CAUCHY – Jurnal Matematika Murni dan Aplikasi
Volume 11 (1) (2026), Pages 1082-1094
p-ISSN: 2086-0382; e-ISSN: 2477-3344

The Triple Total Graph of The Ring $\mathbb{Z}_n$

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Abstract

Let R be a commutative ring and $Z(R)$ denotes the set of zero-divisors of R. The triple total graph of R, denoted by $TT(R)$, is a simple graph with vertex set $R \setminus \{0\}$. Two distinct vertices v_1 and v_2 are adjacent in $TT(R)$ if and only if $v_1 + v_2 \notin Z(R)$ and there exists $v_3 \in R \setminus \{0\}$, with $v_3 \neq v_1$ and $v_3 \neq v_2$, such that $v_1 + v_3 \notin Z(R)$, $v_2 + v_3 \notin Z(R)$, and $v_1 + v_2 + v_3 \in Z(R)$. In this paper we investigate structural properties of the graph $TT(\mathbb{Z}_n)$. We show that if n is even with $n > 2$, then $TT(\mathbb{Z}_n)$ is an empty graph. When n is prime with $2 < n < 11$, the graph $TT(\mathbb{Z}_n)$ is disconnected. In contrast, for prime integers $n \geq 11$, the graph becomes connected with $\mathrm{diam}(TT(\mathbb{Z}_n)) = 2$ and $\mathrm{gr}(TT(\mathbb{Z}_n)) = 3$. Moreover, each vertex has degree $n - 5$, implying that the graph is $(n - 5)$-regular and consequently both Eulerian and Hamiltonian. These results illustrate how the arithmetic nature of n determines the global structure of the triple total graph.

Keywords: Commutative ring; Eulerian graph; Hamiltonian graph; Triple total graph; Zero-divisors.

1. Introduction

The interplay between algebra and graph theory has developed into a lively and productive research area over the past few decades. A particularly influential direction arises from constructing graphs whose vertices and edges reflect the structure of algebraic objects, especially commutative rings. One of the earliest systematic attempts in this spirit was proposed by Beck [1], who associated a graph to a ring to investigate coloring phenomena. His construction opened the door to viewing algebraic properties through combinatorial lenses. Shortly afterward, Anderson and Livingston [2] refined this perspective by introducing what is now widely known as the zero-divisor graph. Their framework established a direct relationship between the algebraic behavior of zero-divisors and graph-theoretic parameters such as connectivity and diameter. Since then, this line of inquiry has grown substantially, with numerous variations and generalizations developed to deepen the connection between ring theory and combinatorics (see, for example, [3–8]).

The significance of algebraic graph theory extends well beyond its internal theoretical elegance. Graphs defined from algebraic structures often display remarkable symmetry and regularity, features that make them attractive in applications. Such graphs have been utilized in coding theory [9], high-performance computing [10], and the design of communication networks [11], where structural predictability is highly valuable. Moreover, spectral techniques originating from algebraic graph theory have found applications in chemistry [12], complex systems [13], control

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Submitted: March 09, 2026 Reviewed: March 31, 2026 Accepted: April 28, 2026
DOI: https://doi.org/10.18860/cauchy.v11i1.41488

theory [14], and computer science [15, 16]. By translating algebraic conditions into combinatorial invariants, researchers gain access to algorithmic and computational tools that may not be readily available within purely algebraic frameworks. This cross-disciplinary flexibility continues to stimulate both theoretical advances and practical innovations.

Among the many constructions that have emerged, the total graph of a commutative ring, introduced by Anderson and Badawi [17], represents an important milestone. In the total graph, every element of the ring serves as a vertex, and adjacency is determined by whether the sum of two elements is a zero-divisor. This additive viewpoint contrasts with earlier product-based constructions and reveals different aspects of ring structure. Further investigations [18] examined structural and regularity properties of total graphs. Subsequent studies introduced extensions such as the generalized total graph [19], explored specialized cases including total graphs of product rings like $\mathbb{Z}_n \times \mathbb{Z}_m$ [20], and provided broader surveys of related developments [21]. More recently, Aitelhadi and Badawi [22] introduced the n-total graph, broadening the notion of adjacency beyond simple pairwise conditions. Collectively, these contributions highlight how additive interactions inside a ring influence global graph-theoretic features such as connectedness, diameter bounds, and regularity.

In parallel with these developments, attention has gradually shifted toward higher-order interactions. Rather than restricting adjacency to relations between two elements, several authors have begun incorporating triple-element conditions. Çelikel [23] introduced the triple zero graph, where adjacency is governed by triple zero-product relations. Computational approaches have also played a role; for instance, triple nilpotent graphs were explored algorithmically in [24] to uncover structural patterns. Other variations, including the triple identity graph [25] and the triple idempotent graph of $\mathbb{Z}_n$ [26], further demonstrate that triple-based constraints can reveal structural phenomena invisible in purely pairwise settings. These works collectively suggest a natural progression toward capturing richer algebraic interactions within graph models.

From zero-divisor graphs to total graphs and then to triple-based constructions, the trajectory of the field reveals an increasing level of structural complexity. Early studies focused primarily on binary relations derived from products or sums, whereas more recent efforts incorporate conditions involving three elements simultaneously. Despite this steady evolution, there remains relatively limited work on constructions that systematically combine the additive philosophy of total graphs with explicit triple-element constraints.

Motivated by this observation, we introduce a new construction called the triple total graph of a commutative ring. Let R be a commutative ring and let $Z(R)$ denote the set of zero-divisors of R. The triple total graph of R, denoted by $TT(R)$, is the simple graph whose vertex set is $R \setminus \{0\}$. Two distinct vertices $v_1, v_2 \in V(TT(R))$ are adjacent if and only if $v_1 + v_2 \notin Z(R)$ and there exists an element $v_3 \in R \setminus \{0\}$, with $v_3 \neq v_1$ and $v_3 \neq v_2$, such that $v_1 + v_3 \notin Z(R)$, $v_2 + v_3 \notin Z(R)$, and $v_1 + v_2 + v_3 \in Z(R)$. This construction extends the additive perspective of the classical total graph. While adjacency in the total graph depends solely on whether the sum of two elements is a zero-divisor, adjacency in $TT(R)$ additionally requires the existence of a third element satisfying a triple additive constraint. Consequently, the resulting graph captures higher-order additive interactions and encodes more subtle information about the additive structure of the ring.

In this paper, we concentrate on the case $R = \mathbb{Z}_n$ and study the graph $TT(\mathbb{Z}_n)$. Our aim is to determine several fundamental graph-theoretic properties, including connectivity, diameter, regularity, girth, and the existence of Hamiltonian and Eulerian cycles. The arithmetic nature of n turns out to play a decisive role. We show that if n is even and $n > 2$, then $TT(\mathbb{Z}_n)$ is an empty graph. For prime integers n with $2 < n < 11$, the graph is disconnected. In contrast, when n is prime and $n \geq 11$, the structure becomes markedly different: $TT(\mathbb{Z}_n)$ is connected and satisfies $\operatorname{diam}(TT(\mathbb{Z}_n)) = 2$. Moreover, in this case every vertex has degree $n - 5$, so the graph is $(n - 5)$-regular. We also prove that for prime $n \geq 11$, the girth of $TT(\mathbb{Z}_n)$ equals 3, and the graph admits both Hamiltonian and Eulerian cycles.

These findings illustrate how strongly the global structure of the triple total graph depends on the arithmetic properties of n. In particular, sufficiently large prime values of n produce graphs with high regularity and robust cycle structures, revealing a striking link between number-theoretic conditions and combinatorial behavior. We anticipate that further investigation of triple total graphs will deepen our understanding of higher-order additive interactions and encourage new connections between ring theory and graph theory.

2. Methods

This research adopts a theoretical–computational framework to analyze the structural characteristics of the triple total graph associated with a commutative ring, with particular emphasis on the ring $\mathbb{Z}_n$. The investigation begins with an extensive review of relevant literature, including textbooks, peer-reviewed journal articles, theses, and dissertations in the areas of commutative algebra and algebraic graph theory. Special attention is given to previous studies on zero-divisor graphs and total graphs of rings. Through this stage, we establish the necessary conceptual background concerning zero-divisors, properties of modular arithmetic, and fundamental graph-theoretic notions such as connectivity, diameter, regularity, girth, Hamiltonian cycles, and Eulerian cycles. This foundation serves as the conceptual basis for the subsequent development.

After consolidating the theoretical groundwork, we introduce the formal definition of the triple total graph $TT(R)$ for a commutative ring R and then focus specifically on the case $R = \mathbb{Z}_n$. The adjacency condition is reformulated in terms of explicit modular arithmetic constraints. By translating the graph-theoretic definition into number-theoretic conditions, we obtain a clearer characterization of when two elements of $\mathbb{Z}_n$ are adjacent in $TT(\mathbb{Z}_n)$. This reformulation not only simplifies the analytical process but also provides a concrete bridge between algebraic structure and combinatorial behavior.

To complement the theoretical analysis, we design a computational procedure to construct $TT(\mathbb{Z}_n)$. The process begins by determining the vertex set $\mathbb{Z}_n \setminus \{0\}$ and identifying all zero-divisors in $\mathbb{Z}_n$. For each pair of distinct vertices, the triple adjacency condition is then verified systematically. The entire procedure is implemented using Python, allowing us to generate explicit graph representations for selected values of n. These computational experiments help reveal structural patterns across different classes of integers, including even integers, prime integers, and prime squares. Through this exploration, we examine features such as connectivity, degree distribution, the presence of isolated vertices, and the existence of cycles.

Guided by patterns observed in the computational stage, we formulate several conjectures regarding the structural properties of $TT(\mathbb{Z}_n)$. These conjectures are subsequently established through rigorous proofs based on modular arithmetic arguments, the characterization of zero-divisors in $\mathbb{Z}_n$, and classical techniques from graph theory. The analysis is organized according to the arithmetic nature of n, which proves to be decisive in determining properties such as emptiness, disconnectedness, connectivity, diameter, girth, regularity, Hamiltonicity, and Eulerian structure.

In the final stage, the theoretical arguments and computational findings are integrated to produce comprehensive conclusions about the behavior of the triple total graph of $\mathbb{Z}_n$. By combining deductive reasoning with algorithmic verification, this methodology ensures both mathematical rigor and structural validation of the results presented in this study.

3. Results and Discussion

In this section, we present the principal results concerning the triple total graph of the ring $\mathbb{Z}_n$. For clarity, the exposition is divided into three interconnected parts, each addressing a different aspect of the analysis.

We begin by formulating the definition of $TT(\mathbb{Z}_n)$ in a precise manner. To make the abstract definition more transparent, several illustrative examples are provided. These examples demonstrate how the adjacency condition can be interpreted explicitly in terms of modular arithmetic and the characterization of zero-divisors in $\mathbb{Z}_n$. Through this step, the underlying algebraic constraints are translated into concrete computational criteria.

Next, we outline the algorithm constructed to generate $TT(\mathbb{Z}_n)$ computationally. The procedure is described in sufficient detail to clarify how vertices and edges are determined from arithmetic conditions. This computational perspective not only supports the theoretical arguments but also helps visualize structural patterns that arise for different values of n.

Finally, we present the main structural properties of $TT(\mathbb{Z}_n)$. These results are established through rigorous mathematical proofs, while computational experiments serve as supporting evidence that guides and confirms the theoretical development. Together, these components provide a coherent description of the behavior of the triple total graph of $\mathbb{Z}_n$.

3.1. Definition of the Triple Total Graph of the Ring $\mathbb{Z}_n$

In this subsection, we introduce the triple total graph of the ring $\mathbb{Z}_n$ as a particular case of the general construction $TT(R)$ for a commutative ring R. Since $\mathbb{Z}_n$ is a finite commutative ring whose structure depends entirely on the arithmetic properties of n, the adjacency relation in $TT(\mathbb{Z}_n)$ can be described explicitly using modular arithmetic.

We first restate the general definition and then specialize it to $\mathbb{Z}_n$. In this setting, the vertices correspond to all nonzero residue classes modulo n, while adjacency is governed by additive conditions involving zero-divisors.

Definition 1. Let $\mathbb{Z}_n$ be the ring of integers modulo n, and let $Z(\mathbb{Z}_n)$ denote the set of zero-divisors of $\mathbb{Z}_n$. The triple total graph of $\mathbb{Z}_n$, denoted by $TT(\mathbb{Z}_n)$, is a simple graph whose vertex set is $V(TT(\mathbb{Z}_n)) = \mathbb{Z}_n \setminus \{0\}$. Two distinct vertices $x, y \in V(TT(\mathbb{Z}_n))$ are adjacent if and only if $x + y \notin Z(\mathbb{Z}_n)$, and there exists $z \in \mathbb{Z}_n \setminus \{0\}$ with $z \neq x$ and $z \neq y$ such that $x + z \notin Z(\mathbb{Z}_n)$, $y + z \notin Z(\mathbb{Z}_n)$, and $x + y + z \in Z(\mathbb{Z}_n)$.

To clarify the definition, we present illustrative examples for specific values of n.

Example 1. Consider the ring $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. The set of zero-divisors is $Z(\mathbb{Z}_6) = \{0, 2, 3, 4\}$. Hence, the vertex set of $TT(\mathbb{Z}_6)$ is $V(TT(\mathbb{Z}_6)) = \{1, 2, 3, 4, 5\}$. The only elements not in $Z(\mathbb{Z}_6)$ are 1 and 5. Thus, adjacency requires $x + y \in \{1, 5\}$. By checking all pairs modulo 6, the candidate pairs are $(1, 4)$, $(2, 3)$, $(2, 5)$, $(3, 4)$. However, for each pair, there is no element $z \in \mathbb{Z}_6 \setminus \{0\}$ with $z \neq x, y$ satisfying the required triple condition. Therefore, $TT(\mathbb{Z}_6)$ is an empty graph. The visualization of $TT(\mathbb{Z}_6)$ is shown in the Fig. 1.

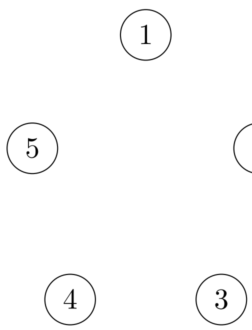

Fig. 1: Visualization of the graph $TT(\mathbb{Z}_6)$

Example 2. Consider $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$. Since 7 is prime, the only zero-divisor is 0, so $Z(\mathbb{Z}_7) = \{0\}$. Thus, $V(TT(\mathbb{Z}_7)) = \{1, 2, 3, 4, 5, 6\}$. For distinct vertices x and y, adjacency holds if and only if $x + y \not\equiv 0 \pmod 7$, and there exists $z \neq 0, x, y$ such that

$x+z \not\equiv 0$, $y+z \not\equiv 0$, $x+y+z \equiv 0 \pmod 7$. From the last condition, we have $z \equiv -(x+y) \pmod 7$. The examination of all pairs of distinct vertices is presented in Table 1. A complete verification shows that the graph decomposes into two connected components, namely $\{1,2,4\}$ and $\{3,5,6\}$. Each component forms a triangle. Hence, $TT(\mathbb{Z}_7) \cong K_3 \cup K_3$, and the graph is disconnected. The visualization of $TT(\mathbb{Z}_7)$ is shown in the Fig. 2.

Table 1: Verification of the adjacency condition in $TT(\mathbb{Z}_7)$

x	y	$(x+y)$ mod 7	$z=-(x+y)$ mod 7	Adjacent?
1	2	3	4	Yes
1	3	4	3	No
1	4	5	2	Yes
1	5	6	1	No
1	6	0	–	No
2	3	5	2	No
2	4	6	1	Yes
2	5	0	–	No
2	6	1	6	No
3	4	0	–	No
3	5	1	6	Yes
3	6	2	5	Yes
4	5	2	5	No
4	6	3	4	No
5	6	4	3	Yes

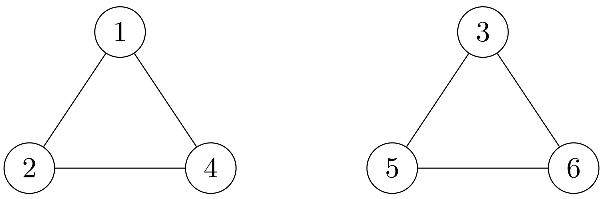

Fig. 2: Visualization of the graph $TT(\mathbb{Z}_7)$

3.2. Algorithm for Constructing $TT(\mathbb{Z}_n)$

To reinforce the theoretical analysis, we construct a systematic algorithm for generating the triple total graph of $\mathbb{Z}_n$. The primary objective of this procedure is to determine explicitly the vertex set and the edge set according to the triple adjacency condition established in the previous subsection.

The algorithm starts by listing all nonzero elements of $\mathbb{Z}_n$, which constitute the vertex set of $TT(\mathbb{Z}_n)$. Next, the set of zero-divisors in $\mathbb{Z}_n$ is identified using their number-theoretic characterization. Once these preliminary components are determined, the algorithm proceeds by examining each pair of distinct vertices. For every such pair, the adjacency condition is checked by verifying the required additive constraints together with the zero-divisor condition involving a third element of the ring.

The construction procedure described above can be formalized as a structured algorithm. For clarity, Algorithm 1 presents the pseudocode for generating the triple total graph $TT(\mathbb{Z}_n)$. The algorithm explicitly specifies the input parameter n, defines the vertex set and adjacency condition, and details the step-by-step procedure used to construct the edge set. All variables appearing in the pseudocode are introduced prior to their use to ensure clarity and reproducibility.

Algorithm 1 Construction of the triple total graph $TT(\mathbb{Z}_n)$

Require: A positive integer n
Ensure: The graph $G = TT(\mathbb{Z}_n)$

Initialize an empty simple graph G
Define $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$
Determine the set of zero-divisors:
$Z(\mathbb{Z}_n) \leftarrow \{a \in \mathbb{Z}_n \mid \exists\, b \in \mathbb{Z}_n \setminus \{0\},\ ab \equiv 0 \pmod{n}\}$
Define the vertex set:
$V(G) \leftarrow \mathbb{Z}_n \setminus \{0\}$
for all distinct vertices $v_1, v_2 \in V(G)$ **do**
    if $v_1 + v_2 \notin Z(\mathbb{Z}_n)$ **then**
        for all $v_3 \in V(G)$ such that $v_3 \neq v_1$ and $v_3 \neq v_2$ **do**
            if $v_1 + v_3 \notin Z(\mathbb{Z}_n)$ **and** $v_2 + v_3 \notin Z(\mathbb{Z}_n)$ **and** $v_1 + v_2 + v_3 \in Z(\mathbb{Z}_n)$ **then**
                Add edge $v_1 v_2$ to $E(G)$
                break
            end if
        end for
    end if
end for
return Graph G

The computational effort of Algorithm 1 mainly arises from examining triples of vertices in $\mathbb{Z}_n \setminus \{0\}$. For each pair of vertices v_1 and v_2, the algorithm searches for a third element v_3 satisfying the triple condition. Consequently, the number of required checks increases rapidly as n grows. In practice, however, the arithmetic structure of $\mathbb{Z}_n$ allows several conditions to be verified efficiently. The algorithm was implemented in Python to visualize the graphs and to verify the theoretical results for representative values of n. Sample outputs of the program are shown in Fig. 3.

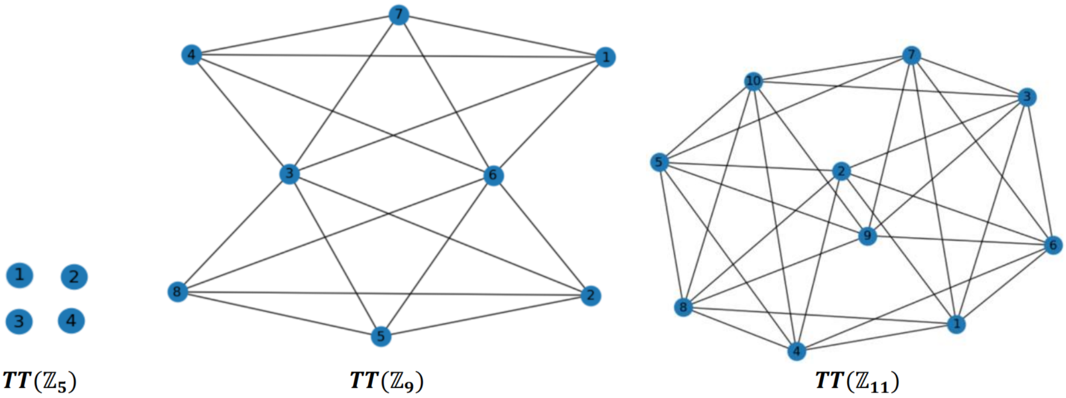

Fig. 3: Graphical representations of $TT(\mathbb{Z}_n)$ generated by the program for $n = 5, 9, 11$

3.3. Properties of the Triple Total Graph of the Ring $\mathbb{Z}_n$

In this subsection, we focus on developing the principal theoretical results related to the structural behavior of $TT(\mathbb{Z}_n)$. Our approach is organized according to the arithmetic nature of n, as the distribution of zero-divisors in $\mathbb{Z}_n$ plays a decisive role in shaping the associated graph. We investigate several fundamental graph-theoretic aspects, such as connectivity, the presence of isolated vertices, diameter, degree distribution, regularity, girth, as well as Hamiltonian and Eulerian properties. The analysis demonstrates that the structure of the graph varies significantly depending on whether n is even, prime, or a prime square. Notably, when n is a sufficiently large prime, the triple total graph displays a remarkably regular structure together with strong cycle properties, reflecting a close relationship between arithmetic conditions and combinatorial features.

Before proceeding further, we recall a basic number-theoretic observation regarding zero-divisors in the ring $\mathbb{Z}_n$. This simple characterization will be used repeatedly in the arguments that follow, especially in the discussion of the case where n is even.

Remark 1. Let $\mathbb{Z}_n$ be the ring of integers modulo n. If n is even, then every even element of $\mathbb{Z}_n$ is a zero-divisor. Consequently, the set of zero-divisors $Z(\mathbb{Z}_n)$ contains all even residue classes modulo n.

Before analyzing the structural behavior of $TT(\mathbb{Z}_n)$ for prime and composite values of n, we first consider the case where n is even. As shown in the previous remark, if n is even, then every even element of $\mathbb{Z}_n$ is a zero-divisor. This property strongly restricts the possibility of satisfying the triple adjacency condition and leads to the following result.

Theorem 1. *Let $\mathbb{Z}_n$ be the ring of integers modulo n. If n is even and $n > 2$, then the triple total graph $TT(\mathbb{Z}_n)$ is an empty graph.*

Proof. Recall that a graph is called an empty graph if it contains no edges. According to the definition of the triple total graph, two distinct vertices x and y are adjacent if and only if $x + y \notin Z(\mathbb{Z}_n)$, and there exists a third vertex $z \in \mathbb{Z}_n \setminus \{0\}$ with $z \neq x$ and $z \neq y$ such that $x + z \notin Z(\mathbb{Z}_n)$, $y + z \notin Z(\mathbb{Z}_n)$, and $x + y + z \in Z(\mathbb{Z}_n)$. Thus, adjacency requires the existence of three distinct vertices x, y, and z satisfying the above additive conditions.
Since n is even, then $2 \mid n$, and consequently every even element of $\mathbb{Z}_n$ is a zero-divisor. Consider any three distinct elements $x, y, z \in \mathbb{Z}_n \setminus \{0\}$. Among these three elements, at least two must have the same parity (both even or both odd). If two of them are even, then their sum is even, hence a zero-divisor. If two of them are odd, then their sum is also even, and therefore a zero-divisor. In either case, at least one of the pairwise sums $x + y$, $x + z$ or $y + z$ is even and thus belongs to $Z(\mathbb{Z}_n)$.
Therefore, the condition that all three pairwise sums lie outside $Z(\mathbb{Z}_n)$ cannot be satisfied. Hence, no pair of vertices in $TT(\mathbb{Z}_n)$ can be adjacent. It follows that $TT(\mathbb{Z}_n)$ contains no edges and is therefore an empty graph. For completeness, note that when $n = 2$, the graph $TT(\mathbb{Z}_n)$ has only one vertex, so it is trivial.
Thus, if n is even and $n > 2$, the triple total graph $TT(\mathbb{Z}_n)$ is an empty graph. □

Before establishing the connectivity properties of $TT(\mathbb{Z}_n)$ for prime n, we first characterize certain pairs of vertices that can never be adjacent. Since $\mathbb{Z}_n$ is an integral domain when n is prime, the set of zero-divisors plays a crucial role in simplifying the adjacency condition.

Lemma 1. *Let n be a prime and let $v_1, v_2 \in V(TT(\mathbb{Z}_n))$ be distinct vertices. Then v_1 is not adjacent to v_2 if and only if $v_1 + v_2 \in \{0, -v_1, -v_2\}$.*

Proof. Since n is prime, $\mathbb{Z}_n$ is a field. It follows that the only zero-divisor is 0, and hence $Z(\mathbb{Z}_n) = \{0\}$. Therefore, two distinct vertices v_1 and v_2 are adjacent in $TT(\mathbb{Z}_n)$ if and only if $v_1 + v_2 \neq 0$ and there exists $v_3 \in \mathbb{Z}_n \setminus \{0\}$ with $v_3 \neq v_1$ and $v_3 \neq v_2$ such that $v_1 + v_3 \neq 0, v_2 + v_3 \neq 0$ and $v_1 + v_2 + v_3 = 0$.

($\Rightarrow$) Suppose that v_1 is not adjacent to v_2.
If $v_1 + v_2 = 0$, then the conclusion holds. Assume now that $v_1 + v_2 \neq 0$. Since $\mathbb{Z}_n$ is a field,

the equation $v_1+v_2+v_3=0$ has a unique solution, namely $v_3=-(v_1+v_2)$. Since $v_1, v_2 \neq 0$, we have $v_1+v_3=v_1-(v_1+v_2)=-v_2\neq 0$ and $v_2+v_3=v_2-(v_1+v_2)=-v_1\neq 0$. Thus, the only possible obstruction to adjacency is that v_3 coincides with v_1 or v_2. Now observe that the candidate vertex $v_3=-(v_1+v_2)$ coincides with v_1 precisely when $-(v_1+v_2)=v_1$, which is equivalent to $v_1+v_2=-v_1$. Similarly, v_3 coincides with v_2 precisely when $-(v_1+v_2)=v_2$, which is equivalent to $v_1+v_2=-v_2$. Therefore, when $v_1+v_2\neq 0$, the non-adjacency occurs precisely when $v_1+v_2=-v_1$ or $v_1+v_2=-v_2$. Hence, if v_1 and v_2 are not adjacent, then $v_1+v_2\in\{0,-v_1,-v_2\}$.

($\Leftarrow$) Suppose that $v_1+v_2\in\{0,-v_1,-v_2\}$.
If $v_1+v_2=0$, then the condition $v_1+v_2\neq 0$ for adjacency is violated, so v_1 and v_2 are not adjacent. Now assume that $v_1+v_2=-v_1$ or $v_1+v_2=-v_2$. Then the equation $v_1+v_2+v_3=0$ has the unique solution $v_3=-(v_1+v_2)$, which equals v_1 or v_2, respectively. This contradicts the requirement that v_3 must be distinct from both v_1 and v_2. Hence, no admissible vertex v_3 exists, and v_1 and v_2 are not adjacent.

Since both implications hold, the result follows. □

To analyze the structural properties of $TT(\mathbb{Z}_n)$ for larger prime values of n, we first investigate the behavior of the graph for small prime integers. It will be shown that the triple total graph is disconnected for every prime n satisfying $2<n<11$. The proof proceeds by direct verification for each prime in this interval.

Theorem 2. *If n is prime and $2<n<11$, then the graph $TT(\mathbb{Z}_n)$ is disconnected.*

Proof. Let n be a prime with $2<n<11$. Then $n\in\{3,5,7\}$. Since n is prime, $\mathbb{Z}_n$ is a field, and hence $Z(\mathbb{Z}_n)=\{0\}$. The result follows by considering each case.

Case 1: $n=3$. We have $V(TT(\mathbb{Z}_3))=\{1,2\}$. Since $1+2=0$, it follows from Lemma 1 that 1 and 2 are not adjacent. Hence, the graph has no edges and is therefore disconnected.

Case 2: $n=5$. We have $V(TT(\mathbb{Z}_5))=\{1,2,3,4\}$. For any distinct vertices v_1, v_2, one verifies that $v_1+v_2\in\{0,-v_1,-v_2\}$. Thus, by Lemma 1, no two distinct vertices are adjacent. Therefore, all vertices are isolated and the graph is disconnected.

Case 3: $n=7$. We have $V(TT(\mathbb{Z}_7))=\{1,2,3,4,5,6\}$. We show that the graph consists of two disjoint cycles.
First, consider the vertices $\{1,2,4\}$. We have $1+2=3$, $1+4=5$, and $2+4=6$, all of which are nonzero. Moreover, $1+2+4=7\equiv 0 \pmod 7$. Hence, each pair among $\{1,2,4\}$ is adjacent, and these vertices form a cycle.
Similarly, for the set $\{3,5,6\}$, we have $3+5=1$, $3+6=2$, and $5+6=4$, and $3+5+6=14\equiv 0 \pmod 7$. Thus, these vertices also form a cycle.
Finally, let $v_1\in\{1,2,4\}$ and $v_2\in\{3,5,6\}$. A direct computation shows that $v_1+v_2\in\{0,-v_1,-v_2\}$. By Lemma 1, such pairs are not adjacent. Therefore, there is no edge between the two sets.
Hence, $TT(\mathbb{Z}_7)$ consists of two disjoint components and is disconnected.

Since in all cases $n=3,5,7$ the graph $TT(\mathbb{Z}_n)$ is disconnected, the result follows. □

We now consider larger prime values of n. While $TT(\mathbb{Z}_n)$ is disconnected for $2<n<11$, the graph becomes connected when $n\geq 11$, with $\operatorname{diam}(TT(\mathbb{Z}_n))=2$. Intuitively, this transition arises from the triple adjacency condition. For small primes, the limited number of elements in $\mathbb{Z}_n\setminus\{0\}$ restricts the formation of admissible triples, which prevents many vertex pairs from being connected. When n becomes larger, the number of possible triples increases, ensuring that suitable elements exist to connect the graph.

Theorem 3. *Let $\mathbb{Z}_n$ be a ring. If n is prime and $n \geq 11$, then the triple total graph $TT(\mathbb{Z}_n)$ is connected and* $\operatorname{diam}(TT(\mathbb{Z}_n)) = 2$.

Proof. Let n be a prime number with $n \geq 11$. Then $\mathbb{Z}_n$ is a field, and hence $Z(\mathbb{Z}_n) = \{0\}$. The vertex set is $V(TT(\mathbb{Z}_n)) = \mathbb{Z}_n \setminus \{0\}$.
Let v_1 and v_2 be any two distinct vertices in $V(TT(\mathbb{Z}_n))$.
Case 1: v_1 and v_2 are adjacent.
In this case, the distance between v_1 and v_2 is 1.
Case 2: v_1 and v_2 are not adjacent.
Assume that v_1 and v_2 are not adjacent. By Lemma 1, this occurs if and only if $v_1 + v_2 \in \{0, -v_1, -v_2\}$. We show that there exists a vertex $v_3 \in V(TT(\mathbb{Z}_n))$ such that v_3 is adjacent to both v_1 and v_2. By Lemma 1, two vertices a and b are adjacent if and only if $a + b \notin \{0, -a, -b\}$. Thus, we require $v_1 + v_3 \notin \{0, -v_1, -v_3\}$ and $v_2 + v_3 \notin \{0, -v_2, -v_3\}$. In addition, by definition of the graph $TT(\mathbb{Z}_n)$, v_3 must satisfy $v_3 \neq 0$, $v_3 \neq v_1$, and $v_3 \neq v_2$. Since we choose $v_3 \in \mathbb{Z}_n \setminus \{0\}$, the condition $v_3 \neq 0$ is automatically satisfied. The remaining constraints exclude only finitely many values of v_3, which we now determine. From $v_3 \neq v_1$ and $v_3 \neq v_2$, we exclude the values v_1 and v_2. From $v_1 + v_3 \neq 0$ and $v_2 + v_3 \neq 0$, we exclude $v_3 = -v_1$ and $v_3 = -v_2$. From $v_1 + v_3 \neq -v_1$, we obtain $v_3 \neq -2v_1$, and from $v_2 + v_3 \neq -v_2$, we obtain $v_3 \neq -2v_2$. Finally, from $v_1 + v_3 \neq -v_3$, we get $v_3 \neq -\frac{1}{2}v_1$, and from $v_2 + v_3 \neq -v_3$, we get $v_3 \neq -\frac{1}{2}v_2$.
Thus, the excluded values are contained in the set $\{v_1, v_2, -v_1, -v_2, -2v_1, -2v_2, -\frac{1}{2}v_1, -\frac{1}{2}v_2\}$, which has at most eight elements. Since $n \geq 11$, the set $\mathbb{Z}_n \setminus \{0\}$ contains $n-1 \geq 10$ elements. Therefore, after excluding at most eight values, at least two elements remain available. In particular, there exists a vertex $v_3 \in \mathbb{Z}_n \setminus \{0\}$ satisfying all the required conditions. Hence, v_3 is adjacent to both v_1 and v_2, and there exists a path $v_1 - v_3 - v_2$ of length 2 connecting them.
Therefore, $\operatorname{diam}(TT(\mathbb{Z}_n)) \leq 2$. Since not all vertices are adjacent, the diameter is equal to 2, and the graph is connected. □

Having established the connectivity and diameter of $TT(\mathbb{Z}_n)$ for prime $n \geq 11$, we now examine its degree distribution. In particular, we show that the graph is regular and compute the degree of every vertex explicitly.

Theorem 4. *If n is prime and $n \geq 11$, then* $\deg(v) = n - 5$ *for every vertex $v \in V(TT(\mathbb{Z}_n))$.*

Proof. Let n be a prime number with $n \geq 11$. Then $\mathbb{Z}_n$ is a field, and hence the set of zero-divisors is $Z(\mathbb{Z}_n) = \{0\}$. The vertex set of $TT(\mathbb{Z}_n)$ is $V(TT(\mathbb{Z}_n)) = \mathbb{Z}_n \setminus \{0\}$, so the graph has $n - 1$ vertices.
Fix a vertex $v \in V(TT(\mathbb{Z}_n))$. To determine $\deg(v)$, we count how many vertices are not adjacent to v, and then subtract this number from $|V(TT(\mathbb{Z}_n))|$.
First, since $TT(\mathbb{Z}_n)$ is a simple graph, the vertex v is not adjacent to itself.
Next, consider a vertex $u \in V(TT(\mathbb{Z}_n))$ with $u \neq v$. By Lemma 1, the vertices u and v are not adjacent if and only if $v + u \in \{0, -v, -u\}$. We analyze these possibilities one by one.
If $v + u = 0$, then $u = -v$.
If $v + u = -v$, then $u = -2v$.
If $v + u = -u$, then $v = -2u$, and hence $u = -\frac{1}{2}v$.
Thus, apart from v itself, the only vertices that are not adjacent to v are $-v$, $-2v$, and

$-\frac{1}{2}v$.
We now verify that these vertices are all distinct. Since n is prime and $v \neq 0$, the equalities $-v = v$, $-2v = v$, or $-\frac{1}{2}v = v$ would imply $2v = 0$, $3v = 0$, or $\frac{3}{2}v = 0$, respectively, which are impossible in a field of prime order $n \geq 11$. Similarly, one checks that $-v$, $-2v$, and $-\frac{1}{2}v$ are pairwise distinct. Therefore, there are exactly four vertices that are not adjacent to v, namely v, $-v$, $-2v$, and $-\frac{1}{2}v$.
Since the graph has $n-1$ vertices, it follows that $\deg(v) = (n-1) - 4 = n - 5$. Because v was arbitrary, the same argument applies to every vertex in $TT(\mathbb{Z}_n)$. □

As a direct consequence of Theorem 4, we obtain the following corollaries.

Corollary 1. *If n is prime and $n \geq 11$, then the graph $TT(\mathbb{Z}_n)$ is an $(n-5)$-regular graph.*

Proof. By Theorem 4, for every vertex $v \in V(TT(\mathbb{Z}_n))$, we have $\deg(v) = n-5$. Since every vertex has the same degree $n-5$, it follows that $TT(\mathbb{Z}_n)$ is an $(n-5)$-regular graph. □

Corollary 2. *If n is prime and $n \geq 11$, then the degree of each vertex in the graph $TT(\mathbb{Z}_n)$ is even.*

Proof. From Theorem 4, we know that $\deg(v) = n - 5$ for every vertex $v \in V(TT(\mathbb{Z}_n))$. Since every prime number greater than 2 is odd and $n \geq 11$, it follows that n is odd. Consequently, $n-5$ is even, and hence $\deg(v)$ is even for every vertex v. Therefore, $\deg(v)$ is even for every vertex v in $TT(\mathbb{Z}_n)$. □

After establishing the connectedness and degree properties of the graph $TT(\mathbb{Z}_n)$ for prime $n \geq 11$, we now determine its girth. Recall that the girth of a graph is the length of its shortest cycle. The following theorem shows that this graph always contains a triangle.

Theorem 5. *Let $\mathbb{Z}_n$ be a ring with n prime and $n \geq 11$. Then* $\mathrm{gr}(TT(\mathbb{Z}_n)) = 3$.

Proof. Assume that n is prime and $n \geq 11$. Then $\mathbb{Z}_n$ is a field and $Z(\mathbb{Z}_n) = \{0\}$. We show that $TT(\mathbb{Z}_n)$ contains a cycle of length 3. Since the graph is connected by Theorem 3, there exist two adjacent vertices v_1 and v_2. By the definition of adjacency, there exists a vertex $v_3 \in \mathbb{Z}_n \setminus \{0\}$, with $v_3 \neq v_1$ and $v_3 \neq v_2$, such that $v_1 + v_3 \neq 0$, $v_2 + v_3 \neq 0$, and $v_1 + v_2 + v_3 = 0$.
We now verify that v_1, v_2, and v_3 are pairwise adjacent. Since $v_1 + v_2 + v_3 = 0$, we have $v_1 + v_3 = -v_2$ and $v_2 + v_3 = -v_1$, both of which are nonzero. Moreover, since v_1, v_2, v_3 are distinct, we have $v_1 + v_3 \notin \{-v_1, -v_3\}$ and $v_2 + v_3 \notin \{-v_2, -v_3\}$. Hence, by Lemma 1, each pair among v_1, v_2, v_3 is adjacent. Therefore, v_1, v_2, v_3 form a cycle of length 3.
Since any cycle in a simple graph has length at least 3, it follows that the girth of $TT(\mathbb{Z}_n)$ is 3. Hence, $\mathrm{gr}(TT(\mathbb{Z}_n)) = 3$. □

After determining the regularity and girth of $TT(\mathbb{Z}_n)$ for prime $n \geq 11$, we now investigate its Hamiltonian property. To establish this result, we use a classical sufficient condition for Hamiltonicity, namely Dirac's Theorem.

Theorem 6. *Let $\mathbb{Z}_n$ be a ring with n prime and $n \geq 11$. Then the triple total graph $TT(\mathbb{Z}_n)$ is Hamiltonian.*

Proof. We apply Dirac's Theorem, which states that a simple graph G with $m \geq 3$ vertices is Hamiltonian if $\deg(v) \geq \frac{m}{2}$ for every vertex $v \in V(G)$.
Let $m = |V(TT(\mathbb{Z}_n))|$. Since $V(TT(\mathbb{Z}_n)) = \mathbb{Z}_n \setminus \{0\}$, we have $m = n - 1$. Since n is prime and $n \geq 11$, by Theorem 4, each vertex has degree $\deg(v) = n - 5 = m - 4$. Thus, $\deg(v) = m - 4 \geq \frac{m}{2}$ whenever $m \geq 8$. Since $n \geq 11$, we have $m = n - 1 \geq 10$, and hence $m \geq 8$. Hence Dirac's condition is satisfied.
Therefore, by Dirac's Theorem, $TT(\mathbb{Z}_n)$ is Hamiltonian. □

Having established that $TT(\mathbb{Z}_n)$ is regular and Hamiltonian for prime $n \geq 11$, we now turn to another fundamental structural property, namely the existence of an Euler cycle. The following theorem confirms that the triple total graph $TT(\mathbb{Z}_n)$ is Eulerian.

Theorem 7. *Let $\mathbb{Z}_n$ be a ring with n prime and $n \geq 11$. Then the triple total graph $TT(\mathbb{Z}_n)$ is an Eulerian graph.*

Proof. By Theorem 3, the graph $TT(\mathbb{Z}_n)$ is connected. Furthermore, by Corollary 2, every vertex has even degree, namely $\deg(v) = n - 5$. Since $TT(\mathbb{Z}_n)$ is a connected graph in which all vertices have even degree, it follows from Euler's Theorem that $TT(\mathbb{Z}_n)$ is Eulerian. □

4. Conclusion

This paper set out to characterize the structural properties of the graph $TT(\mathbb{Z}_n)$, a higher-order construction defined by triple additive relations in the ring $\mathbb{Z}_n$. By formulating adjacency explicitly in terms of modular arithmetic, we established how the global structure of the graph is governed by the arithmetic nature of n. In contrast to classical zero-divisor and total graphs that arise from pairwise interactions, this construction reveals structural phenomena driven by three-element additive constraints.

Our analysis shows a clear transition in the prime case: when n is prime with $2 < n < 11$, the graph $TT(\mathbb{Z}_n)$ is disconnected, whereas for primes $n \geq 11$ it becomes connected and regular. This shift demonstrates that purely number-theoretic conditions determine fundamental graph properties such as connectivity and regularity. Degree formulas, symmetry considerations, and edge enumerations further provide a coherent structural description of this new family of graphs.

It should be noted, however, that the present study focuses primarily on the prime case, and a comprehensive structural analysis for general composite integers n remains open. Extending the investigation to integers of the form $n = p^k$, $n = p_1^{k_1} q_2^{k_2}$, and other composite configurations is expected to reveal richer structural behavior due to the presence of nontrivial zero-divisors, potentially leading to new connectivity patterns and component structures. In addition, further study of graph invariants such as the chromatic number, independence number, and domination number would deepen the understanding of the combinatorial complexity of $TT(\mathbb{Z}_n)$ and clarify more broadly the interplay between ring-theoretic structure and higher-order graph constructions.

CRediT Authorship Contribution Statement

The authors' contributions to this work are specified according to the Contributor Roles Taxonomy (CRediT) as follows.
Vika Yugi Kurniawan: Conceptualization, Methodology, Validation, Writing-Review & Editing.
Syaifudin Zyuhri: Formal Analysis, Software, Writing-Original Draft Preparation.
Santoso Budi Wiyono: Supervision, Project Administration, Funding Acquisition.

Declaration of Generative AI and AI-assisted technologies

Generative AI technology was used during the preparation of this manuscript. Specifically, ChatGPT (version GPT-5.2, OpenAI) was utilized to assist in language refinement, improving clarity, grammar, and overall readability of the text. The AI tool was used solely for writing assistance and proofreading purposes. All scientific content, mathematical results, interpretations, and conclusions remain the responsibility of the authors, who carefully reviewed and verified the final manuscript.

Declaration of Competing Interest

The authors declare no competing interests.

Funding and Acknowledgments

The researchers gratefully acknowledge the financial and institutional support provided by the Institute for Research and Community Service (LPPM) of Universitas Sebelas Maret. This study was carried out under the PKGR-UNS research program and funded through the 2026 RKAT Universitas Sebelas Maret budget allocation (Contract No. 462/UN27.22/PT.01.03/2026).

Data and Code Availability

The code that supports the findings of this study is available from the corresponding author upon reasonable request.

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