The Rainbow Vertex-Connection Number of Star Fan Graphs

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ABSTRACT

A vertex-colored graph \( G = (V(G), E(G)) \) is said to be rainbow vertex-connected, if for every two vertices \( u \) and \( v \) in \( V(G) \), there exists a \( u-v \) path with all internal vertices have distinct colors. The rainbow vertex-connection number of \( G \), denoted by \( rvc(G) \), is the smallest number of colors needed to make \( G \) rainbow vertex-connected. In this paper, we determine the rainbow vertex-connection number of star fan graphs.

Keywords: Rainbow Vertex-Coloring; Rainbow Vertex-Connection Number; Star Fan Graph; Fan

INTRODUCTION

All graph considered in this paper are finite, simple, and undirected. We follow the notation and terminology of Diestel [1]. A vertex-colored graph \( G = (V(G), E(G)) \) is said to be rainbow vertex-connected, if for every two vertices \( u \) and \( v \) in \( V(G) \), there exists a \( u-v \) path with all internal vertices have distinct colors. The rainbow vertex-connection number of \( G \), denoted by \( rvc(G) \), is the smallest number of colors needed to make \( G \) rainbow vertex-connected. It was introduced by Krivelevich and Yuster [2].

Let \( G \) be a connected graph, \( n \) be the size of \( G \), and diameter of \( G \) denoted by \( diam(G) \), then they stated that

\[
diam(G) - 1 \leq rvc(G) \leq n - 2
\]

Besides that, if \( G \) has \( c \) cut vertices, then

\[
rvc(G) \geq c
\]

In fact, by coloring the cut vertices with distinct colors, we obtain \( rvc(G) \geq c \). It is defined that \( rvc(G) = 0 \) if \( G \) is a complete graph.

There are many interesting results about rainbow vertex-connection numbers. Some of them were stated by Li and Liu[3] and Simamora and Salman[4] and Bustan [5]. Li and Liu determined the rainbow vertex-connection number of a cycle \( C_n \) of order \( n \geq 3 \). Based on it, they proved that for a connected graph \( G \) with a block decomposition \( B_1, B_2, ..., B_k \) and \( c \) cut vertices, \( rvc(B_1) + rvc(B_2) + \cdots + rvc(B_k) + t \). In 2015 Simamora and Salman determined the rainbow vertex-connection number of pencil graph. In 2016 Bustan determined the rainbow vertex-connection number of star cycle graph.

In this paper, we introduce a new class of graph that we called star fan graphs and we determine the rainbow vertex-connection number of them. Star fan graphs are divided into two classes based on the selection of a vertex of the fan graph ie a vertex with \( n \) degree and vertex with 3 degree.
RESULT AND DISCUSSION

Definition 1. Let $m$ and $n$ be two integers at least 3, $S_m$ be a star with $m+1$ vertices, $F_n$ be a fan with $n+1$ vertices, $v \in V(F_n)$ and $v$ is a vertex with $n$ degree. A star fan graph is a graph obtained by embedding a copy of $F_n$ to each pendant of $S_m$, denoted by $S(m, F_n, v_{i,1})$ $i \in [1, m]$, such that the vertex set and the edge set, respectively, as follows.

$V(S(m, F_n, v_{i,1})) = \{v_{i,j} | i \in [1, m], j \in [1, m+1]\} \cup \{v_{m+1}\}$

$E(S(m, F_n, v_{i,1})) = \{v_{m+1}v_{i,1} | i \in [1, m]\} \cup \{v_{i,1}v_{i,j} | i \in [1, m], j \in [2, m+1]\} \cup \{v_{i,j}v_{i,j+1} | i \in [1, m], j \in [2, m]\}$
The Rainbow Vertex-Connection Number of Star Fan Graphs

**Theorem 1.** Let $m$ and $n$ be two integers at least 3 and $S(m, F_n, v_{i,1})$ be a star fan graph, then

$$rvc(S(m, F_n, v_{i,1})) = m + 1$$

**Proof.**
Based on equation (2), we have

$$rvc(S(m, F_n, v_{i,1})) \geq c = m + 1 \quad (3)$$

In order to prove $rvc(S(m, F_n, v_{i,1})) \leq m + 1$, define a vertex-coloring

$$\alpha: V(S(m, F_n, v_{i,1})) \rightarrow [1, m + 1]$$

as follows.

$$\alpha(v_{i,j}) = \begin{cases} 
  i, & \text{for } j = 1 \\
  m + 1, & \text{others}
\end{cases}$$

We are able to find a rainbow path for every pair vertices $u$ and $v$ in $V(S(m, F_n, v_{i,1}))$ as shown in Table 1.

**Tabel 1.** The rainbow vertex $u-v$ path for graph $S(m, F_n, v_{i,1})$

<table>
<thead>
<tr>
<th>$u$</th>
<th>$v$</th>
<th>Condition</th>
<th>Rainbow-vertex path</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{ij}$</td>
<td>$v_{kl}$</td>
<td>$i, k \in [1, m]$</td>
<td>$v_{ij}, v_{i1}, v_{m1}, v_{k1}, v_{kl}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$j, l \in [1, m + 1]$</td>
<td></td>
</tr>
</tbody>
</table>

So we conclude that

$$rvc(S(m, F_n, v_{i,1})) \leq m + 1 \quad (4)$$

From equation (3) and (4), we have $rvc(S(m, F_n, v_{i,1})) = m + 1$

**Definition 2.** Let $m$ and $n$ be two integers at least 3, $S_m$ be a star with $m + 1$ vertices. $F_n$ be a fan with $n + 1$ vertices, $v \in V(F_n)$ and $v$ is a vertex with 3 degree. A star fan graph is a graph obtained by embedding a copy of $F_n$ to each pendant of $S_m$, denoted by $S(m, F_n, v_{i,j}) \in [1, m], j \in [2, m]$ such that the vertex set and the edge set, respectively, as follows.

$$V(S(m, F_n, v_{i,j})) = \{v_{i,j} | i \in [1, m], j \in [m + 1]\} \cup \{v_{m+1}\},$$

$$E(S(m, F_n, v_{i,j})) = \{v_{m+1}v_{i,j} | i \in [1, m], j \in [2, m + 1]\} \cup \{v_{i,j}v_{i,j+1} | i \in [1, m], j \in [2, m]\}$$

**Theorem 2.** Let $m$ and $n$ be two integers at least 3 and $S(m, F_n, v_{i,j})$ be a star fan graph, then

$$rvc(S(m, F_n, v_{i,j})) = m + 2$$

**Proof.**
- **Case 1.** $m = 3$

Based on equation (1), we have $rvc(S(3, F_3, v_{i,4})) \geq diam - 1 = 6 - 1 = 5$. We may define a rainbow vertex 5-coloring on $S(m, F, v_{i,5})$ as shown in Figure 2.
\textbf{Case 2.} $m \geq 4$

Based on equation (2), we have $\text{rvc}\left(S(m, F_n, v_{i,j})\right) \geq m + 1$. Suppose that there is a rainbow vertex $m + 1$-coloring on $S(m, F_n, v_{i,j})$. Without loss of generality, color the vertices as follows:

\begin{align*}
\beta'(v_{m+1}) &= m + 1 \\
\beta'(v_{i,m+1}) &= i, i \in [1, m]
\end{align*}

Look at the vertex $v_{1,2}$ and $v_{2,2}$ who can not use the same color. To obtain rainbow vertex path between them, should be passed the path of $v_{1,2}, v_{1,1}, v_{1,m+1}, v_{2,m+1}, v_{2,1}, v_{2,2}$. Certainly $v_{1,1}$ should be colored by the color which used at the cut vertices, beside color $1, m + 1$ and 2. Suppose that $v_{1,1}$ being color with $k$. It's impacted there is no rainbow vertex path between vertex $v_{k,2}$ and $v_{i,2}$ for $i \neq k$. So that, graph $S(m, F_n, v_{i,j})$ cannot be colored with $m + 1$ colors, so we obtain

$$\text{rvc}\left(S(m, F_n, v_{i,j})\right) \geq m + 2 \quad (5)$$

In order to proof $\text{rvc}\left(S(m, F_n, v_{i,j})\right) \leq m + 2$, define a vertex-coloring $\beta : V\left(S(m, F_n, v_{i,j})\right) \rightarrow [1, m + 2]$ as follows.

\begin{align*}
\beta(v_{m+1}) &= m + 2 \\
\beta(v_{i,j}) &= (i + j) \mod (m + 1), i \in [1, m], j \in [1, m + 1]
\end{align*}

We are able to find a rainbow path for every pair vertices $u$ and $v$ in $V\left(S(m, F_n, v_{i,j})\right)$ as shown in table 2.

\begin{table}[h]
\centering
\caption{The rainbow vertex $u - v$ path for graph $S(m, F_n, v_{i,j})$}
\begin{tabular}{c c c}
\hline
$u$ & $v$ & Condition & Rainbow-vertex path \\
\hline
$u$ is adjacent to $v$ & & & \\
$v_{ij}$ & $v_{kl}$ & $i, k \in [1, m]$ & $u$, $v_{i,j+1}, v_{i,j+2}, \ldots, v_{i,m+1}, v_{m+1}v_{k,m+1}, v_k$. \\
& & $j, l \in [1, m + 1]$ & \\
& & $k \neq i + l, k \neq i - l, k$ & \\
& & $\in [2, m - 1]$ & \\
& & If $i = 1, k \neq m$ & \\
& & others & $u$, $v_{i,1}$, $v_{i,m+1}, v_{m+1}v_{k,m+1}, v_{k,1}, v$ \\
\hline
\end{tabular}
\end{table}

So we conclude that...
From equation (5) and (6), we have $rvc\left(S(m,F_n,v_{i,j})\right) = m + 2$

\[ rvc\left(S(m,F_n,v_{i,j})\right) \leq m + 2 \] (6)

**Figure 4.** $S(6,F_6,v_{i,7})$

**REFERENCES**


