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On the Metric Dimension of Some Operation Graphs

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ABSTRACT

Let G be a simple, finite, and connected graph. An ordered set of vertices of a nontrivial connected graph G is $W = \{w_1, w_2, w_3, \dots, w_k\}$ and the k-vector $r(v|W) = \big(d(v, w_1), d(v, w_2), \dots, d(v, w_k)\big)$ represent vertex v that respect to W, where $v \in G$ and $d(v, w_i)$ is the distance between vertex v and w_i for $1 \le i \le k$. The set W called a resolving set for G if different vertex of G have different representations that respect to W. The minimum cardinality of resolving set of G is the metric dimension of G, denoted by $\dim(G)$. In this paper, we give the local metric dimension of some operation graphs such as joint graph $P_n + C_m$, amalgamation of parachute, amalgamation of fan, and $shack(H_2^2, e, m)$.

Keywords: metric dimension, resolving set, operation graphs.

INTRODUCTION

All graphs in this paper are simple, finite and connected, for basic definition of graph we can see in Chartrand [1]. Chartrand [2] define the length of a shortest path between two vertices u and v is the distance $d(u, v)$ between two vertices in a connected graph G. An ordered set of vertices of a nontrivial connected graph G is $W = \{w_1, w_2, w_3, \dots, w_k\}$ and the k-vector $r(v|W) = \big(d(v, w_1), d(v, w_2), \dots, d(v, w_k)\big)$ represent vertex v that respect to W. The set W called a resolving set for G if different vertex of G have different representations that respect to W. The minimum of cardinality of resolving set of G is the metric dimension of G, denoted by $\dim(G)$ [3].

There are many articles explained about metric dimension such as [2], [4], [5], [6], and [7]. [8] defined a shackle graphs $shack(G_1, G_2, \dots, G_k)$ constructed by nontrivial connected graphs $G_1, G_2, \dots, G_k$ such that G_i and G_j have no a common vertex for every $i, j \in [1, k]$ with $|i - j| \ge 2$, and for every $l \in [1, k-1]$, G_l and G_{l+1} share exactly one common vertex (called linkage vertex) and the $k - 1$ linking vertices are all different. [9] defined an amalgamation of graphs constructed from isomorphic connected graphs H and the choice of the vertex v_j as a terminal is irrelevant. For any k positive integer, we denote such an amalgamation by $amal(H, k)$, where k denotes the number of copies of H.

Proposition 1. *[2] Let G be a connected graph or order $n \ge 2$, then the following hold:*

a. $dim(G) = 1$ if and only if graph G is a path graph

b. $dim(G) = n - 1$ if and only if graph G is a complete graph

c. For $n \ge 3$, $dim(C_n) = 2$

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d. *For* $n \geq 4$, $dim(G) = n - 2$ *if and only if* $G = K_{p,q}$ $(p, q \geq 1)$, $G = K_p + \overline{K_q} (p \geq 1, q \geq 2)$.

RESULTS AND DISCUSSION

Theorem 2.1. *For* $n \geq 2$ and $m \geq 7$, the metric dimension of joint graph $P_n + C_m$ *is* $dim(P_n + C_m) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$.

Proof. The joint of path and cycle graph, denoted by $P_n + C_m$ is a connected graph with vertex set $V(P_n + C_m) = \{x_j;\ 1 \leq j \leq n\} \cup \{y_l;\ 1 \leq l \leq m\}$ and edge set $E(P_n + C_m) = \{x_j y_l;\ 1 \leq j \leq n;\ 1 \leq l \leq m\} \cup \{x_j x_{j+1};\ 1 \leq j \leq n-1\} \cup \{y_l y_{l+1};\ 1 \leq l \leq m-1\} \cup \{y_n y_1\}$. The cardinality of vertex set and edge set, respectively are $|V(P_n + C_m)| = n + m$ and $|E(P_n + C_m)| = n(m+1) + m$.

If we show that $dim(P_n + C_m) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$ for $n \geq 2$ dan $m \geq 7$, then we will show the lower bound namely $dim(P_n + C_m) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor - 1$. Assume that $dim(P_n + C_m) < \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$. This can be shown with take resolving set $W = \{x_1, y_1, y_5\}$ so that it obtained the representation of the vertices $x, y \in V(P_2 + C_7)$ respect to W.

It can be seen that there is at least two vertices in $P_n + C_m$ which have the same representation respect to W, one of them is $r(y_4|W) = (1, 2, 1)$ and $r(y_6|W) = (1, 2, 1)$ such that we have the cardinality of resolving set of $dim(P_n + C_m) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$.

Furthermore, we will prove that $dim(P_n + C_m) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$ with determine the resolving set $W = \left\{x_j;\ 1 \leq j \leq 2\left\lceil \frac{n}{2} \right\rceil;\ i \in odd\right\} \cup \left\{y_l;\ 1 \leq l \leq 2\left\lfloor \frac{m-1}{2} \right\rfloor;\ j \in odd\right\}$. So, we have the cardinality of resolving set of $P_n + C_m$ namely $|W| = \frac{2\left\lceil \frac{n}{2} \right\rceil}{2} + \frac{2\left\lceil \frac{m-1}{2} \right\rceil}{2} = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$. The representation of the vertices $y \in F_n$ and $x \in F_n$ respect to W as follows.

$r(x_j|W) = \left\{(a_{ij});\ 1 \leq j \leq n, 1 \leq i \leq \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + \left(\left\lfloor \frac{m-1}{2} \right\rfloor\right)\right\}$, where

$$a_{ij} = \begin{cases} 0; for\ i = \frac{j+1}{2}, 1 \leq j \leq n, j \in odd \\ 1; for\ \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \leq i \leq \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + \left(\left\lfloor \frac{m-1}{2} \right\rfloor\right), 1 \leq j \leq n \\ or\ i = \frac{j}{2}, 2 \leq j \leq n, j \in even\ or\ i = \frac{j}{2} + 1, 2 \leq j \leq n, j \in even \\ 2; for\ i, j = otherwise \end{cases}$$

$r(y_l|W) = \left\{(a_{ij});\ 1 \leq j \leq m, 1 \leq i \leq \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + \left(\left\lfloor \frac{m-1}{2} \right\rfloor\right)\right\}$, where

$$a_{ij} = \begin{cases} 0; for\ i = \left(\left\lceil \frac{n}{2} \right\rceil\right) + \frac{j+1}{2}, 1 \leq j \leq m-2, j \in odd \\ 1; for\ 1 \leq i \leq \left(\left\lceil \frac{n}{2} \right\rceil\right), 1 \leq j \leq m-2 \\ or\ i = \left(\left\lceil \frac{n}{2} \right\rceil\right) + \frac{j}{2}, 2 \leq j \leq m, j \in even \\ or\ i = \left(\left\lceil \frac{n}{2} \right\rceil\right) + \frac{j+1}{2} + 1, 2 \leq j \leq m, j \in even \\ 2; for\ i, j = otherwise \end{cases}$$

It can be seen that every vertex in $P_n + C_m$ have distinct representation respect to W, such that the cardinality of resolvng set in $P_n + C_m$ is $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$ or $dim(F_n) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$. Thus, we conclude that $dim(P_n + C_m) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor$ for $n \geq 2$ and $m \geq 7$. ■

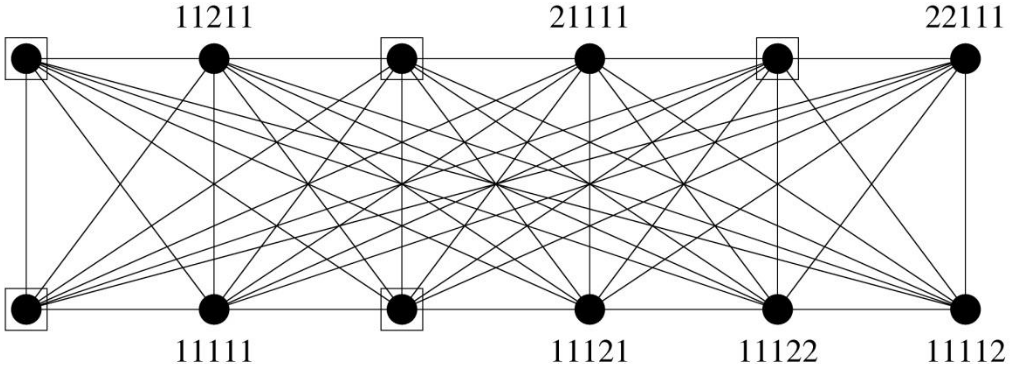

Fig 1. The Metric Dimension of Joint Graph $P_6 + C_4$.

Theorem 2.2. *For $n \geq 7$, the metric dimension of amalgamation of parachute amal (PC_7, v, m) is $dim(amal(PC_7, v = A, m)) = \frac{6m}{2}$.*

Proof. The amalgamation of parasut graph, denoted by $amal(PC_7, v, m)$ is a connected graph with vertex set $V(amal(PC_7, v, m)) = \{x_i^j;\ 1 \leq i \leq 7;\ 1 \leq j \leq m\} \cup \{y_i^j;\ 1 \leq i \leq 7;\ 1 \leq j \leq m\} \cup \{A\}$ and edge set $E(amal(PC_7, v, m)) = \{A\, x_i^j;\ 1 \leq i \leq 7;\ 1 \leq j \leq m\} \cup \{x_i^j\, x_{i+1}^j;\ 1 \leq i \leq 6;\ 1 \leq j \leq m\} \cup \{y_i^j\, y_{i+1}^j;\ 1 \leq i \leq 6;\ 1 \leq j \leq m\} \cup \{x_1^j\, y_1^j;\ 1 \leq j \leq m\} \cup \{x_7^j\, y_7^j;\ 1 \leq j \leq m\}$. The cardinality of vertex set and edge set, respectively are $|V(amal(PC_7, v, m))| = 14\text{m} + 1$ and $|E(amal(PC_7, v, m))| = 21m$.

If we show that $dim(amal(PC_7, v, m)) \geq \frac{6m}{2}$r $n = 7$, then we will show the best lower bound namely $dim(amal(PC_7, v, m)) \geq \frac{7m}{2} - 1$. Assume that $dim(amal(PC_7, v, m)) < \frac{6m}{2}$. This can be shown with take resolving set $W = \{x_1^1, x_4^1, x_6^1, x_1^2, x_4^2, x_6^2, x_1^3, x_4^3, x_6^3, x_1^4, x_4^4, x_6^4\}$ so that it obtained the representation of the vertices $x, y \in V(amal(PC_7, v, m))$ respect to W. It can be seen that there is at least two vertices in $amal(PC_7, v, 4)$ which have the same representation respect to W, one of them is $r(x_3^1|W) = (2,1,2,2,2,2,2,2,2,2,2,2)$ and $r(x_5^1|W) = (2,1,2,2,2,2,2,2,2,2,2,2)$ such that we have the cardinality of resolving set of $(amal(PC_7, v, m)) \geq \left\lceil\frac{6m}{2}\right\rceil$.

Furthermore, we will prove that $dim(amal(PC_7, v, m)) \leq \left\lceil\frac{6m}{2}\right\rceil$ with determine the resolving set $W = \{x_i^j;\ 4 \leq i \leq 7;\ 2 \leq j \leq m;\ i = odd\} \cup \{x_1^j;\ 1 \leq j \leq m\}$. So, we have the cardinality of resolving set of $amal(PC_7, v, m)$ namely $|W| = |\{x_i^j;\ 4 \leq i \leq 7;\ 2 \leq j \leq m; i = odd\} \cup \{x_1^j; 1 \leq j \leq m\}| = \left(\frac{4}{2}m\right) + m = \left(\frac{6m}{2}\right)$. The representation of the vertices $y \in (amal(PC_7, v, m = 4))$ and $x \in (amal(PC_7, v, m = 4))$ respect to W as follows.

$$r(x_i^j|W) = \left\{(a_{ik}^j);\ 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq \tfrac{6m}{2}\right\}, \text{where}$$

$$a_{ik} = \begin{cases} 0; for\ k = 1, k = 2i, 2 \leq i \leq \left(\left\lfloor\frac{n}{2}\right\rfloor\right), 1 \leq j \leq m \\ 1; for\ k = \frac{i+1}{2}, 3 \leq i \leq n, i \in odd \\ 1 \leq j \leq m\ or\ k = \frac{i-1}{2}, 5 \leq i \leq 7, i \in odd, 1 \leq j \leq m \\ 2; for\ 1 \leq j \leq m, k, i = other \end{cases}$$

$r(y|W) = \left\{(a_{ik});\ 1 \le i \le 7, 1 \le k \le \frac{6m+2}{2}\right\}$, where

$$a_{ik} = \begin{cases} 1; for\ k = 3j-2, i = 1, 1 \le j \le m \\ 2; for\ k = \frac{6m+2}{2}, i = 1, 7, 1 \le j \le m\ or\ k = 3j-2, i = 2, \\ 1 \le j \le m\ or\ k = 3j, i = 7, 1 \le j \le m \\ 3;\ for\ k = \frac{6m+2}{2}, i = 2,6, 1 \le j \le m\ or\ k = 3j-2, i = 3, \\ 1 \le j \le m\ or\ k = 3j, i = 6, 1 \le j \le m\ or\ i = 1, \\ k \ne 3j-2\ and\ k = \frac{6m+2}{2}\ or\ i = 7, k \ne 3j\ and\ k = \frac{6m+2}{2} \\ 4;\ for\ k = \frac{6m+2}{2}, i = 3, 5, 1 \le j \le m\ or\ k = 3j-2, i = 4, \\ 1 \le j \le m\ or\ k = 3j, i = 5, 1 \le j \le m\ or\ i = 2, \\ k \ne 3j-2\ and\ k = \frac{6m+2}{2}\ or\ i = 6, k \ne 3j\ and\ k = \frac{6m+2}{2} \\ 5;\ for\ k = \frac{6m+2}{2}, i = 3, 3j\ or\ k \ne 3j-2\ and\ k \ne \frac{6m+2}{2}, i = 3, \\ or\ k \ne 3j\ and\ k \ne \frac{6m+2}{2}, i = 5 \\ 6; for\ i = 4\ and\ i \ne 3j\ and\ i \ne 3j-2\ and\ i \ne \frac{6m+2}{2} \end{cases}$$

It can be seen that every vertex in $amal(PC_7, v, m)$ have distinct representation respect to W, such that the cardinality of resolvng set in $amal(PC_7, v, m)$ is $\frac{6m}{2}$ or $dim(amal(PC_7, v, m)) \le \frac{6m}{2}$. Thus, we conclude that $dim(amal(PC_7, v, m)) = \frac{6m}{2}$. ■

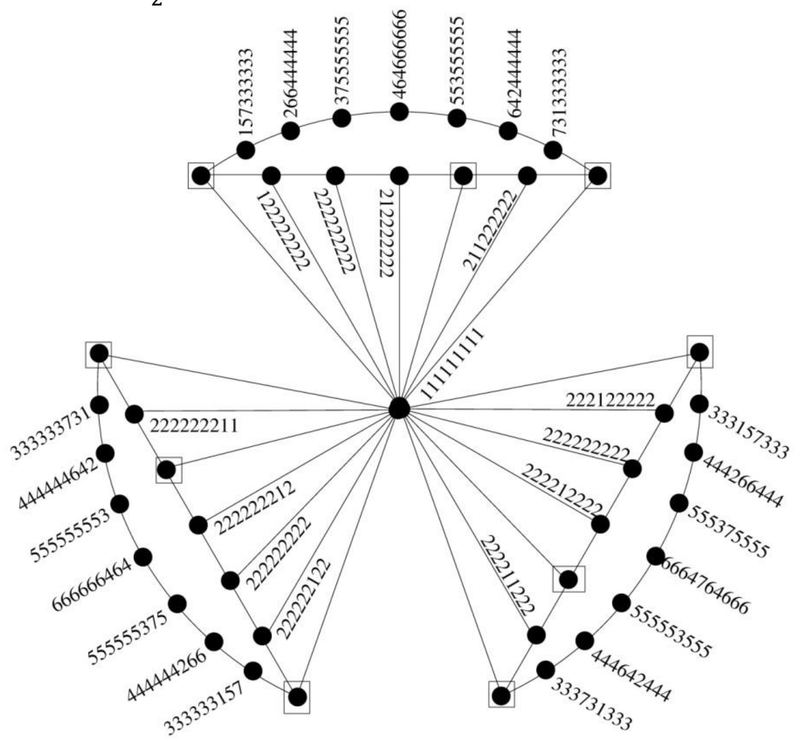

Fig 2. *The Metric Dimension of Amalgamation of Parachute Amal* $(PC_7, v, 3)$.

Theorem 2.3. *For* $n \ge 6$, *the metric dimension of amalgamation of fan graph* $amal(F_n, v = y, m)$ *is:*

$$dim\big(amal(F_n, v = A, m)\big) = \begin{cases} \dfrac{nm}{2} - 1, for\ n \text{ is even} \\ \dfrac{nm - m}{2}, for\ n \text{ is odd} \end{cases}$$

Proof. The amalgamation of fan graph, denoted by $amal(F_n, v = y, m)$ is a connected graph with vertex set $V\big(amal(F_n, v = y, m)\big) = \{x_i^j;\ 1 \le i \le n-1;\ 1 \le j \le m\} \cup \{y_j;\ 1 \le j \le m\} \cup \{x_n^m\}$ and edge set $E\big(amal(F_n, v = y, m)\big) = \{\, x_i^j\, x_{i+1}^j;\ 1 \le i \le n-2;\ 1 \le j \le m\} \cup \{y_j x_i^j;\ 1 \le i \le n-1;\ 1 \le j \le m\} \cup \{\, x_{n-1}^j\, x_1^{j+1};\ 1 \le j \le m-1\} \cup \{\, x_{n-1}^m\, x_n^m\} \cup \{y_j x_1^{j+1};\ 1 \le j \le m-1\} \cup \{y_m x_n^m\}$. The cardinality of vertex set and edge set, respectively are $|V\big(amal(F_n, v = y, m)\big)| = \text{nm} + 1$ and $|E\big(amal(F_n, v = y, m)\big)| = m(2n-1)$.

If we show that $dim\big(amal(F_n, v = y, m)\big) = \frac{nm}{2} - 1$ for$n \ge 7$ and n is even, then we will show the best lower bound namely $dim\big(amal(F_n, v = y, m)\big) \ge \frac{nm}{2} - 1$. Assume that $dim\big(amal(F_n, v = y, m)\big) < \frac{nm}{2} - 1$. This can be shown with take resolving set $W = \{x_1^1, x_4^1, x_1^2, x_4^2, x_6^2, x_1^3, x_4^3, x_6^3, x_1^4, x_4^4\}$ so that it obtained the representation of the vertices $y \in V(amal(F_6, v = y, 4))$ and $x_i^j \in V(amal(F_n, v = 6, 4))$ respect to W. It can be seen that there is at least two vertices in $amal(F_6, v = y, 4)$ which have the same representation respect to W, one of them is $r(x_6^1|W) = (2,2,2,2,2,2,2,2,2,2)$ and $r(x_6^4|W) = (2,2,2,2,2,2,2,2,2,2)$ such that we have the cardinality of resolving set of $dim\big(amal(F_n, v = y, m)\big) \ge \frac{nm}{2} - 1$.

Furthermore, we will prove that $dim\big(amal(F_n, v = y, m)\big) \le \frac{nm}{2} - 1$ with determine the resolving set $W = \{x_i^j;\ 4 \le i \le n;\ 2 \le j \le m;\ i = odd\} - \{x_n^m\} \cup \{x_1^j;\ 1 \le j \le m\}$. So, we have the cardinality of resolving set of $amal(F_n, v = y, m)$ namely $|W| = \big|\{x_i^j;\ 4 \le i \le n;\ 1 \le j \le m; i \text{ is even}\} - \{x_n^m\} \cup \{x_1^j; 1 \le j \le m\}\big| = \left(\frac{n-2}{2}\right)m + m - 1 = \left(\frac{nm}{2} - 1\right)$. The representation of the vertices $y \in F_n$ and $x \in F_n$ respect to W' as follows.

$$r\big(x_i^j|W\big) = \left\{\big(a_{ik}^j\big);\ 1 \le i \le n, 1 \le j \le m, 1 \le k \le \tfrac{nm}{2} - 1\right\}, \text{where}$$

$$a_{ik} = \begin{cases} 0; for\ k = 1, k = 2i, 2 \le i \le \left(\left\lfloor \frac{n}{2} \right\rfloor\right), 1 \le j \le m \\ 1; for\ k = \frac{i+1}{2}, 3 \le i \le n, i \in odd, 1 \le j \le m \\ k = \frac{i-1}{2}, 5 \le i \le n, i \in odd, 1 \le j \le m\ and\ (k \ne m \cap i \ne n) \\ 2; for\ 1 \le j \le m, k, i = others \end{cases}$$

$$r(y|W) = \left\{(a_{ik});\ 1 \le i \le n, 1 \le k \le \tfrac{n-2}{2}\right\}, \text{where}$$

$$a_{ik} = \left\{1; for\ 1 \le k \le \tfrac{nm-n}{2}, i = 1\right.$$

It can be seen that every vertex in $amal(F_6, v, 4)$ have distinct representation respect to W, such that the cardinality of resolvng set in $amal(F_n, v, m)$ is $\frac{nm}{2} - 1$ or $dim\big(amal(F_n, v, m)\big) \le \frac{nm}{2} - 1$. Thus, we conclude that $dim\big(amal(F_n, v, m)\big) = \frac{nm}{2} - 1$. ■

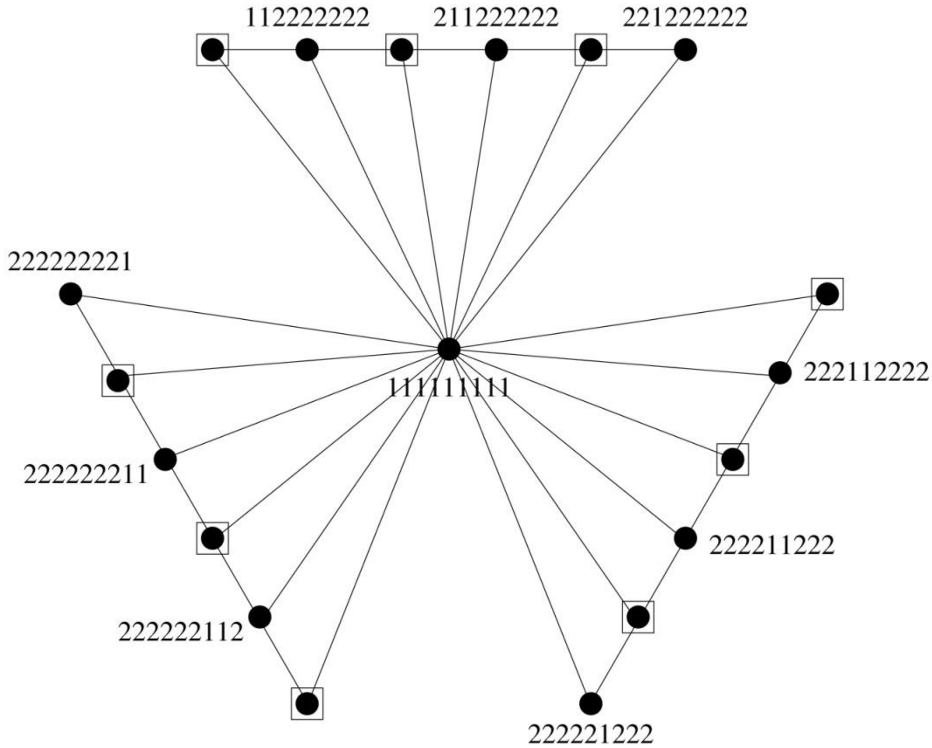

Fig 3. *The Metric Dimension of Amalgamation of Fan Graph* $Amal(F_6, v = y, 3)$.

Theorem 2.4. *For* $m \geq 2$, *the metric dimension of* $shack(H_2^2, e, m)$ *is* $dim(shack(H_2^2, e, m)) = 2$.

Proof. The shackle of fan graph, denoted by $shack(H_2^2, e, m)$is a connected graph with vertex set $V(shack(H_2^2, e, m)) = \{x_j;\ 1 \leq j \leq m+1\} \cup \{y_j;\ 1 \leq j \leq m+1\}$ and edge set $E(shack(H_2^2, e, m)) = \{x_j y_j;\ 1 \leq j \leq m+1\} \cup \{x_j y_{j+1};\ 1 \leq j \leq n\} \cup \{x_{j+1} y_j;\ 1 \leq j \leq m\}$. The cardinality of vertex set and edge set, respectively are $\left|V\left(shack(H_2^2, e, m)\right)\right| = 2m + 2$ and $\left|E\left(shack(H_2^2, e, m)\right)\right| = 3m + 1$.

The proof that the lower bound of $shack(H_2^2, e, m)$ is $dim(shack(H_2^2, e, m)) \geq 2$. Based on Proposition 1, that $dim(G) = 1$ if only if $G \cong P_n$. The graph $shack(H_2^2, e, m)$ does not isomorphic to path P_n such that $dim(shack(H_2^2, e, m)) \geq 2$. Furthermore, we proof that the upper bound of $shack(H_2^2, e, m)$ is $dim(shack(H_2^2, e, m)) \leq 2$, we choose the resolving set $W = \{x_1, y_1\}$.

The representation of the vertices $v \in V(shack(H_2^2, e, m))$ respect to W as follows.

$$r(x_j|W) = (j-1, j); j \in odd \qquad r(y_j|W) = (j, j-1); j \in odd$$
$$r(x_j|W) = (j, j-1); j \in even \qquad r(y_j|W) = (j-1, j); j \in even$$

Vertex $v \in V(shack(H_2^2, e, m))$ are distict. So, we have the cardinality of resolving set W is $|W| = 2$. Thus, the upper bound of $shack(H_2^2, e, m)$ is $dim(shack(H_2^2, e, m)) \leq 2$. It concludes that $dim\left(shack(H_2^2, e, m)\right) = 2$.

■

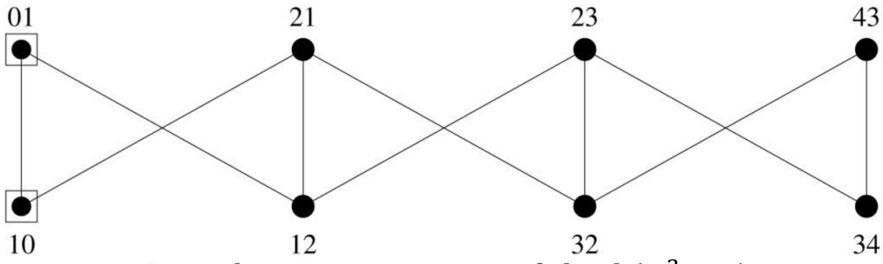

Fig 4. *The Metric Dimension of* $Shack(H_2^2, e, 3)$.

CONCLUSIONS

In this paper, the result show that the local metric dimension of some graph operation such as joint graph $P_n + C_m$, amalgamation of parachute, amalgamation of fan, and $shack(H_2^2, e, m)$.

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