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Bankruptcy Risk Analysis and Estimation of Healthy and Distressed Stocks Using Extended Black-Scholes Model

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Abstract

The classical *Black-Scholes* model assumes that stock prices follow a *Geometric Brownian Motion* (GBM) process, in which stock prices remain strictly positive. However, this assumption restricts the model's capability to represent firms experiencing *financial distress*, where stock prices may approach zero as bankruptcy risk increases. This study employs the *Extended Black-Scholes Model* (EBSM), which introduces a stopping-time mechanism that enables stock prices to reach the zero-price boundary, to analyze and compare bankruptcy risk characteristics between two contrasting stock-price regimes. Adjusted closing price data from *Microsoft Corporation* (MSFT.US) and *Bed Bath & Beyond Inc.* (BBBY.US) were utilized, with model parameters estimated using *quadratic variation* and *Maximum Likelihood Estimation* (MLE) methods. The estimation results revealed different stochastic characteristics between the two stocks, where MSFT.US generated a positive drift parameter of 0.216682, whereas BBBY.US generated a negative drift parameter of -0.248385 along with higher volatility. The bankruptcy risk analysis demonstrated that MSFT.US produced infinite *Expected Bankruptcy Time* (EBT) and *Conditional Expected Bankruptcy Time* (CEBT) values, while BBBY.US yielded a finite EBT of 5.113633 years, with CEBT values updated according to the observed stock price conditions. The benchmark comparison showed that, for BBBY.US, GBM implies an infinite time to reach the zero-price boundary despite its negative drift, whereas EBSM produces a finite EBT by incorporating the possibility of reaching zero. These results suggest that the EBSM can distinguish contrasting bankruptcy risk characteristics by combining stochastic stock-price dynamics with model-implied measures of time to the zero-price boundary.

Keywords: Extended Black-Scholes Model; Bankruptcy Risk; Expected Bankruptcy Time; Conditional Expected Bankruptcy Time; Stochastic Stock Price Dynamics.

1. Introduction

Investing in stocks is one of the primary instruments in financial markets for achieving long-term asset growth. Nevertheless, investment decisions are subject to various uncertainties arising from the continuously evolving and increasingly interconnected global economy. Economic policies and trade disputes among countries have contributed to higher levels of volatility in financial markets [1, 2]. These conditions not only lead to fluctuations in asset prices but also increase systemic risks that investors must take into account when making investment decisions. Consequently, there is a need for quantitative approaches that can capture the dynamics of risk in stock price movements

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in a more accurate and measurable manner. Beyond reflecting changes in market value, stock prices also provide information about a firm's underlying financial condition, including default risk and the potential for future bankruptcy [3].

One of the fundamental models in modern financial theory is the Black-Scholes Model, introduced by Black and Scholes (1973). The model is widely used for pricing European options, which are derivative contracts that can only be exercised at their maturity date [4, 5]. Due to its influential role in option pricing, the Black-Scholes Model has become a cornerstone of both option theory and quantitative finance. Despite its widespread application, the model relies on several assumptions, including constant volatility and stock prices following a log-normal distribution. In practice, however, these assumptions are often violated, particularly during periods of heightened market uncertainty. Previous studies have reported that such conditions may result in substantial differences between theoretical valuations and observed market data [6]. This limitation highlights the need for more advanced models that can better represent the behavior of asset prices under real market conditions.

In general, the Black-Scholes Model is founded on a set of idealized assumptions, such as the absence of transaction costs, no dividend payments throughout the option's life, a constant risk-free interest rate, and stock prices evolving according to a specified stochastic process [7]. These assumptions help simplify the mathematical framework of the model and allow analytical solutions to be derived. Nevertheless, they also reduce the model's capacity to fully reflect the complexity of real financial markets. In actual market environments, financial conditions often exhibit characteristics that deviate from these assumptions, especially during periods of economic downturns or financial distress. Such discrepancies between theoretical assumptions and market realities become increasingly evident during periods of heightened uncertainty, when asset price volatility rises and market movements become more difficult to predict [6]. Moreover, the assumption of a log-normal distribution is often unable to account for the extreme price movements commonly observed in empirical financial data [8]. As a result, there is a continuing need for more flexible models that can better capture the empirical behavior of financial markets.

Under volatile market conditions, stock prices may decline substantially, in some cases approaching zero, which is often associated with a state of financial distress. Such circumstances reflect a weakening of company's financial position and may increase the risk of business failure or eventual bankruptcy [9]. In general, financial distress describes a situation in which a firm encounters financial difficulties that hinder its ability to satisfy both operational and financial obligations. When these difficulties persist without appropriate corrective actions, they may ultimately lead to bankruptcy [10]. This phenomenon has attracted considerable attention in financial analysis because it represents an extreme market condition that is not adequately captured by conventional asset pricing models. Consequently, the modeling of distress conditions has become an important aspect of modern financial risk theory.

In recent years, numerous approaches have been developed for predicting financial distress, ranging from traditional accounting-based and market-based models to structural approaches and more recent data-driven techniques, including machine learning [11–13]. Comparative studies have shown that these methods exhibit different strengths and limitations depending on firm characteristics and the information available, indicating that no single approach consistently outperforms the others across different settings [13]. Recent advances in data-driven bankruptcy prediction have further improved predictive performance by incorporating additional sources of information, highlighting the continued evolution of financial distress prediction methodologies [14]. Furthermore, recent benchmarking studies on machine learning-based bankruptcy prediction models have shown that market-related variables, such as stock return information and firm-specific risk indicators, provide valuable information for assessing corporate financial distress conditions [15]. A recent review has also highlighted the continued development of bankruptcy and financial-distress prediction methods, particularly through improvements in predictive models and the use of relevant financial information [16]. Nevertheless, many existing approaches do not explicitly model the stochastic evolution of stock prices as a direct representation of bankruptcy risk. Consequently, the relationship between the stochastic evolution of stock prices and the occurrence of bankruptcy remains insufficiently represented in existing prediction models. This limitation underscores the need for a more integrated framework that combines asset price dynamics with corporate bankruptcy risk analysis.

An early attempt to establish such an integration was introduced through Merton's structural credit risk model, where corporate default is represented by the relationship between firm value dynamics and debt obligations [17]. Nevertheless, this framework considers default only at a predetermined maturity date, limiting its ability to capture the progressive deterioration process associated with financially distressed firms. Subsequent developments in structural credit risk modeling have introduced more flexible stochastic features, including stochastic default boundaries and jump-diffusion dynamics, to better represent uncertainty and complex default behavior in financial systems [18]. These developments highlight the importance of stochastic asset and firm-value dynamics in bankruptcy-risk analysis and provide motivation for models in which the bankruptcy boundary is reached through the evolution of the stochastic process rather than being imposed at a predetermined maturity.

To address these limitations, Hsu and Wu (2020) introduced the *Extended Black-Scholes Model* (EBSM), an extension of the *Geometric Brownian Motion* framework that incorporates a *stopping-time* mechanism, allowing stock prices to reach zero as a representation of bankruptcy [19]. The EBSM therefore provides a stochastic framework in which bankruptcy risk is incorporated directly into the evolution of the modeled stock-price process. Moreover, the EBSM provides a unified mathematical framework for representing stock-price dynamics before the process reaches the bankruptcy boundary [19]. In their original study, Hsu and Wu [19] established the probabilistic properties of bankruptcy risk, developed statistical inference methods for estimating the unknown model parameters, examined European option pricing within the extended framework, and proposed computational procedures for evaluating the associated financial quantities. Their study also incorporated an empirical analysis using real stock-price data and found that the EBSM provided a better representation of stock-price trends than GBM in their specific empirical setting [19]. As a result, the model not only extends the theoretical foundation of conventional asset pricing models but also offers a more comprehensive framework for bankruptcy risk analysis.

Subsequent research has expanded the application of the EBSM framework to a broader range of financial problems. Hsu, Chen, and Wu [20] examined optimal limit prices for buy and sell orders within an extended geometric Brownian motion framework incorporating bankruptcy risk, deriving expected discounted profit functions and optimal limit prices while also assessing the effects of disregarding bankruptcy risk. Hsu, Chen, and Wu [21] later developed double-barrier option pricing equations under extended geometric Brownian motion with bankruptcy risk by formulating the associated partial differential equation, obtaining a numerical solution, and evaluating the effects of omitting bankruptcy risk through simulation. These studies indicate that the EBSM framework has evolved beyond its original formulation to address applications in trading and derivative pricing while explicitly accounting for bankruptcy risk.

Despite these developments, an empirical gap remains in the application of the EBSM for explicitly characterizing and comparing bankruptcy-risk regimes across stocks with different financial conditions. Previous studies on the EBSM have established its theoretical properties, developed statistical inference procedures, evaluated its empirical representation of stock-price dynamics, and extended the framework to applications in trading and option pricing [19–21]. However, relatively limited attention has been given to using the estimated EBSM dynamics to distinguish contrasting stock-price regimes through time-to-bankruptcy measures. In particular, linking the estimated drift parameter to the resulting *Expected Bankruptcy Time* (EBT), together with the state-conditional *Expected Bankruptcy Time* (CEBT), provides a direct connection between empirically estimated stock-price dynamics and a model-implied time to the zero-price boundary.

This perspective differs from conventional bankruptcy prediction methods, which generally express bankruptcy risk through classification outcomes, default probabilities, or risk scores. Within the EBSM framework, the sign of the drift parameter b determines whether the *Expected Bankruptcy Time* is finite or infinite: when $b < 0$, the stochastic process tends toward the bankruptcy boundary and the EBT is finite, whereas when $b \geq 0$, the EBT becomes infinite [19]. The *Conditional Expected Bankruptcy Time* additionally accounts for the observed state of the stock-price process at a specific time, thereby providing a state-conditional complement to the long-term characterization captured by the EBT. Accordingly, analyzing EBT and CEBT across stocks with contrasting price dynamics offers an empirical time-to-boundary perspective on bankruptcy risk that differs from approaches primarily based on bankruptcy classification or default probability.

Motivated by these considerations, this study applies the *Extended Black-Scholes Model* to estimate and compare bankruptcy risk between two contrasting stock-price cases using the *Expected Bankruptcy Time* (EBT) and *Conditional Expected Bankruptcy Time* (CEBT). More specifically, this study empirically applies the Hsu-Wu EBSM framework to discrete daily adjusted closing price data from Microsoft Corporation (MSFT.US) and Bed Bath & Beyond Inc. (BBBY.US), which are selected as contrasting empirical cases representing relatively stable and distressed stock-price conditions, respectively. The selection of these stocks is intended to provide a focused comparison of contrasting stock-price regimes rather than to represent the broader population of financially healthy and distressed firms. The EBSM parameters are estimated using the quadratic variation-based procedure and *Maximum Likelihood Estimation* proposed by Hsu and Wu [19], after which the estimated parameters are used to evaluate the EBT and CEBT.

The contribution of this study is primarily empirical and analytical rather than introducing a new stochastic process or parameter-estimation procedure. First, the study empirically applies the Hsu-Wu EBSM framework to compare contrasting stock-price regimes using real-world daily market data. Second, it examines how differences in the estimated volatility and drift parameters are reflected in the resulting model-implied measures of bankruptcy time. Third, the study evaluates EBT and CEBT as complementary time-to-boundary measures, connecting the long-term stochastic regime with the observed state of the stock-price process. Thus, this study offers an additional time-to-bankruptcy perspective that complements existing accounting-based, market-based, structural, and data-driven approaches to bankruptcy-risk analysis.

In addition, the study compares the EBSM with the conventional *Geometric Brownian Motion* model using the same normalized stock-price data and observation intervals. This benchmark comparison is intended to highlight the additional value of the EBSM zero-price boundary mechanism, particularly by distinguishing a negative stock-price drift from the model's ability to yield a finite time to the zero-price boundary.

2. Methods

This section describes the methodological framework used to estimate and analyze bankruptcy risk based on the *Extended Black-Scholes Model* (EBSM). The methodology consists of several stages, beginning with the construction and normalization of historical stock price data, followed by the formulation of the EBSM framework and the estimation of the model parameters. The estimated parameters are subsequently used to calculate the *Expected Bankruptcy Time* (EBT) as a quantitative measure of bankruptcy risk. The methodology also incorporates a benchmark specification based on the *Geometric Brownian Motion* (GBM), estimated using the same normalized stock-price observations and time intervals, to enable a direct comparison of the two models in terms of their treatment of the zero-price boundary. This systematic procedure establishes a connection between the observed stock price dynamics and the theoretical stochastic framework used for bankruptcy risk analysis.

2.1. Data and Parameter Construction

This study uses secondary data consisting of adjusted closing prices obtained from the Stooq historical stock database [22]. Historical stock price data for Microsoft Corporation (MSFT.US) covering the period from January 2, 2015, to December 31, 2019, are used as a relatively stable stock-price case, whereas data for Bed Bath & Beyond Inc. (BBBY.US) from January 4, 2021, to November 30, 2022, are used as a distressed stock-price case. The two firms were selected as contrasting empirical cases because of the pronounced differences in their stock-price dynamics during the respective observation periods. Microsoft represents a relatively stable stock-price case, characterized by sustained growth over the selected period, whereas Bed Bath & Beyond represents a distressed stock-price case characterized by substantial deterioration and persistent downward pressure during the respective observation period. The BBBY observation period ends on November 30, 2022, preceding the company's Chapter 11 filing on April 23, 2023 [23], thereby allowing the analysis to focus on the pre-filing period rather than subsequent post-bankruptcy price behavior. Accordingly, the two cases provide a focused empirical comparison of relatively stable and distressed stock-price regimes rather than serving as statistically representative samples of all healthy and distressed firms. The observation periods differ intentionally because the study adopts a case-based comparison of contrasting stock-price regimes rather than a matched-period design under identical market conditions. Accordingly, each observation window is chosen to represent the respective market regime of the stock, while the subsequent trailing-window robustness analysis examines whether the qualitative distinction between the two stock-price regimes remains consistent as earlier observations are progressively excluded.

This study employs adjusted closing prices, which account for corporate actions such as stock splits and dividend distributions. Using adjusted data ensures the consistency of stock price movements throughout the observation period, preventing the estimation process from being affected by price changes resulting solely from administrative adjustments. Consequently, the price movements analyzed more accurately reflect the underlying market dynamics and provide a reliable basis for constructing the stochastic model.

The basic parameters and notation used in this study are summarized in Table 1. These parameters are associated with the observational data and are consistently used throughout the mathematical formulations presented in the subsequent stages of the analysis.

Table 1: Basic Parameters and Notation

Parameter	Description
$S(t_k)$	Adjusted closing price at observation time t_k.
t_k	The k-th observation time.
T	Observation horizon.
n	Number of observations during the study period.
Δt	Observation time interval expressed in years.

Before the modeling process, all stock prices are normalized with respect to their initial values so that each stock starts from the same baseline. This normalization eliminates the effect of differences in price scales across companies, allowing the analysis to focus on the relative changes in stock prices over the observation period rather than their nominal price levels. Accordingly, if $S(t_k)$ denotes the adjusted closing price at observation time t_k, the normalized stock price is defined as

$$\tilde{S}(t_k) = \frac{S(t_k)}{S(0)}, \qquad k = 0, 1, \dots, n-1. \tag{1}$$

The normalization process produces stock price trajectories with a common initial value, allowing price movements across different companies to be compared directly without being affected by differences in their initial price levels. The normalized data are then treated as observations at discrete time points $t_k = k\Delta t, k = 0, 1, \ldots, n-1$, providing a discrete approximation to the underlying continuous stochastic process. This framework enables daily historical stock price data to be linked with the continuous-time formulation of the *Extended Black-Scholes Model* and serves as the foundation for the subsequent parameter estimation procedure.

2.2. Extended Black-Scholes Model

The classical Black-Scholes model assumes that stock prices follow a *Geometric Brownian Motion* (GBM) process, implying that stock prices remain strictly positive throughout the observation period. While this assumption is appropriate for modeling companies with relatively stable financial conditions, it becomes less suitable for firms experiencing *financial distress*. Under such circumstances, stock prices may decline continuously toward zero, reflecting an increasing risk of bankruptcy. To overcome this limitation, this study employs the *Extended Black-Scholes Model* (EBSM) proposed by Hsu and Wu [19]. The EBSM extends the *Geometric Brownian Motion* framework by introducing an additional component that allows stock prices to reach zero within a finite time. As a result, the model provides a unified stochastic framework for representing contrasting stock-price regimes, including relatively stable and distressed conditions. The flexibility of this stochastic framework has been further demonstrated through its application to a broader range of problems in quantitative finance. For example, Hsu et al. [21] applied the framework to derive pricing equations for double-barrier options under bankruptcy risk, highlighting how the incorporation of a stopping-time mechanism extends its applicability to financial models involving default risk.

Let $B(t)$ denote a standard Brownian motion. Given the volatility parameter $a > 0$, the drift parameter b, and the shape parameters satisfying $p > q \geq 0$, the stochastic process describing the stock price prior to bankruptcy is defined as

$$Y_1(t) = pe^{aB(t)+bt} - qe^{-aB(t)-bt}, \qquad t \geq 0. \tag{2}$$

Eq. (2) defines the *Extended Black-Scholes Model* (EBSM) and distinguishes it from the classical *Geometric Brownian Motion* framework. Unlike the classical Black-Scholes model, which consists of a single exponential term that ensures stock prices remain positive, the EBSM incorporates two exponential components with opposite effects. The first component represents the growth of the stock price, whereas the second captures the downward pressure that allows the stock price process to reach zero. Based on this formulation, the first time at which the stock price reaches zero is defined as

$$T_{1,0} = \inf\left\{t > 0 \,|\, Y_1(t) = 0\right\}. \tag{3}$$

Eq. (3) defines the model-implied zero-price boundary hitting time, namely the first time at which the stochastic process $Y_1(t)$ reaches zero. Unlike the *Geometric Brownian Motion* framework, which assumes that stock prices remain strictly positive, the introduction of a *stopping time* in the EBSM enables the modeled stock-price process to reach the zero-price boundary within finite time. In the EBSM framework, this event represents the mathematical manifestation of bankruptcy risk and should not be interpreted as an exact representation of the legal or operational occurrence of corporate bankruptcy.

Since stock prices are no longer observed after bankruptcy occurs, the stochastic process considered in this study is not $Y_1(t)$, but rather the corresponding *stopped process*, defined as

$$Y_2(t) = Y_1\left(t \wedge T_{1,0}\right). \tag{4}$$

where the operator $t \wedge T_{1,0} = \min\{t, T_{1,0}\}$ denotes the minimum of the observation time t and model-implied zero-price boundary hitting time $T_{1,0}$. According to Eq. (4), as long as bankruptcy has not occurred ($t < T_{1,0}$), the observed process coincides with the original process, that is,

$Y_2(t) = Y_1(t)$. Once the observation time reaches or exceeds the zero-price boundary hitting time $(t \geq T_{1,0})$, the process is stopped at $T_{1,0}$, and the stock price no longer evolves after reaching zero. Consequently, the *Extended Black-Scholes Model* provides a unified stochastic framework for describing the evolution of stock prices from the beginning of the observation period until the zero-price boundary is reached.

According to Eq. (4), the observed stochastic process no longer evolves once the bankruptcy time is reached, and the stock price thereafter remains fixed at its bankruptcy value. Consequently, after the stock price reaches zero, the observed process no longer follows the evolution of $Y_1(t)$, but is instead represented by the stopped process $Y_2(t)$. This construction incorporates the zero-price boundary hitting time directly into the dynamics of the stochastic process, allowing the modeled bankruptcy boundary event to emerge naturally from the evolution of the stock-price trajectory rather than being imposed through external assumptions.

The introduction of the *stopped process* represents one of the fundamental distinctions between the *Extended Black-Scholes Model* (EBSM) and the *Geometric Brownian Motion* framework. In the classical Black-Scholes model, stock prices remain strictly positive with probability one, making it impossible to model a finite-time zero-price boundary event explicitly. In contrast, the EBSM allows the stock price process to reach zero in finite time through a *stopping-time* mechanism, enabling both relatively stable and distressed stock-price regimes to be represented within a unified stochastic framework. Accordingly, the zero-price boundary is interpreted as a implied representation of extreme stock-price distress rather than as a direct legal definition of corporate bankruptcy. This formulation serves as the mathematical foundation for the parameter estimation procedure developed in this study.

2.3. Parameter Estimation

Once the *Extended Black-Scholes Model* has been formulated, the next step is to estimate its parameters from the observed stock price trajectories. Since the model parameters are not directly observable, an estimation procedure is required to establish a link between the theoretical stochastic process and the available historical data. In this study, parameter estimation is based on the trajectory of the stopped process $Y_2(t)$, ensuring that all information used in the estimation accounts for the possibility of bankruptcy. Following the approach proposed by Hsu and Wu [19], the parameters are estimated in two complementary stages. The first stage employs *quadratic variation* to estimate the volatility parameter and the product of the shape parameters, which are subsequently used to determine the individual shape parameters. The second stage estimates the drift parameter using the *Maximum Likelihood Estimation* (MLE) method. This two-stage estimation procedure allows each parameter to be estimated using the method most appropriate to its statistical properties, thereby ensuring consistent estimation of all model parameters from the observed process trajectory.

The first stage of parameter estimation is based on the *quadratic variation* approach, which measures the cumulative squared increments of a stochastic process over the observation period. In this study, *quadratic variation* is used to characterize the fluctuations of the observed process $Y_2(t)$, providing the information required to estimate the volatility parameter and the shape parameters in the *Extended Black-Scholes Model*. Based on the observed trajectory of $Y_2(t)$, the first quadratic variation is defined as

$$Q_1(T) = \sum_{i=1}^{n-1} \left(Y_2(t_i) - Y_2(t_{i-1})\right)^2 . \tag{5}$$

Eq. (5) represents the cumulative sum of the squared successive increments of the observed process over the observation period. This quantity characterizes the variability of the stochastic process and contains information about its volatility. In addition to the quadratic variation of $Y_2(t)$, the quadratic variation of the squared process is also considered and is defined as

$$Q_2(T) = \sum_{i=1}^{n-1} \left(Y_2^2(t_i) - Y_2^2(t_{i-1}) \right)^2 . \tag{6}$$

Unlike Eq. (5), Eq. (6) is constructed from the squared process rather than the process itself, thereby providing complementary information that cannot be obtained from $Q_1(T)$ alone. Both quantities are used jointly because the parameter estimators of the *Extended Black-Scholes Model* are derived from the combined information provided by these two quadratic variations. Consequently, they constitute the basis for constructing the parameter estimators in the subsequent stage.

In addition to the *quadratic variation*, the parameter estimation procedure also requires approximations of several integrals arising in the theoretical formulation of the *Extended Black-Scholes Model*. Since the stock price data are observed at discrete time points, these integrals cannot be evaluated directly and are instead approximated using Riemann sums constructed from the observed trajectory. Accordingly, the discrete approximation of the integral of the squared process is defined as

$$I_2(T) = \sum_{i=1}^{n-1} Y_2^2(t_{i-1})\, \Delta t. \tag{7}$$

Eq. (7) gives the discrete approximation to the integral of the squared stochastic process over the observation period. Using the same approach, the approximation to the integral of the fourth power of the process is defined as

$$I_4(T) = \sum_{i=1}^{n-1} Y_2^4(t_{i-1})\, \Delta t. \tag{8}$$

Eq. (8) provides the discrete approximation to the integral of the fourth power of the process. The quantities $I_2(T)$ and $I_4(T)$ complement the information provided by the *quadratic variation*. Together with $Q_1(T)$ and $Q_2(T)$, they constitute the principal components required for constructing the parameter estimators in the subsequent stage.

Once the *quadratic variation* components and the discrete integral approximations have been obtained, the model parameters can be estimated by combining these four quantities according to the formulation of the *Extended Black-Scholes Model*. This combination yields the estimator of the volatility parameter, given by

$$\hat{a}^2 = \frac{4I_2(T)Q_1(T) - Q_2(T)\,(T \wedge T_{1,0})}{4\left(I_2^2(T) - I_4(T)\,(T \wedge T_{1,0})\right)}. \tag{9}$$

Eq. (9) shows that the volatility estimator is constructed not only from the information contained in the *quadratic variation* but also from the discrete integral approximations computed from the observed process trajectory. Consequently, $Q_1(T)$, $Q_2(T)$, $I_2(T)$, and $I_4(T)$ each provide complementary information for constructing this estimator. The product of the shape parameters is then estimated using

$$\widehat{pq} = \frac{Q_2(T) - 4\hat{a}^2 I_4(T)}{16\hat{a}^2 I_2(T)}. \tag{10}$$

The estimator in Eq. (10) provides an estimate of the product pq, which is subsequently used to recover the individual values of the shape parameters p and q. Accordingly, this stage yields two key quantities: the estimate of the volatility parameter and the estimate of the product of the shape parameters, both of which serve as the basis for the parameter recovery procedure in the subsequent stage.

Once the estimate of $\widehat{pq}$ has been obtained, the next step is to recover the individual estimates of the shape parameters p and q. This recovery is based on two quantities that are already available: the estimate of the product $\widehat{pq}$ obtained from Eq. (10) and the initial condition of the stochastic process, $Y_2(0) = p - q$, given in Eq. (2). Together, these relationships form a system of two equations with two unknowns, allowing the estimates of p and q to be determined explicitly as

$$\hat{p} = \frac{Y_2(0) + \sqrt{Y_2^2(0) + 4\widehat{pq}}}{2}. \tag{11}$$

and

$$\hat{q} = \frac{-Y_2(0) + \sqrt{Y_2^2(0) + 4\widehat{pq}}}{2}. \tag{12}$$

Equations (11) and (12) show that the shape parameters are not estimated directly from the observed data. Instead, they are recovered by combining the initial condition of the stochastic process with the estimated product of the parameters. Consequently, all quantities required in the first stage of the estimation procedure namely $\hat{a}^2$, $\widehat{pq}$, $\hat{p}$, and $\widehat{q}$ have been obtained and are subsequently used in the estimation of the drift parameter.

Unlike the volatility and shape parameters, which are estimated using the *quadratic variation* approach, the drift parameter b is estimated using the *Maximum Likelihood Estimation* (MLE) method. This distinction arises because *quadratic variation* captures only the local fluctuations of the stochastic process and is independent of its drift component. Accordingly, following the approach proposed by Hsu and Wu [19], the drift parameter is estimated from the conditional transition density of the observed process prior to bankruptcy. The *Maximum Likelihood* estimator of the drift parameter is given by

$$\hat{b}_{\text{MLE}} = \frac{1}{T \wedge T_{1,0}} \ln\left(\frac{Y_2(T) + \sqrt{Y_2^2(T) + 4\hat{p}\hat{q}}}{2\hat{p}}\right). \tag{13}$$

Eq. (13) shows that the drift estimator depends on the observed process at the terminal observation time, the estimated shape parameters obtained in the previous stage, and the effective observation horizon $T \wedge T_{1,0}$, which accounts for the possibility of bankruptcy during the observation period. Consequently, parameter estimation in the *Extended Black-Scholes Model* is carried out as a two-stage procedure: the volatility and shape parameters are first estimated using the *quadratic variation* approach, followed by estimation of the drift parameter via the *Maximum Likelihood Estimation* method. Once all model parameters have been estimated, they are subsequently used to compute the *Expected Bankruptcy Time* (EBT) and the *Conditional Expected Bankruptcy Time* (CEBT), as described in the following subsection.

2.4. Expected Bankruptcy Time and Conditional Expected Bankruptcy Time

Once all parameters of the *Extended Black-Scholes Model* have been estimated, the next step is to evaluate bankruptcy risk through the expected time to bankruptcy. In this model, bankruptcy risk is characterized not only by the possibility that the stock price reaches zero, but also by the expected time until this event occurs. Accordingly, the *Expected Bankruptcy Time* (EBT) provides a measure of the expected time required for the stochastic process to reach the model-implied zero-price boundary for the first time. Following the formulation proposed by Hsu and Wu [19], if the drift parameter satisfies $b < 0$, the *Expected Bankruptcy Time* is given by

$$E(T_{1,0}) = \frac{1}{b} \ln\left(\sqrt{\frac{q}{p}}\right). \tag{14}$$

Eq. (14) shows that the *Expected Bankruptcy Time* depends on the drift parameter and the shape parameters estimated in the previous stage. The condition $b < 0$ implies that the expected time to reach the model-implied zero-price boundary is finite only when the stochastic process exhibits a tendency to move toward the zero-price boundary. In contrast, if $b \geq 0$, the process has no tendency to approach zero, and therefore the *Expected Bankruptcy Time* is infinite.

Although the *Expected Bankruptcy Time* (EBT) provides the expected time to bankruptcy measured from the initial state of the process, it does not incorporate information available at a specific observation time. In practice, stock prices evolve continuously over time, making it necessary to update bankruptcy risk estimates as new information becomes available. For this purpose, the *Conditional Expected Bankruptcy Time* (CEBT) is introduced, representing the expected bankruptcy time conditional on the observed process being at state $Y_2(t_1) = y$ at observation time t_1. Following the formulation proposed by Hsu and Wu [19], the *Conditional Expected Bankruptcy Time* is given by

$$E(T_{1,0} \mid Y_2(t_1) = y) = t_1 + \frac{1}{b} \ln \left(\sqrt{\frac{4pq}{y + \sqrt{y^2 + 4pq}}} \right). \tag{15}$$

Eq. (15) shows that the expected time to bankruptcy depends not only on the estimated model parameters but also on the state of the observed process at time t_1. Consequently, the CEBT provides an updated estimate of the time to bankruptcy by incorporating the latest information contained in the observed stock price trajectory, thereby serving as a state-conditional complement to the EBT rather than replacing the model-based measure.

2.5. Benchmark Model: Geometric Brownian Motion

As a conventional benchmark for the *Extended Black-Scholes Model*, the *Geometric Brownian Motion* (GBM) is estimated using the same normalized stock-price observations and time intervals as those employed in the EBSM analysis. The GBM is chosen because it represents the standard strictly positive stock-price process from which the EBSM is developed. This benchmark enables the analysis to distinguish the implications of a conventional positive-price stochastic process from those of the EBSM, which permits access to the zero-price boundary through a stopping-time mechanism. The comparison is focused specifically on how the two stochastic specifications characterize bankruptcy risk rather than on their option-pricing performance. Accordingly, the benchmark comparison serves as supporting evidence for interpreting qualitative differences in the model-implied bankruptcy-time results rather than as a formal assessment of EBSM goodness-of-fit or predictive performance.

Under the GBM specification, the normalized stock price $Y(t)$ is assumed to satisfy the stochastic differential equation

$$dY(t) = \mu Y(t)\, dt + \sigma Y(t)\, dB(t) \tag{16}$$

where μ represents the annualized drift parameter, $\sigma > 0$ denotes the volatility parameter, and $B(t)$ is a standard Brownian motion. The solution to this stochastic differential equation is given by

$$Y(t) = Y(0) \exp \left[\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma B(t) \right] \tag{17}$$

For the discrete observations, the continuously compounded return over the (i)-th observation interval is defined as

$$R_i = \ln \left(\frac{Y(t_i)}{Y(t_{i-1})} \right), \qquad i = 1, \ldots, m. \tag{18}$$

where $m = n - 1$ represents the number of observed log returns. Under the GBM assumption, R_i follows a normal distribution with mean $\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t$ and variance $\sigma^2 \Delta t$. Defining $\bar{R} = m^{-1} \sum_{i=1}^{m} R_i$, the maximum likelihood estimators for the drift and volatility parameters are given by

$$\hat{\sigma}^2_{\text{GBM}} = \frac{1}{m\Delta t}\sum_{i=1}^{m}(R_i - \bar{R})^2. \tag{19}$$

and

$$\hat{\mu}_{\text{GBM}} = \frac{\bar{R}}{\Delta t} + \frac{1}{2}\hat{\sigma}^2_{\text{GBM}}. \tag{20}$$

A key distinction between the GBM and EBSM lies in whether the zero-price boundary can be reached. As shown in Eq. (17), when $Y(0) > 0$, the GBM stock price remains strictly positive for any finite t. Accordingly, the GBM zero-boundary hitting time is defined as

$$\tau_0^{\text{GBM}} = \inf\{t > 0 \,|\, Y(t) = 0\}. \tag{21}$$

Since the process cannot attain zero within finite time, it follows that

$$\mathbb{P}\left(\tau_0^{\text{GBM}} < \infty\right) = 0. \tag{22}$$

Consequently, the expected time to reach the zero-price boundary is infinite. Hence, a negative estimated drift under GBM does not, by itself, imply a finite time to the zero-price boundary. This property forms the basis for assessing the practical contribution of the EBSM benchmark, particularly in the case of the distressed stock BBBY.US.

2.6. Robustness Analysis

To assess whether the distinction between the relatively stable and distressed stock-price regimes is sensitive to the observation windows adopted in the main analysis, a trailing-window robustness analysis is performed. The complete observation period is maintained as the baseline specification, while two alternative specifications use the most recent 80% and 60% of observations, respectively. For each stock, the terminal observation date remains unchanged, whereas the starting date is shifted forward in accordance with the selected proportion of observations. This design enables the sensitivity of the estimated EBSM characteristics to earlier observations to be assessed while preserving the terminal market conditions represented in the main analysis. In particular, the analysis assesses the sensitivity of the estimated drift parameter $\hat{b}_{MLE}$, whose sign determines whether the model-implied EBT is finite or infinite. The robustness analysis is therefore intended to provide supporting evidence for the stability of the qualitative findings across alternative observation windows rather than to constitute a formal goodness-of-fit assessment or predictive validation of the EBSM.

For each alternative window, the parameter-estimation procedure applied in the main analysis is retained without modification. Specifically, $Q_1(T)$, $Q_2(T)$, $I_2(T)$, and $I_4(T)$ are recalculated, followed by the estimation of $\hat{a}^2$, $\widehat{pq}$, $\hat{p}$, $\hat{q}$, and $\hat{b}_{MLE}$. The resulting parameter estimates are subsequently used to determine the corresponding EBT. The robustness assessment focuses primarily on parameter admissibility, the sign of the estimated drift parameter $\hat{b}_{MLE}$, and whether the resulting EBT is finite or infinite. Accordingly, variations in the magnitude of $\hat{b}_{MLE}$ across the alternative windows are examined to evaluate the sensitivity of the estimated drift to the selected observation period. Consistency in these qualitative characteristics across the alternative observation windows is interpreted as evidence that the distinction between the two cases is not driven solely by the earliest observations in their respective samples.

For MSFT.US, the 60%, 80%, and 100% specifications consist of 754, 1,006, and 1,258 observations, respectively, covering the periods January 3, 2017-December 31, 2019, January 4, 2016-December 31, 2019, and January 2, 2015-December 31, 2019. For BBBY.US, the corresponding specifications comprise 289, 385, and 482 observations, spanning October 8, 2021-November 30, 2022, May 24, 2021-November 30, 2022, and January 4, 2021-November 30, 2022, respectively. The observation-window sizes are determined mechanically based on 60% and

80% of the total number of observations and are not selected based on the parameter estimates obtained from each specification.

3. Results and Discussion

This section presents the results of applying the *Extended Black-Scholes Model* (EBSM) to the stock price data analyzed in this study and discusses their implications for bankruptcy risk analysis. The presentation follows the methodological framework described in the previous chapter. The analysis begins with the characteristics of the normalized stock price data, followed by the estimation of the model parameters using the proposed estimation procedure. The estimated parameters are then used to evaluate the *Expected Bankruptcy Time* (EBT), enabling a comparison of bankruptcy risk between the two contrasting stock-price cases. The analysis then compares the EBSM results with those obtained under the *Geometric Brownian Motion* benchmark, followed by an assessment of the robustness of the EBSM-based distinction across alternative observation windows.

3.1. Descriptive Analysis of Stock Price Data

The analysis begins with an examination of the normalized stock price trajectories over the observation period. This preliminary step provides an overview of the empirical behavior of each stock before the quantitative analysis based on the *Extended Black-Scholes Model* is carried out. Since all stock prices are normalized with respect to their initial values, the comparison focuses on relative price movements, allowing the behavior of the two stocks to be compared directly without being affected by differences in their nominal price levels. This descriptive analysis also serves to assess whether the selected data exhibit stock-price characteristics consistent with relatively stable and distressed regimes, respectively. The normalized stock price trajectories of *Microsoft Corporation* (MSFT.US) and *Bed Bath & Beyond Inc.* (BBBY.US) are presented in Figures 1 and 2, respectively.

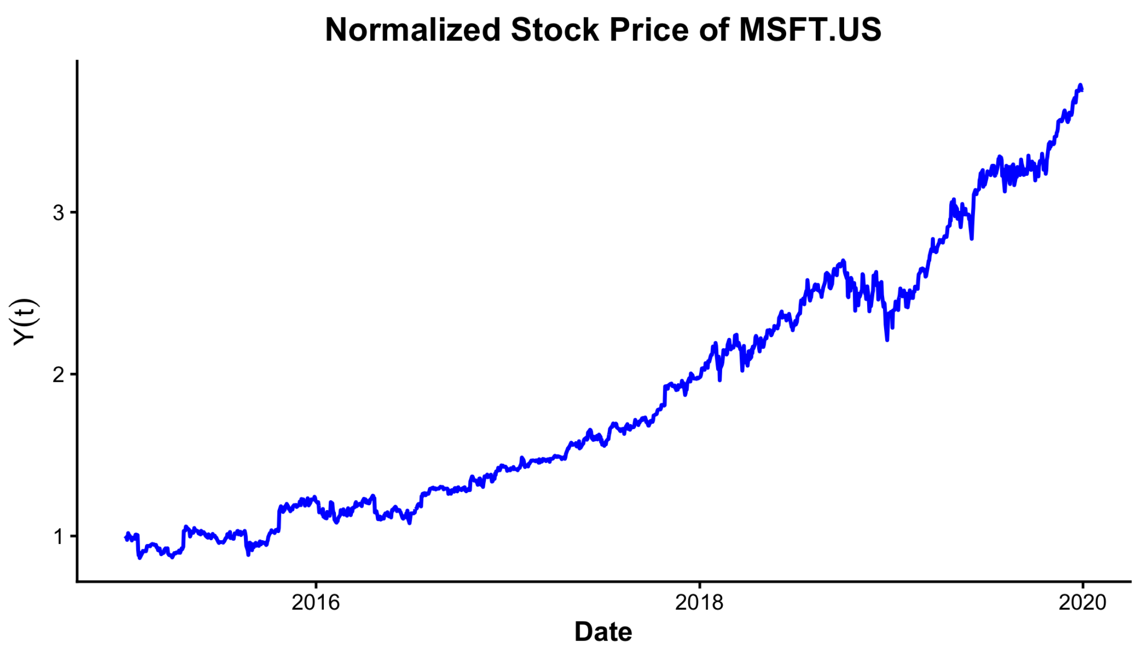

Fig. 1: Normalized stock price of Microsoft Corporation (MSFT.US)

Fig. 1 shows that the stock price of *Microsoft Corporation* (MSFT.US) exhibited an overall upward trend throughout the observation period, despite the presence of daily fluctuations that are characteristic of equity markets. Overall, the normalized price trajectory remained relatively stable, displaying sustained long-term growth without experiencing substantial decline toward zero. These observations indicate that Microsoft exhibited strong stock-price performance during the study period and displayed price dynamics consistent with a relatively stable stock-price regime. Within the framework of the *Extended Black-Scholes Model*, such behavior is consistent with a stock price trajectory that remains well above the bankruptcy boundary, making Microsoft an appropriate empirical case for the relatively stable stock-price regime examined in this study. This interpretation is further examined through the parameter estimation results and the bankruptcy risk measures presented in the following sections.

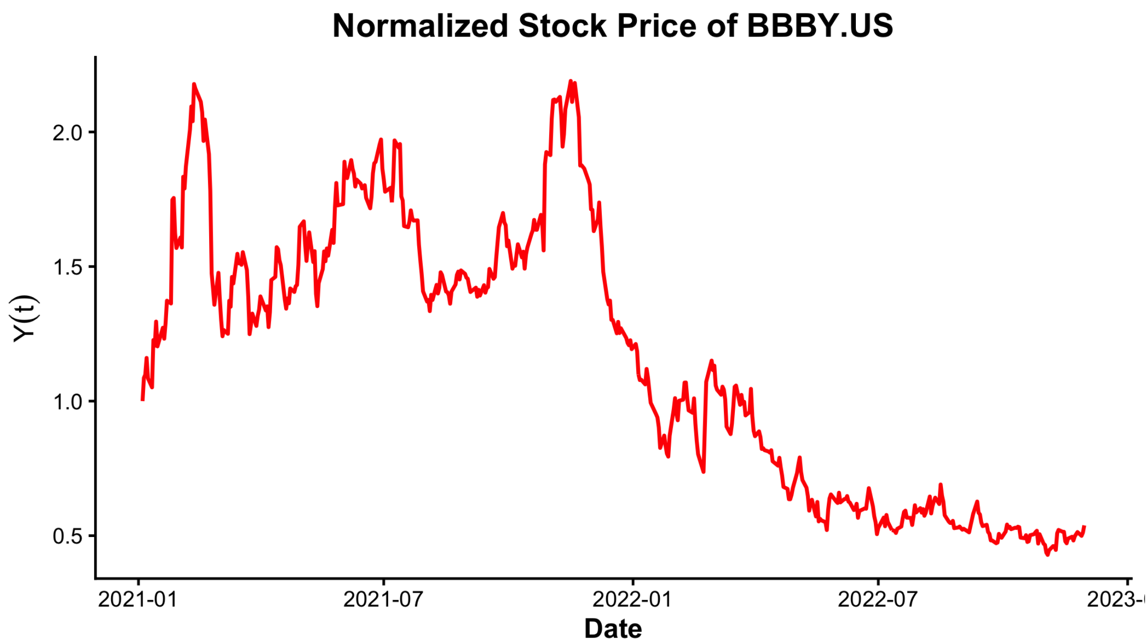

Fig. 2: Normalized stock price of Bed Bath & Beyond Inc. (BBBY.US)

In contrast to Microsoft, Fig. 2 shows that the stock price of *Bed Bath & Beyond Inc.* (BBBY.US) exhibited an overall downward trend throughout the observation period, accompanied by relatively larger fluctuations. Although occasional short-term price increases were observed, these recoveries were not sustained, causing the stock price trajectory to move progressively closer to zero. This behavior is consistent with a distressed stock-price regime, characterized by heightened market uncertainty and persistent downward pressure on its stock price. Within the framework of the *Extended Black-Scholes Model*, such behavior is consistent with a stock price trajectory approaching the bankruptcy boundary, making BBBY.US an appropriate empirical case for the distressed stock-price regime examined in this study. This interpretation is further examined through the parameter estimation results and the bankruptcy risk measures presented in the following sections.

Overall, the two stocks exhibited distinctly different price movement characteristics over the observation period. *Microsoft Corporation* (MSFT.US) maintained a relatively stable upward price trajectory, whereas *Bed Bath & Beyond Inc.* (BBBY.US) exhibited a downward trajectory accompanied by greater price fluctuations. These contrasting patterns support the characterization of the two stocks as relatively stable and distressed stock-price cases, respectively, within the scope of this study. Within the framework of the *Extended Black-Scholes Model*, these differences in stock price behavior are further examined through the parameter estimation procedure. Accordingly, the following section focuses on the parameter estimation results to investigate how these differences are quantitatively reflected in the parameters of the *Extended Black-Scholes Model*.

3.2. Parameter Estimation Results

Following the descriptive analysis of stock price movements, the next step is to estimate the parameters of the *Extended Black-Scholes Model* using the procedure described in Subsection 2.3. The estimation is performed as a two-stage procedure. In the first stage, the *quadratic variation* approach is used to estimate the volatility and shape parameters, followed by the *Maximum Likelihood Estimation* (MLE) method for estimating the drift parameter. Before the model parameters are estimated, the empirical *quadratic variation* and the discrete integral approximations, which constitute the principal components for constructing the parameter estimators, are first computed. The resulting values of the number of observations, the observation horizon, and the quantities $Q_1(T)$, $Q_2(T)$, $I_2(T)$, and $I_4(T)$ for each stock are presented in Table 2.

Table 2: Quadratic Variation and Discrete Integral Results

Stock	n	T	$Q_1(T)$	$Q_2(T)$	$I_2(T)$	$I_4(T)$
MSFT.US	1258	4.988095	0.999543	24.312070	20.434098	140.351405
BBBY.US	482	1.908730	1.782875	16.784894	3.056678	7.727905

Based on Table 2, the two stocks exhibit distinct characteristics in terms of *quadratic variation* and discrete integral approximations over the observation period. The value of $Q_1(T)$ is higher for BBBY.US than for MSFT.US, indicating a greater accumulation of price changes between consecutive observation times. In contrast, the values of $I_2(T)$ and $I_4(T)$ are larger for MSFT.US, reflecting the generally higher magnitude of the observed process throughout the observation horizon. These differences are consistent with the distinct price movement characteristics of the two stocks as well as the different observation periods considered in this study. The quantities reported in Table 2 are subsequently used to construct the parameter estimators of the *Extended Black-Scholes Model*, the results of which are presented in Table 3.

Table 3: Estimated EBSM Parameters

Stock	$\hat{a}^2$	$\widehat{pq}$	$\hat{p}$	$\hat{q}$	$\hat{b}_{\mathrm{MLE}}$
MSFT.US	0.035015	0.406565	1.310287	0.310287	0.216682
BBBY.US	0.473402	0.092916	1.085591	0.085591	-0.248385

Based on Table 3, all parameters of the *Extended Black-Scholes Model* were successfully estimated for both stocks. The estimated variance parameter shows that BBBY.US has a value of $\hat{a}^2 = 0.473402$, which is substantially higher than the corresponding value for MSFT.US, $\hat{a}^2 = 0.035015$. This result indicates that BBBY exhibited greater price fluctuations during the observation period, consistent with the distressed stock-price characteristics identified in the descriptive analysis. Moreover, the estimated shape parameters satisfy $\hat{p} > \hat{q} \geq 0$ for both stocks, indicating that the mathematical conditions required by the *Extended Black-Scholes Model* are fulfilled. The most pronounced difference is observed in the drift parameter, with $\hat{b}_{MLE} = 0.216682$ for MSFT.US and $\hat{b}_{MLE} = -0.248385$ for BBBY.US. The positive drift estimate for MSFT.US suggests that the stochastic process tends to evolve away from the bankruptcy boundary, whereas the negative drift estimate for BBBY.US indicates a tendency toward the bankruptcy boundary. These findings are consistent with the empirical characteristics of the two stocks discussed in Subsection 3.1 and provide the basis for the subsequent analysis of the *Expected Bankruptcy Time* (EBT) and the *Conditional Expected Bankruptcy Time* (CEBT).

Table 4: Feasibility Assessment of the Estimated Parameters

Stock	$\hat{a}^2 > 0$	$\widehat{pq} > 0$	$\hat{p} > \hat{q} \geq 0$	$\hat{b}_{\mathrm{MLE}}$ defined	**Feasibility**
MSFT.US	✓	✓	✓	✓	✓
BBBY.US	✓	✓	✓	✓	✓

The verification results presented in Table 4 show that all estimated parameters satisfy the mathematical conditions required by the *Extended Black-Scholes Model*. For both stocks, the estimated parameters satisfy $\hat{a}^2 > 0$, $\widehat{pq} > 0$, and $\hat{p} > \hat{q} \geq 0$, while the drift parameter is well defined. Consequently, all parameter estimates are considered *feasible*. The satisfaction of these conditions indicates that the estimation procedure produces a set of parameters consistent with the theoretical framework of the model and suitable for subsequent analysis. Accordingly, the estimated parameters are not only mathematically valid but also satisfy the conditions required for computing the *Expected Bankruptcy Time* (EBT) and the *Conditional Expected Bankruptcy Time* (CEBT) as quantitative measures of bankruptcy risk for each stock.

The feasibility checks presented in Table 4 confirm that the estimated parameters satisfy the admissibility conditions required by the EBSM. However, these checks are intended to verify the mathematical validity of the parameter estimates rather than serve as a formal goodness-of-fit assessment. The present study focuses on estimating the EBSM parameters and evaluating their implications for the *Expected Bankruptcy Time* (EBT) and *Conditional Expected Bankruptcy Time* (CEBT), whereas formal model diagnostics and out-of-sample predictive validation are beyond the scope of the current empirical analysis. Such evaluations may be addressed in future research using a broader dataset and additional validation procedures.

3.3. Analysis of Expected Bankruptcy Time

Once all parameters of the *Extended Black-Scholes Model* have been successfully estimated and satisfy the model's feasibility conditions, the next step is to evaluate bankruptcy risk using the *Expected Bankruptcy Time* (EBT). This measure provides an estimate of the expected time until the stochastic process first reaches the bankruptcy boundary based on the estimated model parameters. Accordingly, the EBT serves as a quantitative measure of a stock's long-term tendency to reach bankruptcy under the dynamics described by the *Extended Black-Scholes Model*. The computed *Expected Bankruptcy Time* values for each stock are presented in Table 5.

Table 5: Estimated Expected Bankruptcy Time (EBT)

Stock	$\hat{b}_{\text{MLE}}$	**EBT (years)**	**EBT (trading days)**
MSFT.US	0.216682	∞	∞
BBBY.US	-0.248385	5.113633	1288.64

Based on Table 5, the two stocks yield substantially different *Expected Bankruptcy Time* (EBT) values. For MSFT.US, the estimated drift parameter is $\hat{b}_{\text{MLE}} = 0.216682$, resulting in an infinite EBT (∞). This result indicates that, based on the estimated model parameters, the stochastic process exhibits no tendency to approach the bankruptcy boundary. Consequently, the model does not indicate a finite expected time for the stochastic process to reach the bankruptcy boundary. In contrast, BBBY.US has $\hat{b}_{\text{MLE}} = -0.248385$, yielding an EBT of 5.113633 years, or approximately $1,288.64$ trading days, assuming 252 trading days per year. This result indicates that the stochastic process tends toward the model-implied zero-price boundary, resulting in a finite expected time to reach this boundary. These findings are consistent with the empirical characteristics of the stocks discussed in Subsection 3.1 and the parameter estimation results presented in Subsection 3.2, where Microsoft exhibits a relatively stable growth pattern, whereas BBBY displays characteristics associated with *financial distress*.

Although the *Expected Bankruptcy Time* (EBT) provides a measure of a stock's long-term tendency to reach bankruptcy, it is computed solely from the estimated model parameters without incorporating information about the state of the process at a particular observation time. In practice, the observed stock price at a given time may provide additional information regarding the evolution of bankruptcy risk. Therefore, a measure that updates the expected time to bankruptcy based on the current state of the observed process is required. For this purpose, the subsequent analysis employs the *Conditional Expected Bankruptcy Time* (CEBT), which estimates the expected time to bankruptcy conditional on the state of the process at a given observation time, thereby providing a state-conditional complement to the model-based EBT.

3.4. Analysis of Conditional Expected Bankruptcy Time

In addition to the *Expected Bankruptcy Time* (EBT), the bankruptcy risk of each stock is further assessed using the *Conditional Expected Bankruptcy Time* (CEBT), which estimates the expected time to bankruptcy by incorporating stock price information at a specified observation time. Unlike the EBT, which depends solely on the estimated model parameters, the CEBT updates the bankruptcy time estimate according to the state of the stochastic process at the observation time. Following the approach proposed by Hsu and Wu [19], the CEBT is evaluated at two observation points to illustrate the influence of stock price information on the estimated time to bankruptcy. The first evaluation (CEBT_1) uses the stock price at the beginning of the final year of the observation period, whereas the second evaluation (CEBT_2) uses the stock price at the end of the observation period. The resulting EBT and CEBT values for each stock are presented in Table 6.

Table 6: Comparison of EBT and CEBT

Stock	EBT (years)	CEBT_1 (years)	CEBT_2 (years)
MSFT.US	∞	∞	∞
BBBY.US	5.113633	6.790774	5.113632

Based on Table 6, MSFT.US yields infinite values for the EBT, CEBT_1, and CEBT_2. This result is a direct consequence of the estimated drift parameter satisfying $\hat{b} > 0$. According to Proposition 2 of Hsu and Wu [19], when $b \geq 0$, both the *Expected Bankruptcy Time* and the *Conditional Expected Bankruptcy Time* are infinite, implying that the stochastic process has no finite expected time to bankruptcy. In contrast, BBBY.US yields an EBT of 5.113633 years, while the corresponding CEBT_1 and CEBT_2 values are 6.790774 years and 5.113632 years, respectively. The difference between CEBT_1 and EBT shows that conditioning on the observed stock price can affect the estimated time to bankruptcy. However, the close numerical agreement between CEBT_2 and EBT should not be regarded as independent evidence of convergence in bankruptcy risk. Since CEBT_2 is evaluated at the terminal observation using the same estimated model parameters and the corresponding terminal stock price, its proximity to EBT reflects the conditional estimate under that particular state rather than a separate convergence test. Accordingly, CEBT is interpreted as a state-conditional measure that updates the estimated bankruptcy time based on the observed stock price, whereas EBT represents the corresponding model-based estimate without conditioning on a specific observation-time stock price.

Overall, the results of the *Expected Bankruptcy Time* (EBT) and *Conditional Expected Bankruptcy Time* (CEBT) analyses demonstrate that the *Extended Black-Scholes Model* provides a consistent quantitative framework for evaluating bankruptcy risk based on the characteristics of the observed stochastic process. The EBT characterizes a stock's long-term tendency to reach the bankruptcy boundary using the estimated model parameters, whereas the CEBT extends this assessment by incorporating stock price information at a specific observation time. The results for MSFT.US and BBBY.US show that these two measures complement each other in distinguishing the bankruptcy risk characteristics of the two contrasting stock-price cases. Consequently, the *Extended Black-Scholes Model* provides an unconditional model-based measure through EBT and a state-conditional measure through CEBT, enabling bankruptcy-time estimates to be assessed both without conditioning and conditional on the stock price observed at a specific time.

3.5. Benchmark Comparison with Geometric Brownian Motion

To evaluate the practical advantage of the EBSM over a conventional strictly positive-price process, the GBM is estimated using the same normalized stock-price data and observation intervals employed in the EBSM analysis. The resulting parameter estimates are then compared with those of the EBSM and examined in terms of their implications for the zero-price boundary. The benchmark parameter estimates and corresponding *Expected Bankruptcy Time* (EBT) values are presented in Table 7. This comparison is intended to support the interpretation of the additional implications of the EBSM zero-price boundary mechanism relative to the conventional GBM specification, rather than to constitute a formal goodness-of-fit assessment or predictive validation of the EBSM.

Table 7: Benchmark Comparison between EBSM and GBM

Stock	Model	Drift Parameter	Variance Parameter	EBT (years)
MSFT.US	GBM	0.292313	0.053988	∞
MSFT.US	EBSM	0.216682	0.035015	∞
BBBY.US	GBM	−0.014187	0.620751	∞
BBBY.US	EBSM	−0.248385	0.473402	5.113633

For MSFT.US, the GBM yields a positive drift estimate of $\hat{\mu}_{GBM} = 0.292313$ and a variance parameter estimate of $\hat{\sigma}^2_{GBM} = 0.053988$. In comparison, the EBSM yields a positive drift estimate of $\hat{b}_{MLE} = 0.216682$ and an estimated variance parameter of $\hat{a}^2 = 0.035015$, as reported in Table 3. Although the two models employ different parameterizations, they lead to the same qualitative conclusion concerning the zero-price boundary: the expected time to reach zero is infinite. Thus, for MSFT.US, the EBSM does not alter the qualitative conclusion concerning bankruptcy time relative to GBM, which is consistent with the relatively stable stock-price regime observed during the study period.

For BBBY.US, the benchmark comparison reveals a substantive difference between the two models. The GBM produces a negative drift estimate of $\hat{\mu}_{GBM} = -0.014187$ and a variance estimate of $\hat{\sigma}^2_{GBM} = 0.620751$. Nevertheless, despite the negative estimated drift, the GBM still implies an infinite expected time to reach the zero-price boundary because the process cannot attain zero within finite time. By contrast, the EBSM yields a negative drift estimate of $\hat{b}_{MLE} = -0.248385$ and a finite EBT of 5.113633 years, as the zero-price boundary is explicitly incorporated through the stopping-time mechanism. Therefore, the benchmark comparison indicates that the primary practical advantage of the EBSM in this application lies not simply in its ability to estimate a negative drift, but in its capacity to translate a distressed stochastic regime into a finite model-implied time to the zero-price boundary.

Overall, the benchmark comparison provides supporting evidence that the principal distinction between the two models lies in their treatment of the zero-price boundary. GBM provides a conventional baseline for strictly positive stock-price dynamics, whereas EBSM incorporates a stopping-time mechanism that permits the zero-price boundary to be reached within finite time. This distinction is particularly relevant for distressed stock-price regimes, where a negative drift alone under GBM does not imply a finite time to the zero-price boundary. Therefore, the benchmark results support the interpretation that the EBSM provides an additional model-implied measure of time to the zero-price boundary in this empirical setting. However, this comparison should not be interpreted as a formal goodness-of-fit assessment or predictive validation of the EBSM.

It is important to distinguish the model-implied zero-price boundary from the actual occurrence of corporate bankruptcy. In the EBSM, the stopping time $T_{1,0}$ denotes the first time at which the modeled stock-price process reaches zero, so the resulting EBT represents the expected time to this mathematical boundary rather than the legally defined date of bankruptcy. In practice, corporate bankruptcy is determined through legal and financial processes, including the inability to meet contractual obligations and the initiation of formal bankruptcy proceedings, which do not necessarily coincide with the stock price reaching exactly zero. Conversely, a stock may experience severe financial distress or decline to very low price levels without the firm immediately entering formal bankruptcy proceedings. Therefore, the finite EBT obtained for BBBY.US should be interpreted as the model-implied expected time to the zero-price boundary under the assumptions of the EBSM, rather than as a precise prediction of the actual date of corporate bankruptcy.

3.6. Robustness Analysis

To evaluate the sensitivity of the main findings to the selected observation window, the EBSM parameters are re-estimated using the most recent 60% and 80% of observations, with the full sample retained as the baseline specification. The estimation procedure described in Subsection 2.3 is applied consistently to each alternative window. The corresponding results are presented in Table 8. This analysis also evaluates the sensitivity of the estimated drift parameter, given that its sign determines whether the model-implied EBT is finite or infinite. The resulting comparisons provide supporting evidence for the stability of the qualitative findings across alternative observation windows and are not interpreted as a formal goodness-of-fit assessment or predictive validation of the EBSM.

Table 8: Robustness of EBSM Parameter Estimates across Trailing Windows

Stock	**Window**	**Observation Period**	n	$\hat{a}^2$	$\widehat{pq}$	$\hat{b}_{\text{MLE}}$	**EBT (years)**
MSFT.US	60%	2017-01-03–2019-12-31	754	0.027409	0.546896	0.239799	∞
MSFT.US	80%	2016-01-04–2019-12-31	1,006	0.033808	0.339085	0.234200	∞
MSFT.US	100%	2015-01-02–2019-12-31	1,258	0.035015	0.406565	0.216682	∞
BBBY.US	60%	2021-10-08–2022-11-30	289	0.420169	0.036334	−0.777947	2.174955
BBBY.US	80%	2021-05-24–2022-11-30	385	0.329300	0.043299	−0.581053	2.771771
BBBY.US	100%	2021-01-04–2022-11-30	482	0.473402	0.092916	−0.248385	5.113633

For MSFT.US, the estimated drift parameter remains positive across all three specifications, with $\hat{b}_{MLE}$ ranging from 0.216682 to 0.239799. As a result, the EBT remains infinite across all observation windows. The estimates of $\hat{a}^2$ and $\widehat{pq}$ also remain positive, indicating that the parameter admissibility conditions are maintained even as earlier observations are progressively excluded. The consistent positive drift sign across the alternative windows indicates that the finite-versus-infinite EBT classification for MSFT.US remains unchanged under different observation periods. These findings suggest that the positive-drift regime identified in the full-sample analysis is robust to the exclusion of the earliest observations in the MSFT.US sample.

A consistent qualitative pattern is observed for BBBY.US. The estimated drift parameter remains negative across the 60%, 80%, and full-sample specifications, with values of −0.777947, −0.581053, and −0.248385, respectively. Consequently, the EBT remains finite under all three specifications. Although the estimated EBT ranges from 2.174955 to 5.113633 years, the qualitative behavior of the process remains unchanged, as $\hat{b}_{MLE} < 0$ across all alternative windows. The estimates of $\hat{a}^2$ and $\widehat{pq}$ also remain positive, confirming that the parameter admissibility conditions are maintained across the alternative observation windows.

The robustness analysis provides supporting evidence for the stability of the qualitative distinction between the two empirical cases while also indicating sensitivity in the magnitude of the estimated bankruptcy time. Across all alternative trailing windows, MSFT.US maintains a positive estimated drift and an infinite EBT, whereas BBBY.US consistently exhibits a negative estimated drift and a finite EBT. This consistency indicates that the qualitative difference between the relatively stable and distressed regimes is not attributable solely to the earliest observations in the respective samples. However, the variation in BBBY.US's estimated EBT across the observation windows suggests that the magnitude of the model-implied time to the zero-price boundary is sample-dependent rather than an invariant characteristic of the firm. These results provide evidence of qualitative stability across the examined observation windows but should not be interpreted as evidence of formal model fit or out-of-sample predictive performance.

4. Conclusion

This study applied the *Extended Black-Scholes Model* (EBSM) to examine and compare bankruptcy risk characteristics between two contrasting stock-price cases based on stochastic stock-price dynamics. By introducing a stopping-time mechanism, the EBSM provides a framework in which stock prices are allowed to reach the bankruptcy boundary, enabling bankruptcy risk assessment through the *Expected Bankruptcy Time* (EBT) and *Conditional Expected Bankruptcy Time* (CEBT). The empirical results for *Microsoft Corporation* (MSFT.US) and *Bed Bath & Beyond Inc.* (BBBY.US) demonstrated that the estimated model parameters captured the distinct stochastic characteristics of the two stocks. MSFT.US exhibited a positive drift parameter, reflecting a stochastic process with a tendency to move away from the bankruptcy boundary, whereas BBBY.US showed higher volatility and a negative drift parameter, indicating a tendency toward the bankruptcy boundary.

The bankruptcy-time analysis further revealed differences between the two contrasting stock-price cases. Based on the estimated EBSM parameters, MSFT.US resulted in infinite EBT and CEBT values, suggesting that the stochastic process does not have a finite expected time to reach the bankruptcy boundary. In contrast, BBBY.US resulted in a finite EBT value of 5.113633 years, while the CEBT values varied according to the observed stock price conditions, illustrating the capability of CEBT to incorporate updated information from the stock price trajectory. These findings indicate that the EBSM framework can differentiate bankruptcy risk characteristics by combining stochastic parameter estimation with time-to-bankruptcy measures.

Overall, the findings demonstrate that the *Extended Black-Scholes Model* provides a quantitative framework for connecting stock price dynamics with bankruptcy risk analysis. The comparison with GBM further highlights the practical contribution of the EBSM: although GBM yields a negative drift estimate for BBBY.US, it cannot produce a finite time to the zero-price boundary because its stock-price process remains strictly positive over any finite time horizon. By contrast, the EBSM combines stochastic stock-price dynamics with an accessible zero-price boundary, enabling the distressed BBBY.US regime to be characterized by a finite model-implied EBT of 5.113633 years. The benchmark comparison provides supporting evidence for the qualitative distinction in the treatment of the zero-price boundary, while the robustness analysis supports the stability of the qualitative distinction between the two empirical cases across alternative trailing observation windows, although the magnitude of the estimated EBT for BBBY.US varies across specifications. These analyses are intended to support the interpretation and assessment of the stability of the findings rather than to constitute a formal goodness-of-fit assessment or out-of-sample predictive validation of the EBSM. Nevertheless, the model-implied zero-price boundary should be understood as a mathematical representation of extreme stock-price distress rather than as an exact proxy for the legal occurrence or timing of corporate bankruptcy. However, because the analysis is based on only two contrasting empirical cases, the findings cannot be generalized to the broader population of firms. Future research should therefore extend the framework to a broader cross-section of companies, different market conditions, and additional financial indicators to further evaluate the generalizability of the EBSM for bankruptcy-risk analysis.

CRediT Authorship Contribution Statement

Ram Daniel Andrian Vernando: Conceptualization, Methodology, Software, Formal Analysis, Data Curation, Validation, Visualization, Writing-Original Draft, and Writing-Review & Editing.
Rudianto Artiono: Supervision, Validation, and Project Administration.

Declaration of Generative AI and AI-assisted technologies

During the preparation of this work, the authors used ChatGPT (OpenAI) for assistance in preparing the manuscript. After using this tool, the authors reviewed, verified, and edited the content as needed and take full responsibility for the content of the publication.

Declaration of Competing Interest

The authors declare no competing interests.

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Data and Code Availability

The data used in this study consist of historical daily adjusted closing stock prices of Microsoft Corporation (MSFT.US) and Bed Bath & Beyond Inc. (BBBY.US) obtained from the publicly available Stooq historical market database. The datasets can be accessed directly from the official data source using the observation periods described in this study.

The computational procedures and parameter estimation algorithms used in this study were developed by the authors for research purposes. The source code is available from the corresponding author upon reasonable request.

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