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Characterization of $\mathfrak{g}$-quasi-Frobenius Lie Algebras via Inner Derivations

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Abstract

The structure of a $\mathfrak{g}$-quasi-Frobenius Lie algebra can be realized as a quasi-Frobenius Lie algebra module over a Lie algebra $\mathfrak{g}$. This research discusses a special case of the $\mathfrak{g}$-quasi-Frobenius Lie algebra, namely when $\mathfrak{g}$ acts on itself. In this setting, the construction of a $\mathfrak{g}$-quasi-Frobenius Lie algebra on $\mathfrak{g}$ is characterized by the existence of an inner derivation. The main result provides a criterion: such a structure can be constructed on $\mathfrak{g}$ itself if and only if the inner derivation is zero. This characterization extends and clarifies the framework established in previous work, including the cited study by Pham, by specifying the precise role of inner derivations in the construction. Several concrete examples are provided to illustrate and test the criterion.

Keywords: Frobenius Lie algebra; quasi-Frobenius Lie algebra; Inner derivation; Lie algebra module; $\mathfrak{g}$-quasi-Frobenius Lie algebra.

1. Introduction

A Lie algebra is a vector space over a field $\mathbb{F}$ equipped with a Lie bracket that satisfies certain axioms [1]. Lie algebras have several research areas, such as cohomology [2], symplectic structures [3], and contact structures [4]. Lie algebras can also be classified into various types, including Frobenius Lie algebras and quasi-Frobenius Lie algebras. Several studies related to these types have been conducted, such as the classification of Filiform Lie algebras of dimension ≤ 5 within the structure of quasi-Frobenius Lie algebras [5], the quasi-associative property of a Frobenius Lie algebra [6], and research on the skew-symmetric bilinear form of finite-dimensional Frobenius Lie algebras [7]. Specifically in dimensions ≤ 4, Frobenius Lie algebras can be constructed using non-commutative nilpotent Lie algebras [8].

In Lie algebras, there is also the concept of an inner derivation, which is a mapping $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ where $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is defined as $\mathrm{ad}_x(y) = [x, y] \in \mathfrak{g}$ [9]. This inner derivation possesses the properties of a Lie algebra homomorphism and a derivation on $\mathfrak{g}$ [10, 11]. Inner derivations can be constructed through direct sums, current Lie algebras, and abelian extensions. Consequently, these extensions preserve the properties of the original Lie algebra [12].

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A module over a Lie algebra combines the structure of a vector space with an action of the Lie algebra that satisfies specific conditions [13]. Research on these modules has been extensive, including discussions on modules over Lie algebras on semisimple group schemes [14], modular Lie algebras over commutative algebras [15], and modules of the Lie algebra $G(A)$ [16]. The structure of a $\mathfrak{g}$-quasi-Frobenius Lie algebra is discussed in research by Pham [17]. This research covers the general definition as well as concrete examples in 4-dimensional Frobenius Lie algebras.

Despite these contributions, the existing literature has not yet explored the interaction between inner derivations and $\mathfrak{g}$-quasi-Frobenius Lie algebras, particularly in the special case $\mathfrak{q} = \mathfrak{g}$ with $\rho = \mathrm{ad}$. This case is significant because it allows the construction to be expressed entirely in terms of the algebra's own structure, revealing a direct connection between the quasi-Frobenius property and the existence of a distinguished inner derivation.

This research investigates the relationship between Frobenius and quasi-Frobenius Lie algebras, focusing on the case $\mathfrak{q} = \mathfrak{g}$. The main result provides a criterion: a $\mathfrak{g}$-quasi-Frobenius Lie algebra can be constructed on $\mathfrak{g}$ itself if and only if the inner derivation is zero. This result is nontrivial, as it establishes a precise equivalence between a structural condition and a purely algebraic condition, thereby offering a complete characterization in this setting. The criterion is tested using several concrete examples, including a 4-dimensional quasi-Frobenius Lie algebra, and 2-dimensional and 4-dimensional Frobenius Lie algebras.

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries on Lie algebras, Frobenius and quasi-Frobenius, $\mathfrak{g}$-quasi-Frobenius Lie algebras, and inner derivations. Section 3 introduces the main criterion, proves the key theorem and concrete examples. Section 4 concludes the paper with a summary and suggestions for future work.

2. Preliminaries

This study discusses the $\mathfrak{g}$-quasi-Frobenius Lie algebra with $\mathfrak{g} = \mathfrak{q}$. The methodology of this research refers to several stages, namely:

1. `Construct a quasi-Frobenius Lie algebra.` We construct a quasi-Frobenius Lie algebra $(\mathfrak{g}, \beta)$ with a non-degenerate skew-symmetric bilinear form β.
2. `Define the action.` We define a representation $\rho := \mathrm{ad}$ with $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $x \mapsto \rho_x$, which encodes the action of $\mathfrak{g}$ on $\mathfrak{g}$.
3. `Verify the derivation condition.` We verify that for every $x \in \mathfrak{g}$, the map ρ_x satisfies the derivation property on $\mathfrak{g}$.
4. `Establish` $\mathfrak{g}$`-invariance.` We check the $\mathfrak{g}$-invariance condition, which guarantees that the structure $(\mathfrak{g}, \beta, \rho)$ forms a $\mathfrak{g}$-quasi-Frobenius Lie algebra.

The several stage can be visually represented in the following Fig. 1 below:

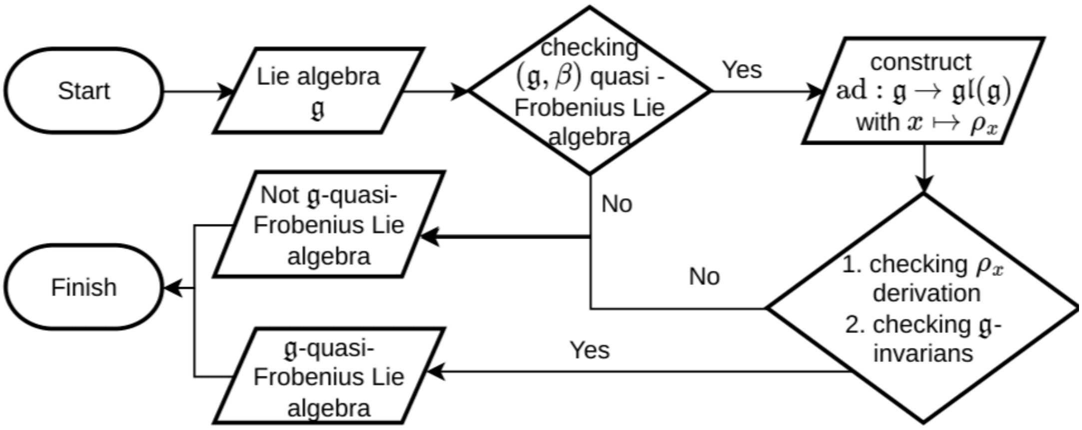

Fig. 1: Diagram of the methodology.

The steps in this study must be based on several related theories. Therefore, several definitions of Lie algebra, inner derivation, Frobenius Lie algebra, quasi-Frobenius Lie algebra, and $\mathfrak{g}$-quasi-Frobenius Lie algebra structures are introduced.

Definition 1 ([1]). *Let $\mathfrak{g}$ be a vector space and let $[.,.] : \mathfrak{g} \times \mathfrak{g} \to (x, y) \mapsto [x, y] \in \mathfrak{g}$ be a map. The pair $(\mathfrak{g}, [.,.])$ is called a Lie algebra if it satisfies the following conditions:*

1. *The bracket $[.,.]$ is a bilinear map.*
2. *It holds that $[x, x] = 0$ for every $x \in \mathfrak{g}$.*
3. *The bracket $[.,.]$ satisfies the Jacobi identity, i.e.,*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{for every } x, y, z \in \mathfrak{g}. \tag{1}$$

Definition 2 ([18]). *A Lie algebra $\mathfrak{g}$ with basis $B = \{x_1, \ldots, x_n\}$ is said to be Frobenius if one of the following equivalent conditions holds:*

1. *The determinant of the matrix $M(\mathfrak{g})$ is non-zero, where $M(\mathfrak{g})$ is the matrix whose entries are determined by the Lie bracket of $\mathfrak{g}$.*
2. $\det(f(M(\mathfrak{g}))) \neq 0$ *for some 1-form $f \in \mathfrak{g}^*$.*

Theorem 1 ([19]). *Let $\mathfrak{g}$ be a Lie algebra with basis $B = \{x_1, x_2, \ldots, x_n\}$. A skew-symmetric 2-form β on $\mathfrak{g}$ is said to be* nondegenerate *if and only if*

$$|M| = \begin{vmatrix} \beta(x_1,x_1) & \beta(x_1,x_2) & \cdots & \beta(x_1,x_j) \\ \beta(x_2,x_1) & \beta(x_2,x_2) & \cdots & \beta(x_2,x_j) \\ \vdots & \vdots & \ddots & \vdots \\ \beta(x_i,x_1) & \beta(x_i,x_2) & \cdots & \beta(x_i,x_n) \end{vmatrix} \neq 0, \quad 1 \leq i, j \leq n \tag{2}$$

where M is the matrix representation of β.

Definition 3 ([17]). A Lie algebra $\mathfrak{g}$ is said to be quasi-Frobenius if there exists a skew-symmetric 2-form β that is both nondegenerate and satifies a 2-cocyle condition in Eq. (3), i.e.,

$$\beta([x,y],z) + \beta([y,z],x) + \beta([z,x],y) = 0, \quad \textit{for every } x, y, z \in \mathfrak{g}. \tag{3}$$

Definition 4 ([9]). Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. Define the mapping $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto \mathrm{ad}(x) := \mathrm{ad}_x$, and the linear map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}, y \mapsto \mathrm{ad}_x(y)$, defined by

$$\mathrm{ad}_x(y) = [x,y], \text{for all } y \in \mathfrak{g}. \tag{4}$$

The mapping $x \mapsto \mathrm{ad}_x$ is called the adjoint mapping or an inner derivation.

Definition 5 ([17]). *A $\mathfrak{g}$-quasi-Frobenius Lie algebra is a tuple $(\mathfrak{q}, \beta, \rho)$ such that $(\mathfrak{q}, \beta)$ is a quasi-Frobenius Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{q}), x \mapsto \rho_x$, is a left $\mathfrak{g}$-module structure on $\mathfrak{q}$ satisfying:*

1. $\rho_x([u,v]) = [\rho_x(u), v] + [u, \rho_x(v)]$ *for every $x \in \mathfrak{g}$ and $u, v \in \mathfrak{q}$ (ρ_x is a derivation on $\mathfrak{q}$),*

2. *$\beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0$ for every $x \in \mathfrak{g}$ and $u, v \in \mathfrak{q}$ ($\mathfrak{g}$-invariance).*

Example 1. Let $\mathfrak{q}$ be a Lie algebra with basis $B = \{x_1, x_2\}$ and a nontrivial bracket

$$[x_1, x_2] = x_2. \tag{5}$$

Based on the determinant $M(\mathfrak{q})$, we obtain

$$\det(M(\mathfrak{q})) = \begin{vmatrix} [x_1, x_1] & [x_1, x_2] \\ [x_2, x_1] & [x_2, x_2] \end{vmatrix} = \begin{vmatrix} 0 & x_2 \\ -x_2 & 0 \end{vmatrix} = x_2^2.$$

Take $\alpha(x_2) = 1$, then $\det(M(\mathfrak{q})) = (1)^2 = 1$. Choose $\beta = x_1^* \wedge x_2^*$ as a bilinear mapping.

$$\beta(x_1, x_1) = x_1^* \wedge x_2^*(x_1, x_1) = \begin{vmatrix} x_1^*(x_1) & x_1^*(x_1) \\ x_2^*(x_1) & x_2^*(x_1) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$
$$\beta(x_1, x_2) = x_1^* \wedge x_2^*(x_1, x_2) = \begin{vmatrix} x_1^*(x_1) & x_1^*(x_2) \\ x_2^*(x_1) & x_2^*(x_2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$
$$\beta(x_2, x_1) = x_1^* \wedge x_2^*(x_2, x_1) = \begin{vmatrix} x_1^*(x_2) & x_1^*(x_1) \\ x_2^*(x_2) & x_2^*(x_1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$
$$\beta(x_2, x_2) = x_1^* \wedge x_2^*(x_2, x_2) = \begin{vmatrix} x_1^*(x_2) & x_1^*(x_2) \\ x_2^*(x_2) & x_2^*(x_2) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0.$$

Since the value of β coincides with α, we obtain $\beta(x, y) = \alpha([x, y])$. Hence, it is proven that $\mathfrak{q}$ is a Frobenius Lie algebra.

Table 1: Verification of the 2-cocycle condition $\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y) = 0$

(x, y, z)	$\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y)$
(x_1, x_1, x_1)	$\beta(0, x_1) + \beta(0, x_1) + \beta(0, x_1) = 0$
(x_1, x_1, x_2)	$\beta(0, x_2) + \beta(x_2, x_1) + \beta(-x_2, x_1) = 0 - 1 + 1 = 0$
(x_1, x_2, x_1)	$\beta(x_2, x_1) + \beta(-x_2, x_1) + \beta(0, x_2) = -1 + 1 + 0 = 0$
(x_1, x_2, x_2)	$\beta(x_2, x_2) + \beta(0, x_1) + \beta(-x_2, x_2) = 0 + 0 + 0 = 0$
(x_2, x_1, x_1)	$\beta(-x_2, x_1) + \beta(0, x_2) + \beta(x_2, x_1) = 1 + 0 - 1 = 0$
(x_2, x_1, x_2)	$\beta(-x_2, x_2) + \beta(x_2, x_2) + \beta(0, x_1) = 0 + 0 + 0 = 0$
(x_2, x_2, x_1)	$\beta(0, x_1) + \beta(-x_2, x_2) + \beta(x_2, x_2) = 0 + 0 + 0 = 0$
(x_2, x_2, x_2)	$\beta(0, x_2) + \beta(0, x_2) + \beta(0, x_2) = 0$

Based on Table 1, it is shown that the 2-cocycle condition is satisfied. Since β is nondegenerate and satisfies the 2-cocycle condition, then $(\mathfrak{q}, \beta)$ is a quasi-Frobenius Lie algebra. Let $\mathfrak{g} = \mathbb{R}^2$ be an abelian Lie algebra with basis $\{e_1, e_2\}$, i.e., $[e_1, e_2] = 0$. Let $q = \mathbb{R}^2$ with basis $\{x_1, x_2\}$. Define a map $\rho : \mathfrak{g} \to \mathfrak{gl}(q)$ by

$$\rho(e_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\mathfrak{g}$ is abelian, we have $[e_1, e_2] = 0$, and thus $\rho([e_1, e_2]) = 0 = [\rho(e_1), \rho(e_2)]$. Hence, ρ defines a representation of $\mathfrak{g}$ on $\mathfrak{q}$. The verification of the derivation condition is presented in the following Table 2.

Table 2: Verification of derivation condition

(e,x,y)	$\rho(e)([x,y])$	$[\rho(e)(y),z]+[y,\rho(e)(z)]$
(e_1,x_1,x_1)	$\rho(e_1)(0)=0$	$[\rho(e_1)(x_1),x_1]+[x_1,\rho(e_1)(x_1)]=[x_2,x_1]+[x_1,x_2]=-x_2+x_2=0$
(e_1,x_1,x_2)	$\rho(e_1)(x_2)=0$	$[\rho(e_1)(x_1),x_2]+[x_1,\rho(e_1)(x_2)]=[x_2,x_2]+[x_1,0]=0+0=0$
(e_1,x_2,x_1)	$\rho(e_1)(-x_2)=0$	$[\rho(e_1)(x_2),x_1]+[x_2,\rho(e_1)(x_1)]=[0,x_1]+[x_2,x_2]=0+0=0$
(e_1,x_2,x_2)	$\rho(e_1)(0)=0$	$[\rho(e_1)(x_2),x_2]+[x_2,\rho(e_1)(x_2)]=[0,x_2]+[x_2,0]=0$
(e_2,x_1,x_1)	$\rho(e_2)(0)=0$	$[\rho(e_2)(x_1),x_1]+[x_1,\rho(e_2)(x_1)]=[0,x_1]+[x_1,0]=0$
(e_2,x_1,x_2)	$\rho(e_2)(x_2)=0$	$[\rho(e_2)(x_1),x_2]+[x_1,\rho(e_2)(x_2)]=[0,x_2]+[x_1,0]=0$
(e_2,x_2,x_1)	$\rho(e_2)(-x_2)=0$	$[\rho(e_2)(x_2),x_1]+[x_2,\rho(e_2)(x_1)]=[0,x_1]+[x_2,0]=0$
(e_2,x_2,x_2)	$\rho(e_2)(0)=0$	$[\rho(e_2)(x_2),x_2]+[x_2,\rho(e_2)(x_2)]=[0,x_2]+[x_2,0]=0$

Based on Table 2, the derivation condition is satisfied. Finally, we verify the $\mathfrak{g}$-invariance condition at the following Table 3.

Table 3: Verification of $\mathfrak{g}$-invariance condition

(e,x,y)	$\beta(\rho(e)(x),y)+\beta(x,\rho(e)(y))$	Result
(e_1,x_1,x_1)	$\beta(x_2,x_1)+\beta(x_1,x_2)$	$-1+1=0$
(e_1,x_1,x_2)	$\beta(x_2,x_2)+\beta(x_1,0)$	$=0$
(e_1,x_2,x_1)	$\beta(0,x_1)+\beta(x_2,x_2)$	0
(e_1,x_2,x_2)	$\beta(0,x_2)+\beta(x_2,0)$	0
(e_2,x_1,x_1)	$\beta(0,x_1)+\beta(x_1,0)$	0
(e_2,x_1,x_2)	$\beta(0,x_2)+\beta(x_1,0)$	0
(e_2,x_2,x_1)	$\beta(0,x_1)+\beta(x_2,0)$	0
(e_2,x_2,x_2)	$\beta(0,x_2)+\beta(x_2,0)$	0

Based on the table above, the $\mathfrak{g}$-invariance condition is satisfied. Therefore, $(\mathfrak{g},\beta,\rho)$ forms a $\mathfrak{g}$-quasi-Frobenius Lie algebra.

3. Results and Discussion

This section presents the main results of $\mathfrak{g}$-quasi-Frobenius Lie algebra structures with various types of Lie algebra $\mathfrak{g}$. Before that, we will present some propositions that are related to the main results of this study.

Proposition 1 ([17]). *A Frobenius Lie algebra over k is a pair $(\mathfrak{g},\alpha)$ where $\mathfrak{g}$ is a Lie algebra and $\alpha:\mathfrak{g}\to k$ is a linear map such that the skew-symmetric bilinear form β on $\mathfrak{g}$ defined by*

$$\beta(x,y):=\alpha([x,y]),\quad \forall x,y\in\mathfrak{g} \tag{6}$$

is non-degenerate.

Proof. Suppose $\mathfrak{g}$ is an n-dimensional Frobenius Lie algebra with basis $\{e_1,e_2,\dots,e_n\}$. This means there exists $\alpha\in\mathfrak{g}^*$ such that $\det\left(\alpha([e_i,e_j])\right)\neq 0$. Define a bilinear form $\beta:\mathfrak{g}\times\mathfrak{g}\to k$ by

$$\beta(x,y)=\alpha([x,y]),\quad \text{for all } x,y\in\mathfrak{g}.$$

By Definition 1, β is clearly bilinear and skew-symmetric since the Lie bracket satisfies $[x,y]=-[y,x]$. The matrix of β with respect to the basis $\{e_1,\dots,e_n\}$ is

$$(\beta(e_i,e_j))=(\alpha([e_i,e_j])).$$

By assumption, the determinant of this matrix is non-zero, hence the matrix is invertible. Consequently, the bilinear form β is non-degenerate, that is, if $\beta(x, y) = 0$ for all $y \in \mathfrak{g}$, then $x = 0$. Therefore, there exists a linear map $\alpha \in \mathfrak{g}^*$ such that the skew-symmetric bilinear form $\beta(x, y) = \alpha([x, y])$ is non-degenerate. The pair $(\mathfrak{g}, \alpha)$ is a Frobenius Lie algebra. ☐

Proposition 2 ([17])**.** *If $(\mathfrak{g}, \beta)$ is a Frobenius Lie algebra, then $(\mathfrak{g}, \beta)$ is a quasi-Frobenius Lie algebra.*

Proof. Suppose $(\mathfrak{g}, \beta)$ is a Frobenius Lie algebra, and define $\beta(x, y) = \alpha([x, y])$. By the definition of a Frobenius Lie algebra, the bilinear form β is non-degenerate. It remains to prove that β satisfies the 2-cocycle condition in Eq. (3). Using the definition of β, we obtain

$$\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y) = \alpha([[x, y], z]) + \alpha([[y, z], x]) + \alpha([[z, x], y]).$$

By the Jacobi identity in Eq. (1), we have

$$\alpha([[x, y], z] + [[y, z], x] + [[z, x], y]) = \alpha(0) = 0.$$

Thus β satisfies the 2-cocycle condition. Since β is non-degenerate and a 2-cocycle, it follows that $(\mathfrak{g}, \beta)$ is a quasi-Frobenius Lie algebra. ☐

Theorem 2. *Let $\mathfrak{g}$ be a quasi-Frobenius Lie algebra. A $\mathfrak{g}$-quasi-Frobenius Lie algebra structure can be formed on $\mathfrak{g}$ itself if and only if the inner derivation is zero.*

Proof. We prove the theorem in two parts. Suppose $\mathfrak{g}$ is a quasi-Frobenius Lie algebra. This means there exists a 2-form β which is nondegenerate and a 2-cocycle. Define $\rho := \mathrm{ad}$ with $\mathrm{ad}_x : \mathfrak{g} \ni y \mapsto \mathrm{ad}_x(y) = [x, y] \in \mathfrak{g}$. First, we verify that ad_x satisfies the derivation property on $\mathfrak{g}$. For any $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} \mathrm{ad}_x([y, z]) &= [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)] = [[x, y], z] + [y, [x, z]] \\ &= [[x, y], z] - [y, [z, x]] = [[x, y], z] + [[z, x], y] \\ &= -[[y, z], x] = [x, [y, z]] = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]. \end{aligned}$$

Thus, the derivation condition holds for all $x \in \mathfrak{g}$. Next, we examine the $\mathfrak{g}$-invariance condition. Using the properties of β and the Jacobi identity,

$$\begin{aligned} \beta(\mathrm{ad}_x(y), z) + \beta(y, \mathrm{ad}_x(z)) &= \beta([x, y], z) + \beta(y, [x, z]) = \beta([x, y], z) - \beta([x, z], y) \\ &= \beta([x, y], z) + \beta([z, x], y) = -\beta([y, z], x) \\ &= \beta(x, [y, z]). \end{aligned}$$

For ad_x to satisfy the $\mathfrak{g}$-invariance condition required for a $\mathfrak{g}$-quasi-Frobenius structure, we must have $\beta(x, [y, z]) = 0, \quad \forall x, y, z \in \mathfrak{g}$. Since β is nondegenerate, this condition forces $[y, z] = 0$ for all $y, z \in \mathfrak{g}$. Hence $\mathfrak{g}$ is abelian. Consequently, for every $x \in \mathfrak{g}$, ad_x is the zero map. This establishes the necessity: if a $\mathfrak{g}$-quasi-Frobenius structure exists on $\mathfrak{g}$ itself with $\rho = \mathrm{ad}$, then the inner derivation must be zero. Conversely, assume that the inner derivation is zero, i.e., $\mathrm{ad}_x = 0$ for all $x \in \mathfrak{g}$. Then for every $x \in \mathfrak{g}$, the map ad_x trivially satisfies the derivation property:

$$\mathrm{ad}_x([y, z]) = 0 = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)].$$

Moreover, the $\mathfrak{g}$-invariance condition also holds:

$$\beta(\mathrm{ad}_x(y), z) + \beta(y, \mathrm{ad}_x(z)) = \beta(0, z) + \beta(y, 0) = 0.$$

Thus, with $\rho = \mathrm{ad}$ and ad_x being the zero map, the triple $(\mathfrak{g}, \beta, \mathrm{ad})$ forms a $\mathfrak{g}$-quasi-Frobenius Lie algebra. This completes the proof. □

Theorem 2 provides a complete characterization of when a $\mathfrak{g}$-quasi-Frobenius Lie algebra can be constructed on $\mathfrak{g}$ itself with the adjoint representation. The result reveals that such a construction is possible *if and only if* $\mathfrak{g}$ is abelian, i.e., the inner derivation is trivial. In other words, any non-zero inner derivation obstructs the existence of a $\mathfrak{g}$-quasi-Frobenius structure under the adjoint action. To illustrate Theorem 2, we consider the following Example 2.

Example 2. Let $\mathbb{C}^4$ have basis $\{x_1, x_2, x_3, x_4\}$. For every $\bar{a}, \bar{b} \in \mathbb{C}^4$, define the bracket

$$[\bar{a}, \bar{b}] = \begin{bmatrix} a_1b_1 - b_1a_1 \\ a_2b_2 - b_2a_2 \\ a_3b_3 - b_3a_3 \\ a_4b_4 - b_4a_4 \end{bmatrix} \in \mathbb{C}^4.$$

We will prove that this bracket is a Lie bracket. Let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{C}^4$ and $k \in \mathbb{C}$.

$$[\bar{a}, \bar{b} + \bar{c}] = \begin{bmatrix} a_1(b_1 + c_1) - (b_1 + c_1)a_1 \\ a_2(b_2 + c_2) - (b_2 + c_2)a_2 \\ a_3(b_3 + c_3) - (b_3 + c_3)a_3 \\ a_4(b_4 + c_4) - (b_4 + c_4)a_4 \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_1c_1 - (b_1a_1 + c_1a_1) \\ a_2b_2 + a_2c_2 - (b_2a_2 + c_2a_2) \\ a_3b_3 + a_3c_3 - (b_3a_3 + c_3a_3) \\ a_4b_4 + a_4c_4 - (b_4a_4 + c_4a_4) \end{bmatrix} = [\bar{a}, \bar{b}] + [\bar{a}, \bar{c}].$$

$$[\bar{a} + \bar{b}, \bar{c}] = \begin{bmatrix} (a_1 + b_1)c_1 - c_1(a_1 + b_1) \\ (a_2 + b_2)c_2 - c_2(a_2 + b_2) \\ (a_3 + b_3)c_3 - c_3(a_3 + b_3) \\ (a_4 + b_4)c_4 - c_4(a_4 + b_4) \end{bmatrix} = \begin{bmatrix} a_1c_1 + b_1c_1 - (c_1a_1 + c_1b_1) \\ a_2c_2 + b_2c_2 - (c_2a_2 + c_2b_2) \\ a_3c_3 + b_3c_3 - (c_3a_3 + c_3b_3) \\ a_4c_4 + b_4c_4 - (c_4a_4 + c_4b_4) \end{bmatrix} = [\bar{a}, \bar{c}] + [\bar{b}, \bar{c}].$$

$$[k\bar{a}, \bar{b}] = \begin{bmatrix} ka_1b_1 - b_1ka_1 \\ ka_2b_2 - b_2ka_2 \\ ka_3b_3 - b_3ka_3 \\ ka_4b_4 - b_4ka_4 \end{bmatrix} = \begin{bmatrix} k(a_1b_1 - b_1a_1) \\ k(a_2b_2 - b_2a_2) \\ k(a_3b_3 - b_3a_3) \\ k(a_4b_4 - b_4a_4) \end{bmatrix} = k \begin{bmatrix} a_1b_1 - b_1a_1 \\ a_2b_2 - b_2a_2 \\ a_3b_3 - b_3a_3 \\ a_4b_4 - b_4a_4 \end{bmatrix} = k[\bar{a}, \bar{b}].$$

$$[\bar{a}, k\bar{b}] = \begin{bmatrix} a_1kb_1 - kb_1a_1 \\ a_2kb_2 - kb_2a_2 \\ a_3kb_3 - kb_3a_3 \\ a_4kb_4 - kb_4a_4 \end{bmatrix} = \begin{bmatrix} k(a_1b_1 - b_1a_1) \\ k(a_2b_2 - b_2a_2) \\ k(a_3b_3 - b_3a_3) \\ k(a_4b_4 - b_4a_4) \end{bmatrix} = k \begin{bmatrix} a_1b_1 - b_1a_1 \\ a_2b_2 - b_2a_2 \\ a_3b_3 - b_3a_3 \\ a_4b_4 - b_4a_4 \end{bmatrix} = k[\bar{a}, \bar{b}].$$

Thus, the Lie bracket is proven to be a bilinear map.

$$[\bar{a}, \bar{a}] = \begin{bmatrix} a_1a_1 - a_1a_1 \\ a_2a_2 - a_2a_2 \\ a_3a_3 - a_3a_3 \\ a_4a_4 - a_4a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is proven $[\bar{a}, \bar{a}] = 0$.

$$[\bar{a}, [\bar{b}, \bar{c}]] = \left[\bar{a}, \begin{bmatrix} b_1c_1 - c_1b_1 \\ b_2c_2 - c_2b_2 \\ b_3c_3 - c_3b_3 \\ b_4c_4 - c_4b_4 \end{bmatrix} \right] = \left[\bar{a}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$[\bar{b},[\bar{c},\bar{a}]] = \left[\bar{b}, \begin{bmatrix} c_1a_1 - a_1c_1 \\ c_2a_2 - a_2c_2 \\ c_3a_3 - a_3c_3 \\ c_4a_4 - a_4c_4 \end{bmatrix}\right] = \left[\bar{b}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$[\bar{c},[\bar{a},\bar{b}]] = \left[\bar{c}, \begin{bmatrix} a_1b_1 - b_1a_1 \\ a_2b_2 - b_2a_2 \\ a_3b_3 - b_3a_3 \\ a_4b_4 - b_4a_4 \end{bmatrix}\right] = \left[\bar{c}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, the Jacobi identity at Eq. (1) is satisfied. Therefore, it is proven that $\mathbb{C}^4$ with the zero bracket is a Lie algebra. Let $\mathbb{C}^4$ have basis $\{x_1, x_2, x_3, x_4\}$ and Lie bracket $[x_i, x_j] = 0$ for $i, j \in \{1, 2, 3, 4\}$. To prove that $\mathfrak{g}$ is a quasi-Frobenius Lie algebra, choose the map $\beta = x_1^* \wedge x_2^* + x_3^* \wedge x_4^*$ so that

Table 4: Calculation of $\beta(x_i, x_j)$

(x_i, x_j)	$\beta(x_i, x_j)$
(x_1, x_1)	$\begin{vmatrix} x_1^*(x_1) & x_1^*(x_1) \\ x_2^*(x_1) & x_2^*(x_1) \end{vmatrix} + \begin{vmatrix} x_3^*(x_1) & x_3^*(x_1) \\ x_4^*(x_1) & x_4^*(x_1) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$
(x_1, x_2)	$\begin{vmatrix} x_1^*(x_1) & x_1^*(x_2) \\ x_2^*(x_1) & x_2^*(x_2) \end{vmatrix} + \begin{vmatrix} x_3^*(x_1) & x_3^*(x_2) \\ x_4^*(x_1) & x_4^*(x_2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 1$
(x_1, x_3)	$\begin{vmatrix} x_1^*(x_1) & x_1^*(x_3) \\ x_2^*(x_1) & x_2^*(x_3) \end{vmatrix} + \begin{vmatrix} x_3^*(x_1) & x_3^*(x_3) \\ x_4^*(x_1) & x_4^*(x_3) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$
(x_1, x_4)	$\begin{vmatrix} x_1^*(x_1) & x_1^*(x_4) \\ x_2^*(x_1) & x_2^*(x_4) \end{vmatrix} + \begin{vmatrix} x_3^*(x_1) & x_3^*(x_4) \\ x_4^*(x_1) & x_4^*(x_4) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$
(x_2, x_1)	$\begin{vmatrix} x_1^*(x_2) & x_1^*(x_1) \\ x_2^*(x_2) & x_2^*(x_1) \end{vmatrix} + \begin{vmatrix} x_3^*(x_2) & x_3^*(x_1) \\ x_4^*(x_2) & x_4^*(x_1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = -1$
(x_2, x_2)	$\begin{vmatrix} x_1^*(x_2) & x_1^*(x_2) \\ x_2^*(x_2) & x_2^*(x_2) \end{vmatrix} + \begin{vmatrix} x_3^*(x_2) & x_3^*(x_2) \\ x_4^*(x_2) & x_4^*(x_2) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$
(x_2, x_3)	$\begin{vmatrix} x_1^*(x_2) & x_1^*(x_3) \\ x_2^*(x_2) & x_2^*(x_3) \end{vmatrix} + \begin{vmatrix} x_3^*(x_2) & x_3^*(x_3) \\ x_4^*(x_2) & x_4^*(x_3) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$
(x_2, x_4)	$\begin{vmatrix} x_1^*(x_2) & x_1^*(x_4) \\ x_2^*(x_2) & x_2^*(x_4) \end{vmatrix} + \begin{vmatrix} x_3^*(x_2) & x_3^*(x_4) \\ x_4^*(x_2) & x_4^*(x_4) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$
(x_3, x_1)	$\begin{vmatrix} x_1^*(x_3) & x_1^*(x_1) \\ x_2^*(x_3) & x_2^*(x_1) \end{vmatrix} + \begin{vmatrix} x_3^*(x_3) & x_3^*(x_1) \\ x_4^*(x_3) & x_4^*(x_1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$
(x_3, x_2)	$\begin{vmatrix} x_1^*(x_3) & x_1^*(x_2) \\ x_2^*(x_3) & x_2^*(x_2) \end{vmatrix} + \begin{vmatrix} x_3^*(x_3) & x_3^*(x_2) \\ x_4^*(x_3) & x_4^*(x_2) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$
(x_3, x_3)	$\begin{vmatrix} x_1^*(x_3) & x_1^*(x_3) \\ x_2^*(x_3) & x_2^*(x_3) \end{vmatrix} + \begin{vmatrix} x_3^*(x_3) & x_3^*(x_3) \\ x_4^*(x_3) & x_4^*(x_3) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$
(x_3, x_4)	$\begin{vmatrix} x_1^*(x_3) & x_1^*(x_4) \\ x_2^*(x_3) & x_2^*(x_4) \end{vmatrix} + \begin{vmatrix} x_3^*(x_3) & x_3^*(x_4) \\ x_4^*(x_3) & x_4^*(x_4) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$
(x_4, x_1)	$\begin{vmatrix} x_1^*(x_4) & x_1^*(x_1) \\ x_2^*(x_4) & x_2^*(x_1) \end{vmatrix} + \begin{vmatrix} x_3^*(x_4) & x_3^*(x_1) \\ x_4^*(x_4) & x_4^*(x_1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$
(x_4, x_2)	$\begin{vmatrix} x_1^*(x_4) & x_1^*(x_2) \\ x_2^*(x_4) & x_2^*(x_2) \end{vmatrix} + \begin{vmatrix} x_3^*(x_4) & x_3^*(x_2) \\ x_4^*(x_4) & x_4^*(x_2) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$
(x_4, x_3)	$\begin{vmatrix} x_1^*(x_4) & x_1^*(x_3) \\ x_2^*(x_4) & x_2^*(x_3) \end{vmatrix} + \begin{vmatrix} x_3^*(x_4) & x_3^*(x_3) \\ x_4^*(x_4) & x_4^*(x_3) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$
(x_4, x_4)	$\begin{vmatrix} x_1^*(x_4) & x_1^*(x_4) \\ x_2^*(x_4) & x_2^*(x_4) \end{vmatrix} + \begin{vmatrix} x_3^*(x_4) & x_3^*(x_4) \\ x_4^*(x_4) & x_4^*(x_4) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$

Based on the Table 4, the representation matrix of β is obtained as $M(\beta)$ which can be expressed as follows

$$M(\beta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Since $|M(\beta)| = 1 \neq 0$, then β is nondegenerate. Meanwhile, to satisfy the 2-cocycle condition, the fulfillment of Eq. (3) will be proven

$$\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y) = \beta(0, z) + \beta(0, x) + \beta(0, y) = 0$$

Thus, the 2-cocycle condition is satisfied. Since β is nondegenerate and satisfies the 2-cocycle condition, then $(\mathfrak{g}, \beta)$ is a quasi-Frobenius Lie algebra. Define $\rho := \mathrm{ad}$ with $\mathrm{ad}_x : \mathfrak{g} \ni y \mapsto \mathrm{ad}_x(y) = [x, y] = 0 \in \mathfrak{g}$. Based on Theorem 2, it follows that $(\mathfrak{g}, \rho, \mathrm{ad})$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra.

Corollary 1. *Let $\mathfrak{g}$ be a quasi-Frobenius Lie algebra. If there exists a nonzero inner derivation, then the structure of a $\mathfrak{g}$-quasi-Frobenius Lie algebra cannot be constructed from itself using inner derivations.*

Example 3. Based on Example 1, it follows that $(\mathfrak{q}, \beta)$ is a quasi-Frobenius Lie algebra. Define $\mathfrak{g} := \mathfrak{q}$ and $\rho := \mathrm{ad}$ with $\mathrm{ad}_x : \mathfrak{g} \ni y \mapsto \mathrm{ad}_x(y) = [x, y] \in \mathfrak{g}$. Therefore, to verify that ad_x is a derivation on $\mathfrak{g}$, we check the condition $\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$ or $\mathrm{ad}_x([y, z]) = [[x, y], z] + [y, [x, z]]$. The verification of the derivation property is presented in the following Table 5.

Table 5: Verification of derivation condition $\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$

(x, y, z)	$[x, [y, z]]$	$[[x, y], z] + [y, [x, z]]$
(x_1, x_1, x_1)	$[x_1, 0] = 0$	$[0, x_1] + [x_1, 0] = 0 + 0 = 0$
(x_1, x_1, x_2)	$[x_1, x_2] = x_2$	$[0, x_2] + [x_1, x_2] = 0 + x_2 = x_2$
(x_1, x_2, x_1)	$[x_1, -x_2] = -x_2$	$[x_2, x_1] + [x_2, 0] = -x_2 + 0 = -x_2$
(x_1, x_2, x_2)	$[x_1, 0] = 0$	$[x_2, x_2] + [x_2, x_2] = 0 + 0 = 0$
(x_2, x_1, x_1)	$[x_2, 0] = 0$	$[-x_2, x_1] + [x_1, -x_2] = x_2 - x_2 = 0$
(x_2, x_1, x_2)	$[x_2, x_2] = 0$	$[-x_2, x_2] + [x_1, 0] = 0 + 0 = 0$
(x_2, x_2, x_1)	$[x_2, -x_2] = 0$	$[0, x_1] + [x_2, -x_2] = 0 + 0 = 0$
(x_2, x_2, x_2)	$[x_2, 0] = 0$	$[0, x_2] + [x_2, 0] = 0 + 0 = 0$

Table 6: Verification of $\mathfrak{g}$-invariance condition $\beta([x_i, x_j], x_k) + \beta(x_j, [x_i, x_k]) = 0$

(x_i, x_j, x_k)	Result
(x_1, x_1, x_1)	$\beta(0, x_1) + \beta(x_1, 0) = 0 + 0 = 0$
(x_1, x_1, x_2)	$\beta(0, x_2) + \beta(x_1, 0) = 0 + 0 = 0$
(x_1, x_2, x_1)	$\beta(x_2, x_1) + \beta(x_2, 0) = -1 + 0 = -1$
(x_1, x_2, x_2)	$\beta(x_2, x_2) + \beta(x_2, x_2) = 0 + 0 = 0$
(x_2, x_1, x_1)	$\beta(-x_2, x_1) + \beta(x_1, 0) = 1 + 0 = 1$
(x_2, x_1, x_2)	$\beta(-x_2, x_2) + \beta(x_1, -x_2) = 0 + (-1) = -1$
(x_2, x_2, x_1)	$\beta(0, x_1) + \beta(x_2, 0) = 0 + 0 = 0$
(x_2, x_2, x_2)	$\beta(0, x_2) + \beta(x_2, 0) = 0 + 0 = 0$

Based on Table 5, it is shown that ad_x is a derivation on $\mathfrak{g}$. Next, we examine the $\mathfrak{g}$-invariance condition based on the equation $\beta(\mathrm{ad}_x(y), z)+\beta(y, \mathrm{ad}_x(z)) = 0$ or $\beta([x, y], z)+\beta(y, [x, z]) = 0$. To illustrate the $\mathfrak{g}$-invariance condition, it is presented in the following table. Based on Table 6, it is shown that some of the $\mathfrak{g}$-invariance conditions are not satisfied. Hence, with the inner derivation $(\mathfrak{g}, \beta, \mathrm{ad})$, it follows that is not a $\mathfrak{g}$-quasi-Frobenius Lie algebra.

Example 4. Let $\mathfrak{g}$ be a 4–dimensional Lie algebra with basis $\{x_1, x_2, x_3, x_4\}$ whose nontrivial Lie brackets are

$$[x_1, x_4] = [x_2, x_3] = -x_1, \qquad [x_2, x_4] = -\tfrac{1}{2}x_2, \qquad [x_3, x_4] = -\tfrac{1}{2}x_3.$$

Construct the matrix $M(\mathfrak{g})$ and compute its determinant

$$\det(M(\mathfrak{g})) = \begin{vmatrix} [x_1,x_1] & [x_1,x_2] & [x_1,x_3] & [x_1,x_4] \\ [x_2,x_1] & [x_2,x_2] & [x_2,x_3] & [x_2,x_4] \\ [x_3,x_1] & [x_3,x_2] & [x_3,x_3] & [x_3,x_4] \\ [x_4,x_1] & [x_4,x_2] & [x_4,x_3] & [x_4,x_4] \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & -x_1 \\ 0 & 0 & -x_1 & -\frac{1}{2}x_2 \\ 0 & x_1 & 0 & -\frac{1}{2}x_3 \\ x_1 & \frac{1}{2}x_2 & \frac{1}{2}x_3 & 0 \end{vmatrix}$$

$$= (-1)(-x_1) \begin{vmatrix} 0 & 0 & -x_1 \\ 0 & x_1 & 0 \\ x_1 & \frac{1}{2}x_2 & \frac{1}{2}x_3 \end{vmatrix} = x_1 \left(-x_1 \begin{vmatrix} 0 & x_1 \\ x_1 & \frac{1}{2}x_2 \end{vmatrix} \right) = -x_1^2(-x_1^2) = x_1^4.$$

Let $\alpha : \mathfrak{g} \to \mathbb{R}$ be a linear map defined by $\alpha(x_i) = 0$ for $i = 2, 3, 4$ and $\alpha(x_1) = 1$. Choose the bilinear map

$$\beta = x_4^* \wedge x_1^* - x_2^* \wedge x_3^*. \tag{7}$$

The values of $\beta(x_i, x_j)$ and $\alpha([x_i, x_j])$ can be presented in the following Table 7.

Table 7: Calculation $\beta(x_i, x_j)$ and $\alpha([x_i, x_j])$

(x_i, x_j)	$\beta(x_i, x_j)$	$\alpha([x_i, x_j])$
(x_1, x_1)	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0$	$\alpha(0) = 0$
(x_1, x_2)	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$	$\alpha(0) = 0$
(x_1, x_3)	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0$	$\alpha(0) = 0$
(x_1, x_4)	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = -1$	$\alpha(-x_1) = -1$
(x_2, x_1)	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$	$\alpha(0) = 0$
(x_2, x_2)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$	$\alpha(0) = 0$
(x_2, x_3)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1$	$\alpha(-x_1) = -1$
(x_2, x_4)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$	$\alpha\left(-\frac{1}{2}x_2\right) = 0$
(x_3, x_1)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$	$\alpha(0) = 0$
(x_3, x_2)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 1$	$\alpha(x_1) = 1$
(x_3, x_3)	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0$	$\alpha(0) = 0$
(x_3, x_4)	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$	$\alpha\left(-\frac{1}{2}x_3\right) = 0$

(x_4, x_1)	$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$= 1$	$\alpha(x_1) = 1$
(x_4, x_2)	$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$= 0$	$\alpha\left(\frac{1}{2}x_2\right) = 0$
(x_4, x_3)	$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$= 0$	$\alpha\left(\frac{1}{2}x_3\right) = 0$
(x_4, x_4)	$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$= 0$	$\alpha(0) = 0$

Based on Table 7, $\beta(\bar{x}, \bar{y}) = \alpha([\bar{x}, \bar{y}])$ for every $\bar{x}, \bar{y} \in \{x_1, x_2, x_3, x_4\}$ and β is nondegenerate from Proposition 1, it follows that $(\mathfrak{g}, \beta)$ is a Frobenius Lie algebra. Based on Proposition 2, $(\mathfrak{g}, \beta)$ is a quasi-Frobenius Lie algebra. Define $\rho := \mathrm{ad}$ with $\mathrm{ad}_x : \mathfrak{g} \ni y \mapsto \mathrm{ad}_x(y) = [x, y] \in \mathfrak{g}$. Therefore, to verify that ad_x is a derivation on $\mathfrak{g}$, we check

$$\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)] \quad \text{or} \quad [x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

The derivations are presented in the Table 8.

Table 8: Verification of derivation condition $\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$

(x, y, z)	$[x, [y, z]]$	$[[x, y], z] + [y, [x, z]]$
(x_1, x_1, x_1)	$[x_1, 0] = 0$	$[0, x_1] + [x_1, 0] = 0$
(x_1, x_1, x_2)	$[x_1, 0] = 0$	$[0, x_2] + [x_1, 0] = 0$
(x_1, x_1, x_3)	$[x_1, 0] = 0$	$[0, x_3] + [x_1, 0] = 0$
(x_1, x_1, x_4)	$[x_1, 0] = 0$	$[0, x_4] + [x_1, 0] = 0$
(x_1, x_2, x_1)	$[x_1, 0] = 0$	$[0, x_1] + [x_2, 0] = 0$
(x_1, x_2, x_2)	$[x_1, 0] = 0$	$[0, x_2] + [x_2, 0] = 0$
(x_1, x_2, x_3)	$[x_1, -x_1] = 0$	$[0, x_3] + [x_2, 0] = 0$
(x_1, x_2, x_4)	$[x_1, -\frac{1}{2}x_2] = 0$	$[0, x_4] + [x_2, -x_1] = 0$
(x_1, x_3, x_1)	$[x_1, 0] = 0$	$[0, x_1] + [x_3, 0] = 0$
(x_1, x_3, x_2)	$[x_1, 0] = 0$	$[0, x_2] + [x_3, 0] = 0$
(x_1, x_3, x_3)	$[x_1, 0] = 0$	$[0, x_3] + [x_3, 0] = 0$
(x_1, x_3, x_4)	$[x_1, -\frac{1}{2}x_3] = 0$	$[0, x_4] + [x_3, -x_1] = 0$
(x_1, x_4, x_1)	$[x_1, -x_1] = 0$	$[0, x_1] + [x_4, 0] = 0$
(x_1, x_4, x_2)	$[x_1, -\frac{1}{2}x_2] = 0$	$[0, x_2] + [x_4, 0] = 0$
(x_1, x_4, x_3)	$[x_1, -\frac{1}{2}x_3] = 0$	$[0, x_3] + [x_4, 0] = 0$
(x_1, x_4, x_4)	$[x_1, 0] = 0$	$[0, x_4] + [x_4, 0] = 0$
(x_2, x_1, x_1)	$[x_2, 0] = 0$	$[0, x_1] + [x_1, 0] = 0$
(x_2, x_1, x_2)	$[x_2, 0] = 0$	$[0, x_2] + [x_1, 0] = 0$
(x_2, x_1, x_3)	$[x_2, 0] = 0$	$[0, x_3] + [x_1, 0] = 0$
(x_2, x_1, x_4)	$[x_2, -x_1] = 0$	$[0, x_4] + [x_1, 0] = 0$
(x_2, x_2, x_1)	$[x_2, 0] = 0$	$[0, x_1] + [x_2, 0] = 0$
(x_2, x_2, x_2)	$[x_2, 0] = 0$	$[0, x_2] + [x_2, 0] = 0$
(x_2, x_2, x_3)	$[x_2, -x_1] = 0$	$[0, x_3] + [x_2, 0] = 0$
(x_2, x_2, x_4)	$[x_2, -\frac{1}{2}x_2] = 0$	$[0, x_4] + [x_2, 0] = 0$
(x_2, x_3, x_1)	$[x_2, -x_1] = 0$	$[-x_1, x_1] + [x_3, 0] = 0$
(x_2, x_3, x_2)	$[x_2, -x_1] = 0$	$[-x_1, x_2] + [x_3, 0] = 0$
(x_2, x_3, x_3)	$[x_2, 0] = 0$	$[-x_1, x_3] + [x_3, 0] = 0$
(x_2, x_3, x_4)	$[x_2, -\frac{1}{2}x_3] = \frac{1}{2}x_1$	$[-x_1, x_4] + [x_3, -\frac{1}{2}x_2] = \frac{1}{2}x_1$
(x_2, x_4, x_1)	$[x_2, -\frac{1}{2}x_2] = 0$	$[-\frac{1}{2}x_2, x_1] + [x_4, 0] = 0$
(x_2, x_4, x_2)	$[x_2, -\frac{1}{2}x_2] = 0$	$[-\frac{1}{2}x_2, x_2] + [x_4, 0] = 0$
(x_2, x_4, x_3)	$[x_2, -\frac{1}{2}x_3] = \frac{1}{2}x_1$	$[-\frac{1}{2}x_2, x_3] + [x_4, -\frac{1}{2}x_3] = \frac{1}{2}x_1$
(x_2, x_4, x_4)	$[x_2, 0] = 0$	$[-\frac{1}{2}x_2, x_4] + [x_4, -\frac{1}{2}x_2] = 0$

(x_3, x_1, x_1)	$[x_3, 0] = 0$	$[0, x_1] + [x_1, 0] = 0$
(x_3, x_1, x_2)	$[x_3, 0] = 0$	$[0, x_2] + [x_1, 0] = 0$
(x_3, x_1, x_3)	$[x_3, 0] = 0$	$[0, x_3] + [x_1, 0] = 0$
(x_3, x_1, x_4)	$[x_3, -x_1] = 0$	$[0, x_4] + [x_1, 0] = 0$
(x_3, x_2, x_1)	$[x_3, x_1] = 0$	$[x_1, x_1] + [x_2, 0] = 0$
(x_3, x_2, x_2)	$[x_3, 0] = 0$	$[x_1, x_2] + [x_2, 0] = 0$
(x_3, x_2, x_3)	$[x_3, 0] = 0$	$[x_1, x_3] + [x_2, 0] = 0$
(x_3, x_2, x_4)	$[x_3, -\frac{1}{2}x_2] = -\frac{1}{2}x_1$	$[x_1, x_4] + [x_2, 0] = -x_1$
(x_3, x_3, x_1)	$[x_3, 0] = 0$	$[0, x_1] + [x_3, 0] = 0$
(x_3, x_3, x_2)	$[x_3, 0] = 0$	$[0, x_2] + [x_3, 0] = 0$
(x_3, x_3, x_3)	$[x_3, 0] = 0$	$[0, x_3] + [x_3, 0] = 0$
(x_3, x_3, x_4)	$[x_3, -\frac{1}{2}x_3] = 0$	$[0, x_4] + [x_3, 0] = 0$
(x_3, x_4, x_1)	$[x_3, -\frac{1}{2}x_3] = 0$	$[-\frac{1}{2}x_3, x_1] + [x_4, 0] = 0$
(x_3, x_4, x_2)	$[x_3, -\frac{1}{2}x_3] = 0$	$[-\frac{1}{2}x_3, x_2] + [x_4, -\frac{1}{2}x_2] = 0$
(x_3, x_4, x_3)	$[x_3, -\frac{1}{2}x_3] = 0$	$[-\frac{1}{2}x_3, x_3] + [x_4, 0] = 0$
(x_3, x_4, x_4)	$[x_3, 0] = 0$	$[-\frac{1}{2}x_3, x_4] + [x_4, -\frac{1}{2}x_3] = 0$
(x_4, x_1, x_1)	$[x_4, 0] = 0$	$[x_1, x_1] + [x_1, 0] = 0$
(x_4, x_1, x_2)	$[x_4, 0] = 0$	$[x_1, x_2] + [x_1, 0] = 0$
(x_4, x_1, x_3)	$[x_4, 0] = 0$	$[x_1, x_3] + [x_1, 0] = 0$
(x_4, x_1, x_4)	$[x_4, -x_1] = -x_1$	$[x_1, x_4] + [x_1, 0] = -x_1$
(x_4, x_2, x_1)	$[x_4, -\frac{1}{2}x_2] = -\frac{1}{4}x_2$	$[-\frac{1}{2}x_2, x_1] + [x_2, 0] = 0$
(x_4, x_2, x_2)	$[x_4, -\frac{1}{2}x_2] = -\frac{1}{4}x_2$	$[-\frac{1}{2}x_2, x_2] + [x_2, 0] = 0$
(x_4, x_2, x_3)	$[x_4, -x_1] = -x_1$	$[-\frac{1}{2}x_2, x_3] + [x_2, -\frac{1}{2}x_3] = x_1$
(x_4, x_2, x_4)	$[x_4, -\frac{1}{2}x_2] = -\frac{1}{4}x_2$	$[-\frac{1}{2}x_2, x_4] + [x_2, 0] = \frac{1}{4}x_2$
(x_4, x_3, x_1)	$[x_4, -\frac{1}{2}x_3] = -\frac{1}{4}x_3$	$[\frac{1}{2}x_3, x_1] + [x_3, 0] = 0$
(x_4, x_3, x_2)	$[x_4, -\frac{1}{2}x_3] = -\frac{1}{4}x_3$	$[\frac{1}{2}x_3, x_2] + [x_3, -\frac{1}{2}x_2] = 0$
(x_4, x_3, x_3)	$[x_4, -\frac{1}{2}x_3] = -\frac{1}{4}x_3$	$[\frac{1}{2}x_3, x_3] + [x_3, 0] = 0$
(x_4, x_3, x_4)	$[x_4, -\frac{1}{2}x_3] = -\frac{1}{4}x_3$	$[\frac{1}{2}x_3, x_4] + [x_3, 0] = -\frac{1}{4}x_3$
(x_4, x_4, x_1)	$[x_4, 0] = 0$	$[0, x_1] + [x_4, 0] = 0$
(x_4, x_4, x_2)	$[x_4, 0] = 0$	$[0, x_2] + [x_4, 0] = 0$
(x_4, x_4, x_3)	$[x_4, 0] = 0$	$[0, x_3] + [x_4, 0] = 0$
(x_4, x_4, x_4)	$[x_4, 0] = 0$	$[0, x_4] + [x_4, 0] = 0$

Based on the table above, it is shown that ad_x is a derivation on $\mathfrak{g}$. Next, the $\mathfrak{g}$-invariance condition will be examined based on the equation

$$\beta(\mathrm{ad}_x(y), z) + \beta(y, \mathrm{ad}_x(z)) = 0 \quad \text{or} \quad \beta([x, y], z) + \beta(y, [x, z]) = 0.$$

To illustrate the $\mathfrak{g}$-invariance condition, it is presented in the following table.

Table 9: Verification of $\mathfrak{g}$-invariance condition $\beta([x_i, x_j], x_k) + \beta(x_j, [x_i, x_k]) = 0$

(x_i, x_j, x_k)	$\beta([x_i, x_j], x_k) + \beta(x_j, [x_i, x_k])$
(x_1, x_1, x_1)	$\beta(0, x_1) + \beta(x_1, 0) = 0 + 0 = 0$
(x_1, x_1, x_2)	$\beta(0, x_2) + \beta(x_1, 0) = 0 + 0 = 0$
(x_1, x_1, x_3)	$\beta(0, x_3) + \beta(x_1, 0) = 0 + 0 = 0$
(x_1, x_1, x_4)	$\beta(0, x_4) + \beta(x_1, 0) = 0 + 0 = 0$
(x_1, x_2, x_1)	$\beta(0, x_1) + \beta(x_2, 0) = 0 + 0 = 0$
(x_1, x_2, x_2)	$\beta(0, x_2) + \beta(x_2, 0) = 0 + 0 = 0$
(x_1, x_2, x_3)	$\beta(0, x_3) + \beta(x_2, 0) = 0 + 0 = 0$
(x_1, x_2, x_4)	$\beta(0, x_4) + \beta(x_2, 0) = 0 + 0 = 0$

(x_1, x_3, x_1)	$\beta(0, x_1) + \beta(x_3, 0) = 0 + 0 = 0$
(x_1, x_3, x_2)	$\beta(0, x_2) + \beta(x_3, 0) = 0 + 0 = 0$
(x_1, x_3, x_3)	$\beta(0, x_3) + \beta(x_3, 0) = 0 + 0 = 0$
(x_1, x_3, x_4)	$\beta(0, x_4) + \beta(x_3, 0) = 0 + 0 = 0$
(x_1, x_4, x_1)	$\beta(-x_1, x_1) + \beta(x_4, 0) = 0 + 0 = 0$
(x_1, x_4, x_2)	$\beta(-x_1, x_2) + \beta(x_4, 0) = 0 + 0 = 0$
(x_1, x_4, x_3)	$\beta(-x_1, x_3) + \beta(x_4, 0) = 0 + 0 = 0$
(x_1, x_4, x_4)	$\beta(-x_1, x_4) + \beta(x_4, 0) = -1 + 0 = -1$
(x_2, x_1, x_1)	$\beta(0, x_1) + \beta(x_1, 0) = 0 + 0 = 0$
(x_2, x_1, x_2)	$\beta(0, x_2) + \beta(x_1, 0) = 0 + 0 = 0$
(x_2, x_1, x_3)	$\beta(0, x_3) + \beta(x_1, 0) = 0 + 0 = 0$
(x_2, x_1, x_4)	$\beta(0, x_4) + \beta(x_1, 0) = 0 + 0 = 0$
(x_2, x_2, x_1)	$\beta(0, x_1) + \beta(x_2, 0) = 0 + 0 = 0$
(x_2, x_2, x_2)	$\beta(0, x_2) + \beta(x_2, 0) = 0 + 0 = 0$
(x_2, x_2, x_3)	$\beta(0, x_3) + \beta(x_2, 0) = 0 + 0 = 0$
(x_2, x_2, x_4)	$\beta(0, x_4) + \beta(x_2, 0) = 0 + 0 = 0$
(x_2, x_3, x_1)	$\beta(-x_1, x_1) + \beta(x_3, 0) = 0 + 0 = 0$
(x_2, x_3, x_2)	$\beta(-x_1, x_2) + \beta(x_3, 0) = 0 + 0 = 0$
(x_2, x_3, x_3)	$\beta(-x_1, x_3) + \beta(x_3, 0) = 0 + 0 = 0$
(x_2, x_3, x_4)	$\beta(-x_1, x_4) + \beta(x_3, 0) = -1 + 0 = -1$
(x_2, x_4, x_1)	$\beta(-\frac{1}{2}x_2, x_1) + \beta(x_4, 0) = 0 + 0 = 0$
(x_2, x_4, x_2)	$\beta(-\frac{1}{2}x_2, x_2) + \beta(x_4, 0) = 0 + 0 = 0$
(x_2, x_4, x_3)	$\beta(-\frac{1}{2}x_2, x_3) + \beta(x_4, -\frac{1}{2}x_3) = 0 + 0 = 0$
(x_2, x_4, x_4)	$\beta(-\frac{1}{2}x_2, x_4) + \beta(x_4, -\frac{1}{2}x_2) = 0 + 0 = 0$
(x_3, x_1, x_1)	$\beta(0, x_1) + \beta(x_1, 0) = 0 + 0 = 0$
(x_3, x_1, x_2)	$\beta(0, x_2) + \beta(x_1, 0) = 0 + 0 = 0$
(x_3, x_1, x_3)	$\beta(0, x_3) + \beta(x_1, 0) = 0 + 0 = 0$
(x_3, x_1, x_4)	$\beta(0, x_4) + \beta(x_1, 0) = 0 + 0 = 0$
(x_3, x_2, x_1)	$\beta(x_1, x_1) + \beta(x_2, 0) = 0 + 0 = 0$
(x_3, x_2, x_2)	$\beta(x_1, x_2) + \beta(x_2, 0) = 0 + 0 = 0$
(x_3, x_2, x_3)	$\beta(x_1, x_3) + \beta(x_2, 0) = 0 + 0 = 0$
(x_3, x_2, x_4)	$\beta(x_1, x_4) + \beta(x_2, 0) = 1 + 0 = 1$
(x_3, x_3, x_1)	$\beta(0, x_1) + \beta(x_3, 0) = 0 + 0 = 0$
(x_3, x_3, x_2)	$\beta(0, x_2) + \beta(x_3, 0) = 0 + 0 = 0$
(x_3, x_3, x_3)	$\beta(0, x_3) + \beta(x_3, 0) = 0 + 0 = 0$
(x_3, x_3, x_4)	$\beta(0, x_4) + \beta(x_3, 0) = 0 + 0 = 0$
(x_3, x_4, x_1)	$\beta(-\frac{1}{2}x_3, x_1) + \beta(x_4, 0) = 0 + 0 = 0$
(x_3, x_4, x_2)	$\beta(-\frac{1}{2}x_3, x_2) + \beta(x_4, -\frac{1}{2}x_2) = 0 + 0 = 0$
(x_3, x_4, x_3)	$\beta(-\frac{1}{2}x_3, x_3) + \beta(x_4, 0) = 0 + 0 = 0$
(x_3, x_4, x_4)	$\beta(-\frac{1}{2}x_3, x_4) + \beta(x_4, -\frac{1}{2}x_3) = 0 + 0 = 0$
(x_4, x_1, x_1)	$\beta(x_1, x_1) + \beta(x_1, 0) = 0 + 0 = 0$
(x_4, x_1, x_2)	$\beta(x_1, x_2) + \beta(x_1, 0) = 0 + 0 = 0$
(x_4, x_1, x_3)	$\beta(x_1, x_3) + \beta(x_1, 0) = 0 + 0 = 0$
(x_4, x_1, x_4)	$\beta(x_1, x_4) + \beta(x_1, -x_1) = 1 + 0 = 1$
(x_4, x_2, x_1)	$\beta(\frac{1}{2}x_2, x_1) + \beta(x_2, 0) = 0 + 0 = 0$
(x_4, x_2, x_2)	$\beta(\frac{1}{2}x_2, x_2) + \beta(x_2, 0) = 0 + 0 = 0$
(x_4, x_2, x_3)	$\beta(\frac{1}{2}x_2, x_3) + \beta(x_2, -\frac{1}{2}x_3) = 0 + 0 = 0$
(x_4, x_2, x_4)	$\beta(\frac{1}{2}x_2, x_4) + \beta(x_2, -\frac{1}{2}x_2) = 0 + 0 = 0$
(x_4, x_3, x_1)	$\beta(\frac{1}{2}x_3, x_1) + \beta(x_3, 0) = 0 + 0 = 0$
(x_4, x_3, x_2)	$\beta(\frac{1}{2}x_3, x_2) + \beta(x_3, -\frac{1}{2}x_2) = 0 + 0 = 0$
(x_4, x_3, x_3)	$\beta(\frac{1}{2}x_3, x_3) + \beta(x_3, 0) = 0 + 0 = 0$

(x_4, x_3, x_4)	$\beta(\frac{1}{2}x_3, x_4) + \beta(x_3, -\frac{1}{2}x_3) = 0 + 0 = 0$
(x_4, x_4, x_1)	$\beta(0, x_1) + \beta(x_4, 0) = 0 + 0 = 0$
(x_4, x_4, x_2)	$\beta(0, x_2) + \beta(x_4, 0) = 0 + 0 = 0$
(x_4, x_4, x_3)	$\beta(0, x_3) + \beta(x_4, 0) = 0 + 0 = 0$
(x_4, x_4, x_4)	$\beta(0, x_4) + \beta(x_4, 0) = 0 + 0 = 0$

Based on the Table 9, it is proven that several conditions of $\mathfrak{g}$-invariance are not satisfied. Consequently, with the inner derivation, $(\mathfrak{g}, \beta, \mathrm{ad})$ is not a $\mathfrak{g}$-quasi-Frobenius Lie algebra.

4. Conclusion

In this study, a formal definition of the $\mathfrak{g}$-quasi-Frobenius Lie algebra has been introduced, and a special case is taken, namely $\mathfrak{g} = \mathfrak{q}$. In this case, the $\mathfrak{g}$-quasi-Frobenius Lie algebra can be constructed over itself with a zero-valued inner derivation. Subsequently, several concrete examples are provided to illustrate the concept, namely a 4-dimensional quasi-Frobenius Lie algebra, and 2-dimensional and 4-dimensional Frobenius Lie algebras.

The significance of this characterization lies in its ability to unify the notion of quasi-Frobenius Lie algebras with the framework of inner derivations, thereby offering a clearer structural understanding of how such algebras can be realized. In particular, the examples demonstrate that the presence of a zero-valued inner derivation serves as a distinguishing feature in the construction of $\mathfrak{g}$-quasi-Frobenius Lie algebras when $\mathfrak{g} = \mathfrak{q}$.These results provide new insights into the use of inner derivations in the structure of $\mathfrak{g}$-quasi-Frobenius Lie algebras. For future work, this research can be further explored by using other types of mappings, such as outer derivations or more general homomorphisms, to construct cases where $\mathfrak{g} \neq \mathfrak{q}$.

CRediT Authorship Contribution Statement

Muhammad Arief Budiman: Conceptualization, Methodology, Formal Analysis, Writing-Original Draft Preparation. **Edi Kurniadi**: Methodology, Writing-Review and Editing, Investigation, Project Administration, Funding Acquisition. **Sukono**: Validation, Supervision.

Declaration of Generative AI and AI-assisted technologies

The author utilized generative AI technologies, such as OpenAI's ChatGPT and DeepSeek, to assist with formatting and linguistic structuring.The author confirms that the use of generative AI did not influence the scientific content, analysis, or conclusions of this research. The author takes full responsibility for the integrity and accuracy of the work presented in this paper.

Declaration of Competing Interest

The authors declare no competing interests.

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