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Inclusion Properties of The Homogeneous Herz-Morrey

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ABSTRACT

In this paper, we have discussed the inclusion properties of the homogeneous Herz-Morrey spaces and the homogeneous weak homogeneous spaces. We also studied the inclusion relation between those spaces.

Keywords: homogeneous Herz-Morrey spaces; homogeneous weak Herz-Morrey spaces; inclusion properties.

INTRODUCTION

The subject discussion about inclusion properties of any spaces or inclusion relation between spaces has interested to study. Some authors have studied about inclusion relation in some spaces (see [1], [2], [3], [4] and [5]). It guided us to discuss the inclusion properties of other spaces.

Regarding C.B. Morrey in [6] who introduced Morrey spaces, many authors have defined the generalization of Morrey spaces and combined with other spaces. Lu and Xu [7] introduce one of the homogeneous Herz-Morrey spaces. These spaces are the generalization of Morrey spaces and Herz spaces. Let $\alpha \in \mathbb{R}, 0 < p \leq \infty, 0 < q < \infty$, and $0 \leq \lambda < \infty$, the homogeneous Herz-Morrey spaces $\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ are defined by

$$\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(R^n) := \left\{f \in L_{loc}^q\,(\mathbb{R}^n/\{0\}) :\ \|f\|_{\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha p}\|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right)^{\frac{1}{p}}$$

with $B_k = B(0,2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $A_k = B_k/B_{k-1}$ for $k \in \mathbb{Z}$ and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$ be the characteristic function of the set A_k.

Lu and Xu also defined the homogeneous weak Herz-Morrey spaces. For $\alpha \in \mathbb{R}$, let $0 < p \leq \infty, \lambda \geq 0$ and $0 < q < \infty$, the homogeneous weak Herz-Morrey spaces $\left(W\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)\right)$ is a set of measurable $f \in L_{loc}^q\,(\mathbb{R}^n/\{0\})$ which completed by norm such that

$$\|f\|_{W\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{\gamma>0}\gamma \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha p} m_k(\gamma,f)^{\frac{p}{q}}\right)^{\frac{1}{p}} < \infty,$$

where $m_k(\gamma,f) = |\{x \in A_k : |f(x)| > \gamma\}|$.

Some authors have studied those spaces in different term of discussion (see [7], [8], [9], [10]). Meanwhile, in this article, the authors would like to discuss the inclusion properties and inclusion relation of the homogeneous Herz-Morrey spaces and the homogeneous weak Herz-Morrey spaces.

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RESULTS AND DISCUSSION

Now, we formulate our main results of this paper as follows:

Theorem 1.1. *Let* $1 \le p_1 \le p_2 < q < \infty$*, then the following inclusion holds:*

$$\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_2,q}(R^n) \subseteq \mathcal{M}\dot{K}^{\alpha,\lambda}_{p_1,q}(R^n).$$

Generally, by Theorem 1.1, we have the following inclusions of the homogeneous Herz-Morrey spaces.

Theorem 1.2. *Let* $1 \le p_1 \le p_2 < q < \infty$*, then the following inclusion holds:*

$$L^q(R^n) = \mathcal{M}\dot{K}^{\alpha,\lambda}_{q,q}(R^n) \subseteq \mathcal{M}\dot{K}^{\alpha,\lambda}_{p_2,q}(R^n) \subseteq \mathcal{M}\dot{K}^{\alpha,\lambda}_{p_1,q}(R^n).$$

Besides, we have the inclusion property of the homogeneous weak Herz-Morrey spaces, also the inclusion relation of the homogeneous Herz-Morrey spaces.

Theorem 1.3. *Let* $1 \le p_1 \le p_2 \le q < \infty$*, the following inclusion holds:*

$$W\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_2,q}(\mathbb{R}^n) \subseteq W\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_1,q}(\mathbb{R}^n).$$

Theorem 1.4. *Let* $1 \le p \le q$*. Then the inclusion* $\mathcal{M}\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n) \subseteq W\mathcal{M}\dot{K}^{\alpha,\lambda}_{p,q}(R^n)$ *is proper.*

The proof of each theorem will be described in the following section.

THE PROOF OF THEOREM 1.1.

For proofing Theorem 1.1., we shall show that $\|f\|_{\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_1,q}(R^n)} \le \|f\|_{\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_2,q}(R^n)}$ by applying Hölder inequality.

Proof of Theorem 1.1. Let we first take for any $f \in \mathcal{M}\dot{K}^{\alpha,\lambda}_{p_1,q}(R^n)$, then by using Hölder inequality and $p_1 \le p_2$ we obtain that

$$\begin{aligned}
\|f\|_{\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_1,q}(R^n)} &= \sup_{L\in Z} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha p_1}\|f\chi_k\|^{p_1}_{L^q(\mathbb{R}^n)}\right)^{\frac{1}{p_1}} \\
&\le \sup_{L\in Z} 2^{-L\lambda}\left(\left(\sum_{k=-\infty}^{L}(2^{k\alpha p_1})^{\frac{p_2}{p_1}}\right)^{\frac{p_1}{p_2}}\left(\sum_{k=-\infty}^{L}\left(\|f\chi_k\|^{p_1}_{L^q(\mathbb{R}^n)}\right)^{\frac{p_2}{p_2-p_1}}\right)^{1-\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} \\
&\le \sup_{L\in Z} 2^{-L\lambda}\left(\left(\sum_{k=-\infty}^{L}2^{k\alpha p_2}\right)^{\frac{p_1}{p_2}}\left(\sum_{k=-\infty}^{L}\|f\chi_k\|^{\frac{p_1p_2}{p_2-p_1}}_{L^q(\mathbb{R}^n)}\right)^{1-\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} \\
&\le \sup_{L\in Z} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha p_2}\left(\sum_{k=-\infty}^{L}\|f\chi_k\|^{\frac{p_1p_2}{p_2-p_1}}_{L^q(\mathbb{R}^n)}\right)^{\frac{p_2-p_1}{p_1}}\right)^{\frac{1}{p_2}} \\
&\le \sup_{L\in Z} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha p_2}\|f\chi_k\|^{p_2}_{L^q(\mathbb{R}^n)}\right)^{\frac{1}{p_2}} \\
&\le \|f\|_{\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_2,q}(R^n)}.
\end{aligned}$$

By this observation, we know that $f \in \mathcal{M}\dot{K}^{\alpha,\lambda}_{p_2,q}(R^n)$. Hence it concludes that $\mathcal{M}\dot{K}^{\alpha,\lambda}_{p_2,q}(R^n) \subseteq \mathcal{M}\dot{K}^{\alpha,\lambda}_{p_1,q}(R^n)$.

THE PROOF OF THEOREM 1.2.

Since it has been stated in Theorem 1.1 that $\mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(R^n) \subseteq \mathcal{M}\dot{K}_{p_1,q}^{\alpha,\lambda}(R^n)$, therefore in proving Theorem 1.2, we need to prove that $L^q(R^n) = \mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n) \subseteq \mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(R^n)$.

Proof of Theorem 1.2. To prove that $L^q(R^n) = \mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n)$, we need to show that $\|f\|_{\boldsymbol{L^q(R^n)}} = \|f\|_{M\dot{K}_{q,q}^{\alpha,\lambda}(R^n)}$. Let take for any $f \in M\dot{K}_{q,q}^{\alpha,\lambda}(R^n)$, by applying Hölder inequality for the norm. We obtain that

$$\|f\|_{M\dot{K}_{q,q}^{\alpha,\lambda}(R^n)} \leq \sup_{L\in Z} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha q} \left(\left(\int_{B(0,2^k)} |f(x)|^q \, dy \right)^{\frac{1}{q}} \left(\int_{B(0,2^k)} |\chi_k|^q dy \right)^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}}$$

$$\leq \sup_{L\in Z} 2^{-L\lambda} \sum_{k=-\infty}^{L} 2^{k\alpha} \left(\int_{B(0,2^k)} |f(x)|^q \, dy \right)^{\frac{1}{q}} (2^{kd})^{\frac{1}{q}}$$

$$\leq \sup_{L\in Z} 2^{-L\lambda} \sum_{k=-\infty}^{L} 2^{k\alpha + \frac{kd}{q}} \left(\int_{B(0,2^k)} |f(x)|^q \, dy \right)^{\frac{1}{q}}$$

$$\leq C \left(\int_{B(0,2^k)} |f(x)|^q \, dy \right)^{\frac{1}{q}}$$

$$\leq \|\boldsymbol{f}\|_{\boldsymbol{L^q(R^n)}},$$

it means that $f \in L^q(R^n)$. Then $L^q(R^n) \subseteq \mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n)$. Meanwhile, for any $f \in L^q(R^n)$, we can find any constant C such that $C = \sup_{L\in Z} 2^{-L\lambda} \sum_{k=-\infty}^{L} 2^{k\alpha + \frac{kd}{q}}$, then it shows that $f \in \mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n)$ which means $\mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n) \subseteq L^q(R^n)$. Hence, it concludes that $L^q(R^n) = \mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n)$.

Next, we have to prove that $\mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n) \subseteq \mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(R^n)$ by showing $\|f\|_{\mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(R^n)} \leq \|f\|_{\mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n)}$. By using a similar way for proving Theorem 1.1., and since $q > p_2$, it is clear that $\|f\|_{\mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(R^n)} \leq \|f\|_{\mathcal{M}\dot{K}_{q,q}^{\alpha,\lambda}(R^n)}$. Therefore, the proof is complete.

THE PROOF OF THEOREM 1.3.

One way for proving Theorem 1.3. is showed that $\|f\|_{W\mathcal{M}\dot{K}_{p_1,q}^{\alpha,\lambda}(R^n)} \leq \|f\|_{W\mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(R^n)}$.

Proof of Theorem 1.3. Let take for any $f \in \|f\|_{W\mathcal{M}\dot{K}_{p_1,q}^{\alpha,\lambda}(R^n)}$, then by observing the norm of f we obtain that

$$\|f\|_{W\mathcal{M}\dot{K}_{p_1,q}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{\gamma>0} \gamma \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha p_1} m_k(\gamma, f)^{\frac{p_1}{q}} \right)^{\frac{1}{p_1}}$$

$$\leq \sup_{\gamma>0} \gamma \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha p_2} m_k(\gamma, f)^{\frac{p_2}{q}} \right)^{\frac{1}{p_2}}$$

$$\leq \|f\|_{W\mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(R^n)}.$$

By the observation, it concludes that $W\mathcal{M}\dot{K}_{p_2,q}^{\alpha,\lambda}(\mathbb{R}^n) \subseteq W\mathcal{M}\dot{K}_{p_1,q}^{\alpha,\lambda}(\mathbb{R}^n)$.

THE PROOF OF THEOREM 1.4.

Proving Theorem 1.4 is used a similar idea as previous theorems which shall show that $\|f\|_{W\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}$.

Proof of Theorem 1.4. Let $f \in \mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$, $a \in \mathbb{R}^n$, and $\gamma > 0$. We observe that

$$|\{x \in A_k : |f(x)| > \gamma\}|^{\frac{p}{q}} \leq \left(\int_{B(0,2^k)} |f(x)\chi_k|^q \, dx\right)^{\frac{p}{q}} = \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p$$

Multiplying both side by $\sum_{k=-\infty}^{L} 2^{k\alpha p}$, then we obtain that

$$\sum_{k=-\infty}^{L} 2^{k\alpha p} |\{x \in A_k : |f(x)| > \gamma\}|^{\frac{p}{q}} \leq \sum_{k=-\infty}^{L} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p.$$

It says merely that $\|f\|_{W\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}$, therefore $f \in W\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$. Hence, it is proved that $\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) \subseteq W\mathcal{M}\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$.

CONCLUSIONS

By this result, the author can conclude that the homogeneous Herz-Morrey spaces have inclusion properties as stated above. This result will be useful to be used in proving fractional integral on the homogeneous Herz-Morrey spaces.

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