable by the discrete Fourier matrix, we have*

$$e^{\alpha Lt} = F^{-1}\operatorname{diag}\left(e^{\alpha\left(-2+\omega^{k-1}\right)t}\right)_{k=1}^{n} F,$$

*where $F$ is the Fourier matrix.*

*Proof.* Write $A = \alpha(I_m \otimes L + N \otimes I_n)$, where

$$N = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

is a transpose of nilpotent Jordan matrix of size $m$ satisfying $N^m = 0$. Since

$$(I_m \otimes L)(N \otimes I_n) = (N \otimes I_n)(I_m \otimes L),$$

the two matrices commute, and therefore by Lemma 1

$$e^{At} = e^{\alpha t(I_m \otimes L)}\, e^{\alpha t(N \otimes I_n)}.$$

By properties of the Kronecker product in Lemma 1, we obtain $e^{\alpha t(I_m \otimes L)} = I_m \otimes e^{\alpha L t}$. Because $N$ is nilpotent,

$$e^{\alpha t(N \otimes I_n)} = \sum_{k=1}^{m} \frac{(\alpha t)^{k-1}}{(k-1)!}(N^{k-1} \otimes I_n).$$

Multiplying the two expressions yields a lower block–triangular matrix whose diagonal blocks are $e^{\alpha L t}$ and whose $k$-th subdiagonal blocks are $\dfrac{(\alpha t)^{k-1}}{(k-1)!}e^{\alpha L t}$, giving

$$e^{At} = \begin{bmatrix} e^{\alpha L t} & 0 & \cdots & 0 \\ (\alpha t)e^{\alpha L t} & e^{\alpha L t} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{(\alpha t)^{m-1}}{(m-1)!}e^{\alpha L t} & \cdots & (\alpha t)e^{\alpha L t} & e^{\alpha L t} \end{bmatrix}.$$

Finally, since the cyclic permutation matrix $P$ is diagonalizable by the discrete Fourier matrix $F$, we have

$$L = F^{-1}\,\mathrm{diag}(-2 + \omega^{k-1})F,$$

and hence

$$e^{\alpha L t} = F^{-1}\,\mathrm{diag}\left(e^{\alpha(-2+\omega^{k-1})t}\right)F.$$

Therefore,

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0,$$

which gives the explicit solution. □

**Corollary 2.** *The system (3) is exponentially stable and* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. *Furthermore, because the maximum real part of the eigenvalues is governed by* $\max_k Re(\lambda_k) = -\alpha$, *there exists a constant* $M > 0$ *such that the system state satisfies the explicit decay estimate:*

$$\|\mathbf{x}(t)\| \le M t^{m-1} e^{-\alpha t}$$

*for* $t \ge 0$.

A numerical simulation of a two-layer tank system with five interconnected tanks in each layer is presented to illustrate the analytical results obtained previously. All tanks are assumed to have identical volume and flow rate, with the initial concentration introduced only in the first tank of the first layer. The simulation is computed using the matrix exponential solution of the model. The results not only describe the diffusion and transport dynamics within each layer but also reflect the underlying spectral structure of the system, where the temporal evolution is governed by the eigenvalues of the system matrix and the coupling between layers.

Figure 4 illustrates the concentration dynamics in a two-layer cyclic tank system, with peak points marked on each trajectory to highlight the temporal evolution of the maxima.

In Layer 1 (left subplot), the peak time increases progressively from one tank to the next. This behavior can be explained by the sequential transport mechanism, in which each tank receives inflow from its predecessor. The peak time has been approximated using Equation 2.

In Layer 2 (right subplot), the same increasing trend is observed, but all peak times are shifted

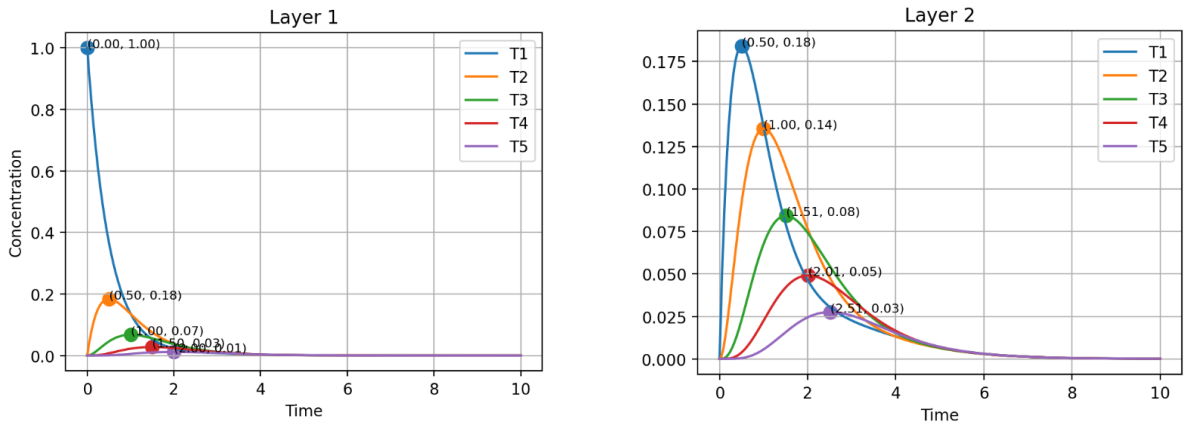


**Fig. 4:** Spectral Solution with Initial condition $(1, 0, \cdots, 0)^T$ and the peak time

to the right. This shift is due to the additional transport delay introduced by the inter-layer coupling. In particular, the peak time in Layer 2 can be interpreted as the sum of the intra-layer delay and an additional delay caused by the transfer from Layer 1.

From a mathematical perspective, this delayed peak is a direct consequence of the vertical inter-layer coupling. Specifically, the vertical transfer is governed by the transpose of the nilpotent Jordan matrix $N$. The matrix exponential of this nilpotent structure introduces polynomial terms of the form $\frac{(\alpha t)^k}{(k-1)!}$. When coupled with the exponentially decaying modes from the horizontal circulant mixing, it naturally produces the polynomial-exponential transient dynamics and the delayed peaks observed in Layer 2.

From Theorem 2, we obtain the exact solution from the system

$$\mathbf{x}(t) = \frac{1}{5}\sum_{k=1}^{5} e^{(-2+\omega^{k-1})t}\begin{bmatrix}\mathbf{v}_k \\ t\,\mathbf{v}_k\end{bmatrix},$$

where $\mathbf{v}_k = (1, \omega^{-(k-1)}, \omega^{-2(k-1)}, \omega^{-3(k-1)}, \omega^{-4(k-1)})^T$. Simple to check that $\mathbf{x}(0) = (1, 0, \cdots, 0)^T$.

Similar to analysis of Fig. 3. The peak time in Layer 2 can be approximated using the same approach as for Layer 1. Using the iterative integral approximation, the concentration is expressed as

$$x_{k,2}(t) \approx \frac{t^k}{(k-1)!}e^{-2t}, \quad k = 1, 2, 3, \ldots$$

We compare the exact solution in the following graph.

Fig. 5 shows the comparison between the exact solution and its analytical approximation for a two-layer cyclic tank system with $n = 5$ and $\alpha = 1$. It can be observed that the approximation agrees well with the exact solution in both layers, particularly in capturing the peak behavior. In each tank the predicted peak time is very close to the exact solution. This shows that the approximation captures the transient dynamics well.

In Layer 2 a delayed peak appears due to inter-layer coupling. The approximation still reproduces its timing with only small deviations.

We use first derivative to get peak time of $x_{k,2}(t)$ with respect to $t$

$$\frac{d}{dt}x_{k,2}(t) = \frac{t^{k-1}e^{-2t}}{(k-1)!}(k-2t)$$

Setting the derivative to zero provides the peak condition, $\dot{x}_{k,2}(t^{(2)}_{\text{peak,k}}) = 0$, which leads to $t^{(2)}_{\text{peak,k}} \approx \frac{k}{2}$.

This result shows that, similar to Layer 1, the peak time in Layer 2 increases progressively from one tank to the next, reflecting the sequential transport mechanism in the cyclic system.

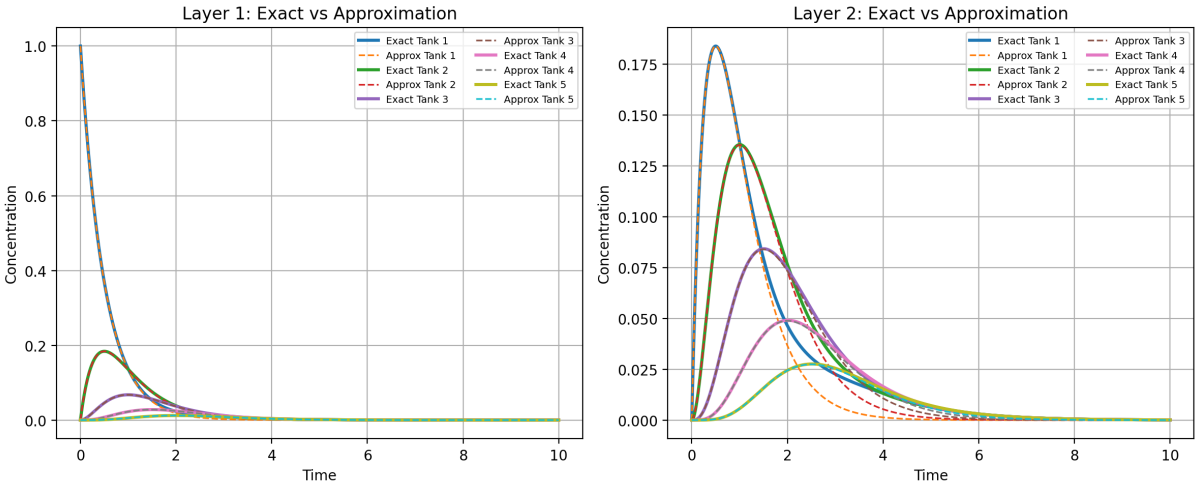


**Fig. 5:** Exact vs Approximation Solution in 2 Layer

Each downstream tank reaches its maximum concentration later than the tank above it, and the simple approximation provides a computationally efficient estimate of these peak times.

**Table 2:** Comparison between the analytical approximation and the numerical solution in layer 2, showing the maximum absolute error and RMS error for each tank.

| Tank | Max Absolute Error | RMS Error |
|---|---|---|
| 1 | $1.52 \times 10^{-2}$ | $7.21 \times 10^{-3}$ |
| 2 | $8.23 \times 10^{-3}$ | $4.06 \times 10^{-3}$ |
| 3 | $4.41 \times 10^{-3}$ | $2.25 \times 10^{-3}$ |
| 4 | $2.34 \times 10^{-3}$ | $1.23 \times 10^{-3}$ |
| 5 | $1.24 \times 10^{-3}$ | $6.65 \times 10^{-4}$ |

Table 2 shows the comparison between the numerical solution and the approximation. It can be observed that the maximum error occurs in Tank 1, while the RMS errors remain relatively small. This indicates that the approximation provides a reasonably accurate estimate of the concentration dynamics in Layer 2. Overall, these results confirm that the iterative integral approximation successfully captures the sequential transport mechanism and the trend of increasing peak times across the tanks.

## 4. Conclusion

This paper investigated the mathematical structure and dynamic behavior of the $m$-layer cyclic tank model as an extension of the classical single-layer system. The main objective was to derive an explicit representation of the system dynamics and to understand how inter-layer and intra-layer interactions influence concentration evolution.

The study demonstrated that the system admits a structured decomposition that separates horizontal mixing effects from vertical coupling effects, leading to a closed-form expression of the solution. This formulation provides a clearer theoretical understanding of how stratification modifies mixing behavior and transient dynamics compared to the single-layer case. The results establish a systematic analytical approach for studying multilayer transport systems.

Despite these contributions, the analysis is based on several simplifying assumptions. In particular, all tanks are assumed to have identical volumes and constant flow rates, and the system is modeled under perfect mixing conditions with pure water inflow. Moreover, the interactions between tanks are assumed to be linear. These assumptions restrict the applicability of the model to idealized settings and may not fully capture more complex physical or chemical processes in practical systems.

To overcome these limitations, the future research may extend to heterogeneous parameters, nonlinear mixing effects, external inputs, or control strategies, thereby providing a more realistic and comprehensive description of multilayer mixing dynamics.

## CRediT Authorship Contribution Statement

**Pebrudal Zanu:** Conceptualization, Methodology, Writing–Original Draft. **Sri Wahyu:** Formal Analysis, Writing–Review & Editing. **Mohammad Januar Ismail Burhan:** Proof Reader, Review & Editing.

## Declaration of Generative AI and AI-assisted technologies

The author used ChatGPT (version 5, developed by OpenAI) solely to refine language and grammar during the preparation of this manuscript. All ideas, methodologies, results, and interpretations are entirely the original work of the author. No AI-generated content contributed to the conceptual, analytical, or substantive aspects of the research.

## Declaration of Competing Interest

The authors declare no competing interests,

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## Data and Code Availability

No external data were used in this research. All results are based on original theoretical analysis conducted by the author.

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