



The Metric Dimension and Local Metric Dimension of Relative Prime Graph

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ABSTRACT

This study aims to determine the metric dimensions and local metric dimensions of relative prime graphs formed from modulo n integer rings, namely $G_{\mathbb{Z}_n}$ with $n \geq 2$. As a vertex set is $\mathbb{Z}_n \setminus \{0\}$ and $uv \in G_{\mathbb{Z}_n}$ if u and v are relatively prime. By finding the pattern elements of resolving set and local resolving set, it can be shown the value of the metric dimension and the local metric dimension of $G_{\mathbb{Z}_n}$ are $n - 2 + k$ and $|P_0| - 1 + k$ respectively, where k is the number of vertices groups that formed multiple $2, 3, \dots, k$ and $|P_0|$ is the cardinality of set P_0 . This research can be developed by determining the fractional metric dimension, local fractional metric dimension and studying the advanced properties of graphs related to their forming rings.

Key Words : Metric Dimension; Modulo n ; Relative Prime Graph; Resolving Set; Rings.

INTRODUCTION

Graph G is defined as a non-empty and finite set of $V(G)$ whose elements are called vertices and set $E(G)$ (maybe empty) whose elements are called edges which are the unordered pair of different vertices in $V(G)$ [1]. Any problems whose objects can be described as vertices and edges can be solved by the concept of graph theory, this has become one of the supporting factors in the field of graph theory research developing very rapidly today.

The notion of metric dimensions was introduced firstly by [2] and independently by [3] (in [4]). In [3], the concepts of bases and dimensions have been built on the graph. A bases on a graph is a set of vertices with minimal cardinality that implies in each vertex of graph having a different representation (resolving set) to the bases, while the dimensions are number elements of the bases. Meanwhile, the local metric dimension was introduced by [5] by considering different representations of two adjacent vertices so that the local metric dimension of a graph is obtained. The research related to the metric dimensions and local metric dimensions of a graph has been carried out by many researchers no exception to the graph of the results of operations (comb, corona, joint, etc.).

The development of research in the field of graph theory is also supported by the expansion of research objects in algebraic systems, namely groups or rings. In [6], the zero divisor graph is introduced from any commutative ring by defining the vertices on the graph are elements of the ring, while the two vertices are adjacent if the product is zero. By using

the definition in [6], the research was also carried out in [7,8,9] and succeeded in finding several properties related to the diameter, girth, isomorphism of the graph, radius, and domination set of zero divisor graphs constructed from the commutative and non-commutative rings. In addition to the zero divisor of graph, research has also been developed on the Jacobson graph formed from commutative rings that have nonzero unit elements [10,11]. The definition of vertices and edges is built from the radical Jacobson and the unit of the ring. Specifically, in [11], Jacobson graph was formed from the commutative ring Z_{3^n} and obtained several properties that related to the graph with its forming ring.

Some of the research above inspired the author to examine the metric dimensions and local metric dimensions on graphs constructed from commutative rings. As an object of research, we choose a ring of modulo n integer with the adjacency between two vertices was chosen based on the relative prime properties between the two vertices. The following is given the definition of the greatest common divisor and the relative prime of two positive integers.

Definition 1 [12]. Let a and b be two positive integers. The positive integer d that satisfies $d = pa + qb$ for some integer p and q is called greatest common divisor (abbreviated gcd) of a and b , denoted by $d = \text{gcd}(a, b)$.

Definition 2 [12]. Two positive integers a and b are relatively prime if $\text{gcd}(a, b) = 1$.

The purpose of this study is to determine the metric dimensions and local metric dimensions of relative prime graphs. The definition of metric dimensions and local metric dimensions which refer to [13] and [5] is given as follows.

Definition 3 [13]. Suppose that G is a connected graph of order n and $W = \{w_1, w_2, \dots, w_j\} \subseteq V(G)$, $1 \leq j \leq n$ is an ordered set of j -tuples of vertices in G . Representation of vertice $v \in V(G)$ with respect to W is an ordered pair j -tuples $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_j))$, where $d(v, w_i)$ representing the distance between vertex v and vertex w_i , $1 \leq i \leq j$. The set W is called the resolving set of G if each vertex in G has different representation with respect to W . The resolving set with a minimum cardinality is called a bases, whereas number of elements on the bases are called dimensions of the graph G , denoted by $\text{dim}(G)$. Since the calculation of dimensions in a graph is built using the concept of distance (metric), it is called the metric dimension.

Definition 4 [5]. Let G be the connected graph and $W \subseteq V(G)$. The set W is called the local resolving set of a graph G if each of two adjacent vertices in G has a different representation with respect to W , i.e. $uv \in E(G)$ implies $r(u|W) \neq r(v|W)$. The set of local resolving set with minimal cardinality is called a local bases, while number of elements on a bases are called the local metric dimensions of graph G , denoted by $\text{dim}_l(G)$.

METHODS

This study aims to determine the metric dimensions and local metric dimensions of relative prime graph G_{Z_n} . The most important step in determining the metric dimensions and local metric dimensions is to determine the pattern of resolving set and local resolving set that have a minimum cardinality as a bases set. The main difference between them is that

on the local metric dimension the different representation are only required at adjacent vertices.

RESULTS AND DISCUSSION

In this section, we will discuss the definition of a relative prime graph built from a ring of modulo n integer and an exploration of the basic properties of a relative prime graph related to the characteristics of a graph. The discussion begins with the definition of a relative prime graph.

Definition 5. Let \mathbb{Z}_n be a rings of modulo n integer \mathbb{Z}_n , where n positive integer and $n \neq 1$. We defined a new graph G with $V(G) = \mathbb{Z}_n \setminus \{0\}$ and $E(G) = \{uv; u \text{ relatively prime with } v\}$. Graph G which formed from ring of modulo n integer with add relatively prime as condition for adjacency two vertices are called relative prime graph, denoted by $G_{\mathbb{Z}_n}$. The number of vertices of $G_{\mathbb{Z}_n}$ is denoted $|V(G_{\mathbb{Z}_n})|$ and the number of edges is denoted $|E(G_{\mathbb{Z}_n})|$.

Example:

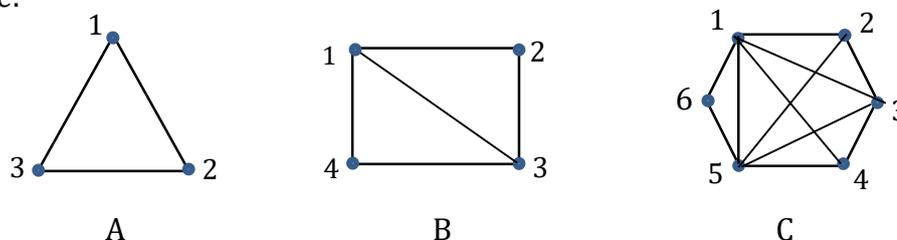


Figure 1. Some relative prime graphs
A: $G_{\mathbb{Z}_4}$; B: $G_{\mathbb{Z}_5}$; C: $G_{\mathbb{Z}_7}$

The graph on Figure 1:

- A: $G_{\mathbb{Z}_4}$ consists of three vertices, namely 1,2,3 which are mutually relative prime. Hence vertex 1 adjacent to 2, vertex 2 adjacent to 3, and vertex 3 adjacent to 1.
- B: $G_{\mathbb{Z}_5}$ consists of four vertices, namely 1,2,3,4. The adjacency of vertices 1,2,3 similar with condition of $G_{\mathbb{Z}_4}$. Furthermore, vertex 4 is relatively prime to 1 and 3, while vertex 2 is not relatively prime to 4. Hence vertex 2 is not adjacent to 4.
- C: $G_{\mathbb{Z}_7}$ consists of six vertices, namely 1,2,3,4,5,6. Vertex 1 is relatively prime to vertices 2,3,4,5,6; vertex 2 is relatively prime to vertices 1,3,5; vertex 3 is relatively prime to vertices 1,2,4,5; vertex 4 is relatively prime to vertices 1,3,5; vertex 5 is relatively prime to vertices 1,2,3,4,6; and vertex 6 is relatively prime to vertices 1,3,5. Since the three vertices (i.e. 2,4,6) are not relatively prime to each other, then they are not adjacent to each other. Likewise, vertex 3 is not adjacent to 6 because 3 is not relatively prime to 6.

The results of this study begin with an exploration the basic properties of $G_{\mathbb{Z}_n}$, furthermore determine the metric dimensions and local metric dimensions of $G_{\mathbb{Z}_n}$. Based on Definition 5 above, there are some basic properties of $G_{\mathbb{Z}_n}$ for $n \geq 2$, as follows:

- a. The set of vertices in $G_{\mathbb{Z}_n}$ is $V(G_{\mathbb{Z}_n}) = \{1,2,3, \dots, n - 1\}$, hence $|E(G_{\mathbb{Z}_n})| = n - 1$.
- b. $G_{\mathbb{Z}_n}$ is not an empty graph.
- c. $G_{\mathbb{Z}_n}$ is a trivial graph for $n = 2$ because it consist only one vertex.

- d. $G_{\mathbb{Z}_n}$ is a connected graph.
- e. $G_{\mathbb{Z}_n}$ is a complete graph for $n = 2,3,4$. Specifically for $n = 3$ it is a path graph and for $n = 4$ it is a sikel graph.
- f. $G_{\mathbb{Z}_n}$ is a regular graph for $n = 3,4$ because every vertex has the same degree.
- g. There is a vertex 1 that adjacent to every vertex in $G_{\mathbb{Z}_n}$ for $n \geq 3$.

Based on Definition 5, a vertex u is adjacent to vertex v if u is relatively prime with v . In $G_{\mathbb{Z}_n}$, there are vertices that are adjacent to each vertex in $G_{\mathbb{Z}_n}$, included in this category are vertex 1 and several vertices which are prime numbers. Based on the definition of two adjacent vertices on $G_{\mathbb{Z}_n}$, the adjacency of vertices are divided into two groups, namely (1) vertices that are adjacent to every vertex, and (2) vertices that are not adjacent to each other. The vertices that not adjacent are the vertices that not relatively prime to each other, i.e. the vertices that have a common divisor other than 1. Furthermore, the vertices that have a common divisor other than 1 can be divided into multiple of 2, multiple of 3, multiple of 5, ..., multiple of 7, Suppose $A = \{2,3,5,7, \dots\} = \{p_1, p_2, \dots, p_k\}$ represent a set of ordered prime numbers, then the non adjacent vertices are grouped together in multiples of 2, multiples of 3, ... to multiples of prime numbers p_k , where $2p_k \leq n - 1$. In this case, p_k is the largest prime numbers and k represents number of groups (multiples 2, multiples 3, etc.). The vertices that adjacent to every vertices in $G_{\mathbb{Z}_n}$ are categorized into group p_0 .

Example: in $G_{\mathbb{Z}_{15}}$, $V(G_{\mathbb{Z}_{15}}) = \{1,2,3,\dots, 14\}$. The grouping of vertices in $G_{\mathbb{Z}_{15}}$ are:

The vertices in group p_0 are 1,11,13.

The vertices in group of multiple $p_1 = 2$ are 2,4,6,8,10,12,14.

The vertices in group of multiple $p_2 = 3$ are 3,6,9,12.

The vertices in group of multiple $p_3 = 5$ are 5,10.

The vertices in group of multiple $p_4 = 7$ are 7,14.

In this case, there are four groups of multiples, where 7 is the largest prime number that satisfy $2 \times 7 \leq 14$. The vertices 1,11,13 are adjacent to every vertex in $G_{\mathbb{Z}_{15}}$ so that it is categorized in group p_0 . It appears that there are vertices that are in more than one group of multiples, for example vertices 6 and 12 are in groups of multiple 2 and multiple 3. Likewise, vertex 15 is in groups of multiple 3 and multiple 5. Furthermore, if P_i is the set of vertices in the group of multiples p_i , where $i = 1,2, \dots, k$ and $\lfloor x \rfloor$ represent the largest integer that same or less than x , then $|P_i| = \lfloor \frac{n-1}{p_i} \rfloor$. On $G_{\mathbb{Z}_{15}}$, $|P_1| = \lfloor \frac{14}{2} \rfloor = 7$; $|P_2| = \lfloor \frac{14}{3} \rfloor = \lfloor 4,67 \rfloor = 4$; $|P_3| = \lfloor \frac{14}{5} \rfloor = \lfloor 2,8 \rfloor = 2$; $|P_4| = \lfloor \frac{14}{7} \rfloor = 2$. So, in general in $G_{\mathbb{Z}_n}$ the number of vertices in the group of multiples p_i is $\lfloor \frac{n-1}{p_i} \rfloor$, while the number of vertices in the group of multiples p_i and p_j is $\lfloor \frac{n-1}{p_i \cdot p_j} \rfloor$ where $1 \leq i, j \leq k$. The lemma that expressed the number of edge of $G_{\mathbb{Z}_n}$ is given as follows.

Lemma 1. $|E(G_{\mathbb{Z}_n})| = \frac{(n-1)(n-2)}{2} - \sum_{i=1}^k \binom{\lfloor \frac{n-1}{p_i} \rfloor}{2}$, where $\lfloor \frac{n-1}{p_i} \rfloor$ is the number of vertices in

groups of multiples p_i , $i = 1,2, \dots, k$ and $2p_k \leq n - 1$.

Proof. As explained above, the vertices on $G_{\mathbb{Z}_n}$ are divided into 2 major groups, namely the group p_0 and the groups of multiples p_i . Due to the specific properties of each group, the number of edges on graph $G_{\mathbb{Z}_n}$ can be calculated by assumption the number of edges of the complete graph with $n - 1$ vertices, then edge reduction is done based on the non relatively prime properties between any two vertices. As we know, the number of edge of the complete graph with $n - 1$ vertices is $\frac{(n-1)(n-2)}{2}$. Since an edge is formed by two different vertices,

then there is a reduction in the edge for each group of multiples p_i as many as $\binom{\lfloor \frac{n-1}{p_i} \rfloor}{2}$. Thus,

$$|E(G_{\mathbb{Z}_n})| = \frac{(n-1)(n-2)}{2} - \sum_{i=1}^k \binom{\lfloor \frac{n-1}{p_i} \rfloor}{2} \quad \square$$

Table 1. Number of edges on graph $G_{\mathbb{Z}_n}$

| n | $ V(G_{\mathbb{Z}_n}) $ | Groups of vertices | Reduction of edge | $ E(G_{\mathbb{Z}_n}) $ |
|-----|-------------------------|---|---|---------------------------------|
| 2 | 1 | - | - | 0 |
| 3 | 2 | $p_0 : 1,2$ | - | $\frac{2(2-1)}{2} = 1$ |
| 4 | 3 | $p_0 : 1,2,3$ | - | $\frac{3(3-1)}{2} = 3$ |
| 5 | 4 | $p_0 : 1,3$ $p_1 : 2,4$ | - $\binom{2}{2} = 1$ | $\frac{4(4-1)}{2} - 1 = 5$ |
| 6 | 5 | $p_0 : 1,3,5$ $p_1 : 2,4$ | - $\binom{2}{2} = 1$ | $\frac{5(5-1)}{2} - 1 = 9$ |
| 7 | 6 | $p_0 : 1,5$ $p_1 : 2,4,6$ $p_2 : 3,6$ | - $\binom{3}{2} = 3$ $\binom{2}{2} = 1$ | $\frac{6(6-1)}{2} - 3 - 1 = 11$ |
| 8 | 7 | $p_0 : 1,5,7$ $p_1 : 2,4,6$ $p_2 : 3,6$ | - $\binom{3}{2} = 3$ $\binom{2}{2} = 1$ | $\frac{7(7-1)}{2} - 3 - 1 = 17$ |
| 9 | 8 | $p_0 : 1,5,7$ $p_1 : 2,4,6,8$ $p_2 : 3,6$ | - $\binom{4}{2} = 6$ $\binom{2}{2} = 1$ | $\frac{8(8-1)}{2} - 6 - 1 = 21$ |

The degree of each vertex on any graph is the number of the vertices that adjacent to that vertex. Thus, the degree of vertex $u \in V(G_{\mathbb{Z}_n})$ is determined based on their adjacency to $(n - 2)$ other vertices as in Observation 2. Furthermore, Lemma 3 states the minimum and maximum degree of the vertex on $G_{\mathbb{Z}_n}$.

Observation 2. If $\deg_{G_{\mathbb{Z}_n}}(u)$ denoted the degree of any vertex $u \in V(G_{\mathbb{Z}_n})$, then

$$\deg_{G_{\mathbb{Z}_n}}(u) = |\{v \in V(G_{\mathbb{Z}_n}) : v \text{ relatively prime with } u\}|.$$

Lemma 3. If $u \in V(G_{\mathbb{Z}_n})$, then $\deg_{G_{\mathbb{Z}_n}}(u) = \begin{cases} 0, & n = 2 \\ 1, & n = 3 \\ 2 \leq \deg_{G_{\mathbb{Z}_n}}(u) \leq n - 2, & n \geq 4 \end{cases}$

Proof. Let $u \in V(G_{\mathbb{Z}_n})$.

- For $n = 2$, then $u = 1$ and $G_{\mathbb{Z}_n}$ is the trivial graph, hence $\deg_{G_{\mathbb{Z}_n}}(u) = 0$.
- For $n = 3$, then $u \in \{1,2\}$. The number 1 is relatively prime with 2, hence vertex 1 adjacent to vertex 2. Thus, $\deg_{G_{\mathbb{Z}_n}}(u) = 1$.
- For $n \geq 4$, it means $|V(G_{\mathbb{Z}_n})| \geq 3$.

To show the minimum degree of any vertex is 2, we use Claim: every vertex in $G_{\mathbb{Z}_n}$ is adjacent to at least two other vertices.

Proof of Claim: Since the criterion for two adjacency vertices are relative prime and the vertices in $G_{\mathbb{Z}_n}$ are natural numbers, it is sufficient to show that any natural number other than 1 must be relative prime to the natural number before and after it. Suppose any natural number a where $a \neq 1$, it will be shown that a relative prime with $a - 1$ and a also relative prime with $a + 1$. Based on Definition 2, the natural number a is relative prime with $a - 1$ if there are integers p and q such that $1 = pa + q(a - 1)$. By choosing $p = 1$ and $q = -1$, equality hold. Hence a relative prime with $a - 1$. In a similar way, it can be shown that a relative prime with $a + 1$. Thus, a adjacent to $a - 1$ and also a adjacent to $a + 1$. Therefore, for any $u \in V(G_{\mathbb{Z}_n})$, there are at least two vertices that adjacent to u , so $\deg_{G_{\mathbb{Z}_n}}(u) \geq 2$. On the other hand, based on the property of the group p_0 , where each vertex is adjacent to every vertex in $G_{\mathbb{Z}_n}$, so for any $u \in P_0$, $\deg_{G_{\mathbb{Z}_n}}(u) = |V(G_{\mathbb{Z}_n})| - 1 = n - 1 - 1 = n - 2$ and this is the maximum degree of any vertex in $G_{\mathbb{Z}_n}$. Thus, it is proven that $2 \leq \deg_{G_{\mathbb{Z}_n}}(u) \leq n - 2$, for $n \geq 4$ and the whole lemma is proven. \square

For example, on graph $G_{\mathbb{Z}_7}$ we have $V(G_{\mathbb{Z}_7}) = \{1,2,3,4,5,6\}$. The degree of each vertex is: $\deg_{G_{\mathbb{Z}_7}}(1) = |\{2,3,4,5,6\}| = 5$; $\deg_{G_{\mathbb{Z}_7}}(2) = |\{1,3,5\}| = 3$; $\deg_{G_{\mathbb{Z}_7}}(3) = |\{1,2,4,5\}| = 4$; $\deg_{G_{\mathbb{Z}_7}}(4) = |\{1,3,5\}| = 3$; $\deg_{G_{\mathbb{Z}_7}}(5) = |\{1,2,3,4,6\}| = 5$; $\deg_{G_{\mathbb{Z}_7}}(6) = |\{1,5\}| = 2$. It can be seen that $2 \leq \deg_{G_{\mathbb{Z}_7}}(u) \leq 5$, for every $u \in V(G_{\mathbb{Z}_7})$.

In $G_{\mathbb{Z}_n}$, there is vertex 1 which is adjacent to each vertex in $G_{\mathbb{Z}_n}$. Several other vertices are also similar. Vertices of this property will form a complete subgraph of $G_{\mathbb{Z}_n}$ as shown in Theorem 4.

Theorem 4. There is a complete subgraph formed by vertices of $G_{\mathbb{Z}_n}$.

Proof. A complete graph is a graph where every two vertices are adjacent. Based on the property of the vertices on $G_{\mathbb{Z}_n}$, there are vertices that are adjacent to each vertex, namely the vertices in the group p_0 . These vertices will form the complete subgraph K_m , where $m = |\{u \in V(G_{\mathbb{Z}_n}) : \gcd(u, v) = 1, \forall v \in V(G_{\mathbb{Z}_n})\}|$. \square

From Theorem 4, the vertices in $G_{\mathbb{Z}_n}$ that form the complete subgraph are vertices in group p_0 , where $m = |\{u \in V(G_{\mathbb{Z}_n}) : \gcd(u, v) = 1, \forall v \in V(G_{\mathbb{Z}_n})\}| = |P_0|$. On the other hand, the vertices in the group of multiples $p_i, i = 1, 2, \dots, k$ have the same property that they are not adjacent to each other. This inspires that the vertices in the group of multiples $p_i, i = 1, 2, \dots, k$ form the multipartite subgraph of $G_{\mathbb{Z}_n}$ which is stated in the Theorem 5.

Theorem 5. For $n \geq 5$, $G_{\mathbb{Z}_n}$ is isomorphic with a $K_m + H$ graph where H is the k -partite graph, K_m is a complete graph with m vertices and $m = |P_0|$.

Proof. Based on Theorem 4, the vertices in the group p_0 form the complete subgraph K_m of $G_{\mathbb{Z}_n}$. Meanwhile, the vertices in the group of multiples $p_i, i = 1, 2, \dots, k$ are not adjacent to

each other for each i , so the vertices in the group of multiples p_i can be partitioned into k partitions with each partition consisting of vertices in the same group of multiples. On the same partition, there are no adjacent vertices according to the properties of the vertices in the group of multiples p_i . The connectedness of the vertices between partitions is based on the relatively prime properties of these vertices. Since every vertex in K_m is adjacent to every vertex in the group of multiples p_i , this means that every vertex in K_m is adjacent to every vertex on all partitions (as many as k partitions) that are formed. Suppose the k partitions formed along with adjacency their vertices are called graph H , then H is a k -partite graph formed from vertices in the group of multiples p_i . Every vertex in the group p_0 is adjacent to every vertex in $G_{\mathbb{Z}_n}$, especially in the group of multiples p_i . This means that every vertex in K_m is adjacent to every vertex in H . Therefore, there is a bijective function f from $V(G_{\mathbb{Z}_n})$ to $V(K_m + H)$ which preserves the adjacency between vertices in $G_{\mathbb{Z}_n}$. Thus, $G_{\mathbb{Z}_n} \cong K_m + H$. \square

Theorem 5 applies for $n \geq 5$, specifically for $n = 2,3,4$ then $G_{\mathbb{Z}_n} \cong K_{n-1}$ (complete graph with $n - 1$ vertices). For example: on $G_{\mathbb{Z}_8}$ where $V(G_{\mathbb{Z}_8}) = \{1,2,3,4,5,6,7\}$.

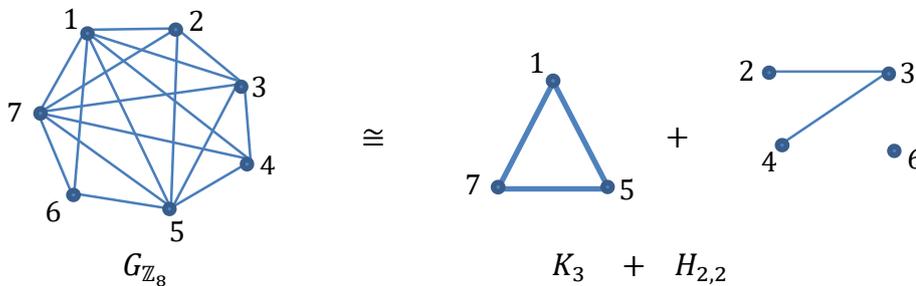


Figure 2. Isomorphism $G_{\mathbb{Z}_8}$ with $K_3 + H_{2,2}$

Figure 2 illustrates the isomorphism $G_{\mathbb{Z}_8}$ with $K_3 + H_{2,2}$ where $H_{2,2}$ is 2-partite graph. In $G_{\mathbb{Z}_8}$, vertices 1,5,7 is adjacent to each vertex such that $m = 3$. Meanwhile, the vertices in the group of multiples 2 are 2,4,6; the vertices in the group of multiples 3 are 3,6 and no vertex are multiples of 5, so there are 2 partitions. In the example above, the vertices on the first partition are 2 and 4, while the vertices on the second partition are 3 and 6. Since the number of vertices on the first partition is 2 and the number of vertices on the second partition is 2, it is written $H_{2,2}$. The existence of graph $H_{2,2}$ as a 2-partite graph is not unique. Another alternative is that if three vertices are selected on the first partition (i.e. vertices 2,4,6) and on the second partition one vertex is chosen (i.e. vertex 3), then the 2-partite graph in this case is written $H_{3,1}$. Thus, $G_{\mathbb{Z}_8} \cong K_3 + H_{2,2} \cong K_3 + H_{3,1}$.

Next, we will be determined the value of the metric dimension and the local metric dimension of $G_{\mathbb{Z}_n}$ for $n \geq 2$. The determination of metric dimensions is calculated using the concept of the distance between two vertices, namely the length of the shortest path connecting the two vertices.

Theorem 6. If $u, v \in V(G_{\mathbb{Z}_n})$, then $d(u, v) \leq 2$.

Proof. Let $u, v \in V(G_{\mathbb{Z}_n})$. Based on Theorem 5, there are three possibilities for u and v , namely (i) $u, v \in V(K_m)$, (ii) $u, v \in V(H)$, dan (iii) $u \in V(K_m), v \in V(H)$.

- (i) Suppose $u, v \in V(K_m)$. Since K_m is a complete graph, then distance between two different vertices is 1, so that $d(u, v) = \begin{cases} 0, & u = v \\ 1, & u \neq v \end{cases}$.

- (ii) Suppose $u, v \in V(H)$, there are two condition, i.e. u and v on the same partitions or u and v on the different partitions.
- If u and v are vertex on the same partition, then there is $z \in V(K_m)$ so that $d(u, z) = 1$ and $d(z, v) = 1$. Therefore, $d(u, v) = 2$. So $d(u, v) = \begin{cases} 0, & u = v \\ 2, & u \neq v \end{cases}$.
 - If u and v are vertices on the different partition, then the distance is 1 if u adjacent to v . For vertex u which is not adjacent to v , there is $x \in V(K_m)$ so that $d(u, x) = 1$ and $d(x, v) = 1$. Therefore $d(u, v) = 2$. Thus $d(u, v) = \begin{cases} 1, & \gcd(u, v) = 1 \\ 2, & \gcd(u, v) \neq 1 \end{cases}$.
- (iii) Suppose $u \in V(K_m)$, $v \in V(H)$, then $d(u, v) = 1$ because every vertex in K_m is adjacent to all vertices in H .
- From all possibilities (i), (ii), and (iii) it is proven that $d(u, v) \leq 2$. \square

Corollary 7. Let $u, v \in V(G_{\mathbb{Z}_n})$, $d(u, v) = 2$ if and only if u is not relatively prime to v .

Proof. It is clear, is a direct result of Theorem 6.

The vertices in the group p_0 apart from forming a complete subgraph of $G_{\mathbb{Z}_n}$ are also adjacent to all vertices in $G_{\mathbb{Z}_n}$. Due to this specific property, in determining the elements of the resolving set, only one vertex is allowed. On the other hand, the vertices in the group of multiples p_i also have a specific property. Two groupings of vertices in $G_{\mathbb{Z}_n}$, each group have specific properties, so it is impossible for the metric bases of $G_{\mathbb{Z}_n}$ consist of vertices in the group p_0 or in the group of multiples p_i only. This condition is illustrated in Theorem 8 below by observing the vertices in the group p_0 .

Theorem 8. Every subset of K_m is not a resolving set.

Proof. Referring to Theorem 4, the vertices in group p_0 form a complete subgraph with m vertices and the metric dimension is $m - 1$. That is, the representation of other vertices in the group p_0 against the $m - 1$ vertices is the same as the vertex representation outside the group p_0 for the $m - 1$ verices. As a result, the set consisting of $m - 1$ vertices is not a resolving set. The same condition also applies to sets whose element are less than $m - 1$ vertices. Based on Theorem 5 which states $G_{\mathbb{Z}_n} \cong K_m + H$, it is obtained that each vertex in K_m is adjacent to every vertex in H , it means that the distance is 1. For the same reason, any vertex taken from K_m is not a resolving set. Evidently, every subset of K_m is not a resolving set. \square

Based on the proof of Theorem 8, the resolving set can not contain only vertices in K_m . On the other hand, the vertices in groups of multiples p_1, p_2, \dots, p_k has a similar property that are not adjacent to each other. Representation of two different vertices of a certain group of multiple to another vertices in the same group of multiple and to different groups of multiple are presented in the following Lemma 9 and Lemma 10.

Lemma 9. If $u, v \in P_i$, where $u \neq v$, then $r(u|P_i \setminus \{u, v\}) = r(v|P_i \setminus \{u, v\})$.

Proof. The vertices in the group of multiples p_i have a similar characteristic, namely they are not adjacent. Based on the proof of Theorem 6 (ii) a, the distance between different vertices is 2. Thus, $r(u|P_i \setminus \{u, v\}) = (2, 2, \dots, 2)$ and $r(v|P_i \setminus \{u, v\}) = (2, 2, \dots, 2)$. Therefore, $r(u|P_i \setminus \{u, v\}) = r(v|P_i \setminus \{u, v\})$. \square

Lemma 10. If $u, v \in P_{j_1}$, then $r(u|P_{j_2}) = r(v|P_{j_2})$, where $1 \leq j_1, j_2 \leq k$ and $u, v \notin P_{j_1} \cap P_{j_2}$.

Proof. Based on the proof of Theorem 6 (ii) b, the distance between two vertices on different partitions is 1 or 2. Furthermore, $r(u|P_{j_2}) = (d(u, p_{j_2}), d(u, 2p_{j_2}), d(u, 3p_{j_2}), \dots, d(u, tp_{j_2}))$ and $r(v|P_{j_2}) = (d(v, p_{j_2}), d(v, 2p_{j_2}), d(v, 3p_{j_2}), \dots, d(v, tp_{j_2}))$, where $tp_{j_2} \leq n - 1$. The distance between two vertices in different groups of multiples is 1 if the two vertices are relatively prime and the distance is 2 if they are not relatively prime. Since the properties of the group of multiples p_{j_1} and multiples p_{j_2} are similar, namely that the vertices are not adjacent in each group, while u and v are vertices in the same multiple group, then $r(u|P_{j_2}) = r(v|P_{j_2})$. \square

Based on the results of Lemma 9 and Lemma 10, the vertices in the group of multiple p_i become element of the resolving set leaving only one vertex in each group. This also applies to the vertices in the group p_0 , which leaves only one vertex. Suppose that the vertex that must be left in the group of multiples p_i are a_i where $1 \leq i \leq k$ and the vertex that must be left in the group p_0 are a_0 , the following is given the metric dimension of $G_{\mathbb{Z}_n}$.

Theorem 11. $Dim(G_{\mathbb{Z}_n}) = n - k - 2$, where k represents number of groups of multiples $p_i, i = 1, 2, \dots, k$.

Proof. Suppose $W = Q_0 \cup Q_1 \cup Q_2 \cup \dots \cup Q_k$, where

$Q_0 = P_0 \setminus \{a_0\}$, a_0 is an arbitrary vertex left from the group p_0 ,

$Q_1 = P_1 \setminus \{a_1\}$, a_1 is an arbitrary vertex left from the group of multiple p_1 ,

$Q_2 = P_2 \setminus \{a_2\}$, a_2 is an arbitrary vertex left from the group of multiple p_2 ,

\vdots

$Q_k = P_k \setminus \{a_k\}$, a_k is an arbitrary vertex left from the group of multiple p_k .

The representation of every vertex in $V(G_{\mathbb{Z}_n} \setminus W)$ with respect to W is:

$$r(a_0|W) = (\underbrace{1,1,1, \dots, 1}_{\text{as many as } |P_0|-1}, \underbrace{1,1,1, \dots, 1}_{\text{as many as } |P_1|-1}, \underbrace{1,1,1, \dots, 1}_{\text{as many as } |P_2|-1}, \dots, \underbrace{1,1,1, \dots, 1}_{\text{as many as } |P_k|-1})$$

$$r(a_1|W) = (\underbrace{1,1,1, \dots, 1}_{\text{as many as } |P_0|-1}, \underbrace{2,2,2, \dots, 2}_{\text{as many as } |P_1|-1}, d(a_1, Q_2), \dots, d(a_1, Q_k))$$

$$r(a_2|W) = (\underbrace{1,1,1, \dots, 1}_{\text{as many as } |P_0|-1}, d(a_2, Q_1), \underbrace{2,2,2, \dots, 2}_{\text{as many as } |P_2|-1}, \dots, d(a_2, Q_k))$$

\vdots

$$r(a_k|W) = (\underbrace{1,1,1, \dots, 1}_{\text{as many as } |P_0|-1}, d(a_k, Q_1), d(a_k, Q_2), \dots, \underbrace{2,2,2, \dots, 2}_{\text{as many as } |P_k|-1})$$

$$\text{where } d(a_i, Q_j) = \begin{cases} 1, & \text{if } a_i \notin P_j \\ 2, & \text{if } a_i \in P_j \end{cases}, \quad 1 \leq i, j \leq k.$$

It appears that the representation of every vertex with respect to W is different, so that W is a resolving set. Next, it will be shown that the cardinality of W is minimal. Suppose that any set X is taken whose cardinality is one less than W , that is, $|X| = |W| - 1$. There are three possibilities for elements of set X , namely (i) all vertices in group p_0 are element of X ; (ii) all vertices in the group of multiples p_i are element of X ; and (iii) all vertices of the group of multiple p_j (certain j), $1 \leq j \leq k$ being element of X .

(i) If all vertices in the group p_0 are element of X , then $k + 1$ of the vertices in $G_{\mathbb{Z}_n}$ must be left in the group of multiples p_i with the number of groups being k .

a. Suppose that as many as $k - 1$ groups leave each one vertex, then there are two vertices that must be left by the group of multiple p_j (certain j), $1 \leq j \leq k$. Due to

the similar properties of vertices in the group of multiple p_j , the representation of the two vertices with respect to X will be the same. Thus X is not being a resolving set.

- b. Suppose that all vertices in the $k - 1$ group are element of X , then there is one particular group, name the group of multiple p_j (certain j), $1 \leq j \leq k$ leaving $k + 1$ vertex. For the same reason as (i)a, these $k - 1$ vertices will have the same representation of X because the vertices in the group of multiples p_j are non-mutually adjacent. So X is not a resolving set.
- (ii) If all the vertices in the group of multiples p_i are element of X , then $|X| = |W| - 1 + k = (n - 2 - k) - 1 + k = n - 3 \geq n - 2 - k$ for $k \geq 1$. The equity hold only to $k = 1$. So, for $k \geq 2$ then $|X| > |W|$. This contradicts the cardinality of the set X .
- (iii) If all vertices in a group of multiples p_i , namely p_j dengan $1 \leq j \leq k$ as element of X , then from the vertices in $G_{\mathbb{Z}_n}$ must be left $k + 1$ vertices in $k - 1$ group of multiples p_i (other than p_j) and in the group p_0 .
 - a. Suppose that all vertices in the group p_0 are element of X , then from $k - 1$ group of multiples p_i must be left k vertices. This is similar to case (i).
 - b. Suppose all vertices in the group of multiples p_i (other than p_j) are element of X , it means that all vertices in the group of multiples p_i are element of X . This is similar to the case (ii).

From all of the above possibilities it can be concluded that X is not a resolving set. If all vertices in one group are selected as element of set X , then X is not a resolving set. Likewise, if there is a group p_0 or a group of multiples p_i that leaves more than one vertex, it will result that X not being a resolving set. Since $|X| = |W| - 1$ and W are resolving set, it means that W is the resolving set with minimal cardinality or the metric bases of $G_{\mathbb{Z}_n}$. Since each group leaves one vertex and the number of groups are $k + 1$, the metric dimension of $G_{\mathbb{Z}_n}$ is $(n - 1) - (k + 1) = n - k - 2$. It is proven that $dim(G_{\mathbb{Z}_n}) = n - k - 2$. \square

Based on Theorem 11, the metric bases of $G_{\mathbb{Z}_n}$ consists of a combination of vertices in the group p_0 and the group of multiples p_i with the conditions each leaving only one vertex. The final part of this research is to determine the value of the local metric dimension of $G_{\mathbb{Z}_n}$ by first determining the local resolving set. In this case the vertex representation may be the same as long as the vertices are not adjacent.

Theorem 12. $Dim_l(G_{\mathbb{Z}_n}) = |P_0| + k - 1$, where k representing number of groups of multiples $p_i, i = 1, 2, \dots, k$ and $|P_0|$ is the cardinality of set P_0 .

Proof. Based on the property of each group of multiples $p_i, i = 1, 2, \dots, k$, which the vertices are not mutually adjacent, the property of the group p_0 that all elements are adjacent to every vertex, and according to the definition of the local metric dimension, then we choose the set $W_l = \{1, p_{01}, \dots, p_{0(m-2)}, p_{11}, p_{21}, \dots, p_{k1}\}$, where

$1, p_{01}, \dots, p_{0(m-2)}$ are $m - 1$ vertices in the group p_0 ,

p_{11} is one vertex in the group of multiples p_1 ,

p_{21} is one vertex in the group of multiples p_2 ,

p_{k1} is one vertex in the group of multiples p_k .

The representation of every vertex in $G_{\mathbb{Z}_n}$ with respect to W_l is:

$$r(1|W_l) = (0, 1, \dots, 1, 1, 1, \dots, 1); \quad r(p_{01}|W_l) = (1, 0, \dots, 1, 1, 1, \dots, 1);$$

$$r(p_{0(m-2)}|W_l) = (1, 1, \dots, 0, 1, 1, \dots, 1); \quad r(p_{11}|W_l) = (1, 1, \dots, 1, 0, 1, \dots, 1);$$

$$r(p_{21}|W_l) = (1, 1, \dots, 1, 1, 0, \dots, 1); \quad r(p_{k1}|W_l) = (1, 1, \dots, 1, 1, 1, \dots, 0);$$

$$\begin{aligned} r(p_{02}|W_l) &= (1,1, \dots, 1,1,1, \dots, 1); & r(p_{03}|W_l) &= (1,1, \dots, 1,1,1, \dots, 1); \\ r(p_{12}|W_l) &= (1,1, \dots, 1,2,2, \dots, 1); & r(p_{13}|W_l) &= (1,1, \dots, 1,2,2, \dots, 1); \\ r(p_{14}|W_l) &= (1,1, \dots, 1,2,2, \dots, 1); & r(p_{1s}|W_l) &= (1,1, \dots, 1,2,2, \dots, 1). \end{aligned}$$

It appears that $r(p_{13}|W_l) = r(p_{13}|W_l) = r(p_{14}|W_l) = r(p_{1s}|W_l)$, but $p_{12}, p_{13}, p_{14}, p_{1s}$ are vertices in the group of multiples p_1 which is not mutually adjacent. In the concept of local metric dimensions, the above still meets the criteria, meaning that W_l is a local resolving set. Next, it will be shown that the cardinality of W_l is minimal. Suppose any set X_l whose cardinality is reduced to one from the set W_l , i.e. $|X_l| = |W_l| - 1$. There are three possibilities for element of X_l , namely (i) all vertices in group p_0 become element of X_l ; (ii) at least one vertex from each group of multiples p_i become element of X_l ; and (iii) the group p_0 leaves more than one vertex.

- (i) Suppose that all vertices in the group p_0 are members of X_l , then there are $k - 2$ vertices from the group of multiple p_i as many as k that can be element of X_l . Assuming at least one vertex in each group of multiple p_i becomes a element of X_l , it means that there are at least two groups whose vertices are not represented as element of X_l . It is sufficient to show that there are two vertices from two different groups, these two vertices are adjacent and have the same representation respect to X_l . Suppose that the two groups not represented in the element of the set X_l are the group of multiples p_{j_1} and p_{j_2} where $1 \leq j_1, j_2 \leq k$. Vertex p_{j_1} is adjacent to p_{j_2} because both are prime numbers and are the first element in their respective group of multiples. In addition, vertices p_{j_1} and p_{j_2} are adjacent to all vertices in the group p_0 , so that $r(p_{j_1}|X_l) = r(p_{j_2}|X_l)$. Consequently, X_l is not a resolving set.
- (ii) Suppose that at least one vertex from each group of multiples p_i is a element of X_l , then there are at least two vertices in the group p_0 that are not element of X_l . The two vertices in the group p_0 will have the same representation respect to X_l , because the two vertices in the group p_0 are adjacent to every vertex in $G_{\mathbb{Z}_n}$. So X_l is not a resolving set.
- (iii) Suppose that the group p_0 leaves more than one vertex, then
 - a. If the group p_0 leaves two vertices, then whatever the condition for selecting vertices in the group of multiples p_i , the remaining two vertices in the group p_0 will have the same representation respect to X_l . Consequently, X_l is not a resolving set.
 - b. If the group p_0 leaves more than two vertices, then there is a group of multiples p_i where all the vertices are not X_l element. In this case, the remaining vertices in the group p_0 will have the same representation of X_l regardless of the vertex conditions in the group of multiples p_i . As a result, X_l is not a resolving set.

From all of the above possibilities, it can be concluded that X_l is not a local resolving set. If there are more than one vertex left in group p_0 , then the representation of the remaining vertices respect to X_l will be the same. Likewise, if there is one or more groups of multiples p_i that are not represented in the element of the set X_l , it can always be found the vertices of the group of multiples p_i which have the same representation respect to set X_l eventhough they are adjacent. Since $|X_l| = |W_l| - 1$ and W_l are local resolving set, it can be concluded that W_l is a local resolving set with minimal cardinality or local metric bases of $G_{\mathbb{Z}_n}$. The cardinality of the set W_l is the local metric dimension of $G_{\mathbb{Z}_n}$.

It is proven that $dim_l(G_{\mathbb{Z}_n}) = |P_0| - 1 + k$. \square

Based on Theorem 12, the local metric bases consists of vertices in the group p_0 and the group of multiples p_i , provided that the group p_0 leaves only one vertex while in each group of multiples p_i are represented by only one vertex.

Example: Suppose given $G_{\mathbb{Z}_{17}}$, where $V(G_{\mathbb{Z}_{17}}) = \{1,2,3,\dots, 16\}$. It will determine the metric dimension and the local metric dimension of $G_{\mathbb{Z}_{17}}$ and some metric bases and local metric bases. Firstly, the vertices of $G_{\mathbb{Z}_{17}}$ were grouped as follows:

- Group p_0 : 1,11,13 then $|P_0| = 3$.
- Group of multiple 2 are 2,4,6,8,10,12,14,16.
- Group of multiple 3 are 3,6,9,12,15.
- Group of multiple 5 are 5,10,15.
- Group of multiple 7 are 7,14.

There are four groups of multiples, namely multiples of 2, multiples of 4, multiples of 5, and multiples of 7, so $k = 4$.

(1) $dim(G_{\mathbb{Z}_{17}}) = 17 - 4 - 2 = 11$. Some of the metric bases of $G_{\mathbb{Z}_{17}}$ are:

- $W_1 = \{1,11,2,3,4,6,8,10,12,14,15\}$, then
 $r(5|W_1) = (1,1,1,1,1,1,2,1,1,2)$; $r(7|W_1) = (1,1,1,1,1,1,1,1,2,1)$;
 $r(9|W_1) = (1,1,1,2,1,2,1,1,2,1,2)$; $r(13|W_1) = (1,1,1,1,1,1,1,1,1,1,1)$;
 $r(16|W_1) = (1,1,2,1,2,2,2,2,2,2,1)$.
- $W_2 = \{1,13,2,3,4,6,10,12,14,15,16\}$, then
 $r(5|W_2) = (1,1,1,1,1,1,2,1,2,2,1)$; $r(7|W_2) = (1,1,1,1,1,1,1,1,2,1,1)$;
 $r(8|W_2) = (1,1,2,1,2,2,2,2,2,1,2)$; $r(9|W_2) = (1,1,1,2,1,2,1,2,1,2,1)$;
 $r(11|W_2) = (1,1,1,1,1,1,1,1,1,1,1)$.
- $W_3 = \{11,13,2,4,6,9,10,12,14,15,16\}$, then
 $r(1|W_3) = (1,1,1,1,1,1,1,1,1,1,1)$; $r(3|W_3) = (1,1,1,1,1,2,2,1,2,2,1)$;
 $r(5|W_3) = (1,1,1,1,1,1,2,1,1,2,1)$; $r(7|W_3) = (1,1,1,1,1,1,1,1,2,1,1)$;
 $r(8|W_3) = (1,1,2,1,2,1,2,2,2,1,2)$.

(2) $dim_l(G_{\mathbb{Z}_{17}}) = 3 - 1 + 4 = 6$. Some of the local metric bases of $G_{\mathbb{Z}_{17}}$ are:

- $W_{l_1} = \{1,11,2,3,5,7\}$, then
 $r(4|W_{l_1}) = (1,1,2,1,1,1)$; $r(6|W_{l_1}) = (1,1,2,2,1,1)$; $r(8|W_{l_1}) = (1,1,2,1,1,1)$;
 $r(9|W_{l_1}) = (1,1,1,2,1,1)$; $r(10|W_{l_1}) = (1,1,2,1,2,1)$; $r(12|W_{l_1}) = (1,1,2,2,1,1)$;
 $r(13|W_{l_1}) = (1,1,1,1,1,1)$; $r(14|W_{l_1}) = (1,1,2,1,1,2)$;
 $r(15|W_{l_1}) = (1,1,1,2,2,1)$; $r(16|W_{l_1}) = (1,1,2,1,1,1)$.

It appears that $r(4|W_{l_1}) = r(8|W_{l_1})$ and $r(6|W_{l_1}) = r(12|W_{l_1})$, but vertex 4 is not adjacent to 8 and vertex 6 is not adjacent to 12.

- $W_{l_2} = \{1,13,4,9,5,7\}$, then
 $r(2|W_{l_2}) = (1,1,2,1,1,1)$; $r(3|W_{l_2}) = (1,1,1,1,1,1)$; $r(6|W_{l_2}) = (1,1,2,2,1,1)$;
 $r(8|W_{l_2}) = (1,1,2,1,1,1)$; $r(10|W_{l_2}) = (1,1,2,1,2,1)$; $r(11|W_{l_2}) = (1,1,1,1,1,1)$;
 $r(12|W_{l_2}) = (1,1,2,2,1,1)$; $r(14|W_{l_2}) = (1,1,2,1,1,2)$;
 $r(15|W_{l_2}) = (1,1,1,2,2,1)$; $r(16|W_{l_2}) = (1,1,2,1,1,1)$.

It appears that $r(2|W_{l_2}) = r(8|W_{l_2})$ and $r(6|W_{l_2}) = r(12|W_{l_2})$, but vertex 2 is not adjacent to 8 and vertex 6 is not adjacent to 12.

- $W_{l_3} = \{1,11,8,12,5,14\}$, then
 $r(2|W_{l_3}) = (1,1,2,2,1,2)$; $r(3|W_{l_3}) = (1,1,1,2,2,1)$; $r(4|W_{l_3}) = (1,1,2,2,1,2)$;
 $r(6|W_{l_3}) = (1,1,2,2,1,2)$; $r(7|W_{l_3}) = (1,1,1,1,1,1)$; $r(9|W_{l_3}) = (1,1,1,2,1,1)$;
 $r(10|W_{l_3}) = (1,1,2,2,2,2)$; $r(13|W_{l_3}) = (1,1,1,1,1,1)$;
 $r(15|W_{l_3}) = (1,1,1,2,2,1)$; $r(16|W_{l_3}) = (1,1,2,2,1,2)$.

It appears that $r(2|W_{l_3}) = r(4|W_{l_3}) = r(6|W_{l_3}) = r(16|W_{l_3})$, but the vertices 2,4,6,16 are not adjacent to each other.

CONCLUSIONS

This research focuses on determining the metric dimension and local metric dimension of relative prime graphs G_{Z_n} . Based on the previous discussion, it can be concluded that:

$$(1) \dim(G_{Z_n}) = n - k - 2;$$

$$(2) \dim_l(G_{Z_n}) = |P_0| - 1 + k,$$

where k is the number of group multiples p_1, p_2, \dots, p_k , and $|P_0|$ is the cardinality of set P_0 .

In the future, the research can be extended to other topics such as fractional metric dimensions, local fractional metric dimensions, domination numbers/set, graph coloring, and graph labeling as well as expansion of research objects in special rings.

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