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# Trace of Positive Integer Power of Squared Special Matrix

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**ABSTRACT**

The rectangle matrix to be discussed in this research is a special matrix where each entry in each line has the same value which is notated by $A_n$. The main aim of this paper is to find the general form of the matrix trace of $A_n$ powered positive integer $m$ or notated by $Tr(A_n)^m$. To prove whether the general form of the matrix trace of $A_n$ powered positive integer can be confirmed, mathematics induction and direct proof are used. The main results present the general formula of $(A_n)^m$ and $Tr(A_n)^m$ with observing the pattern of power matrix for $2 \leq m \leq 11, n \geq 2$, and $m \in \mathbb{Z}^+$.

**Keywords**: direct proof; mathematics induction; matrix *trace*; squared matrix

## INTRODUCTION

The calculation of trace of power of square matrix has become attention. According to Brezinski [1], trace of power of matrix is often used in some fields of mathematics, especially Network Analysis, Number Theory, Dynamic Systems, Matrix Theory, and Differential Equations. The discussion about trace matrix has been widely studied by several researchers before. Datta et.al [2], has obtained algorithm of trace of power of squared matrix $Tr(A^k)$, with $k$ is an integer and $A$ is Hassenberg matrix with a codiagonal unit. There is also discussion of trace in several applications in matrix theory and numerical linear algebra. For example in determining the eigenvalue of a symmetric matrix, the basic procedure in estimating a trace $(A^n)$ and trace $(A^{-n})$ with $n$ integers, this is explained in Pan [3]. Chu. Mt [4] discussed symbolic calculations on the power of squared tridiagonal of matrix trace. For example, $A$ a symmetric positive definite matrix, and for example $\{\lambda_k\}$ notated its eigen value. For $q \in \mathbb{R}$, $A^q$ also symmetric definite matrices, and are listed in Hignam [5] with formula

$$Tr(A^q) = \sum_k \lambda_k^q .$$

According to Zarelua [6] in quantum and combinatorial theory, the trace matrix is a whole number in relation to the Euler equations

$$Tr\left(A^{p^r}\right) = Tr\left(A^{p^{r-1}}\right) mod(p^r)$$

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For all matrix $A$ integers, $p$ is the prime number and $r$ original number. Then this article also discuss about invariant in dynamic system which is illustrate as form of trace of integer squared matrix, for example the number Lefschetz. Next, Pahade and Jha [7], discuss about the formation of general form of trace matrix ordo $2 \times 2$ square with powered positive integer. In that article there are two general forms of order trace $2 \times 2$ with integer square $n$. First, the general form of order trace matrix trace 2 x 2 with even number square $n$, is

$$Tr\ (A^n) = \sum_{r=0}^{n/2} \frac{(-1)^r}{r\,!}\ n\ [n - (r+1)]\ [n - (r+2)] \cdots \big[n - \big(r + (r - 1)\big)\big]\ \big(\text{Det}\ (A)\big)^r \big(Tr\ (A)\big)^{n-2r}.$$

Second, the main form of trace matrix $2 \times 2$ with odd number square $n$, is

$$Tr\ (A^n) = \sum_{r=0}^{(n-1)/2} \frac{(-1)^r}{r\,!}\ n\ [n - (r+1)]\ [n - (r+2)] \cdots \big[n - \big(r + (r - 1)\big)\big]\ \big(\text{Det}\ (A)\big)^r \big(Tr\ (A)\big)^{n-2r}.$$

In the network analysis field, especially on triangle counting in a graph, based on Avron [8], when analyzing a complex network, the important problem is calculating the total numbers of triangle on the simple connected graph. This number is equal to $Tr(A^3)/6$, where $A$ is adjacency matrix from the graph. Then, in 2017, Pahade and Jha [9] discuss about trace of squared adjacency matrix on positive integers. In the paper, there is $A$ symmetrical adjacency matrix on a complete simple graph with vertex $n$, for even number $k$ is formulated

$$Tr(A^k) = \sum_{r=1}^{\frac{n}{2}} s(k,r)n(n-1)^r(n-2)^{k-2r}$$

and for odd number $k$ is formulated

$$Tr(A^k) = \sum_{r=1}^{\frac{n-1}{2}} s(k,r)n(n-1)^r(n-2)^{k-2r}$$

with $s(k,r)$ is a number thats depend on $k$ and $r$, and defined as

$$s(k,1) = 1, s\left(k,\frac{k}{2}\right) = 1, s\left(k,\frac{k-1}{2}\right) = \frac{k-1}{2}, \text{ and } s(k,r) = s(k-1,r) + s(k-2,r-1).$$

Next, by this research, it will be decided the trace of rectangle matrix with the real number entries which for every entry row has an equal value. In this research, there are some related definitions and theorems.

**Definition 1.1 (Anton** [10]**)** If $A$ is a rectangle matrix, then the definition of squared of powered non negative integers of $A$ is

$$A^0 = I\ ,\ A^n = \underbrace{AA \dots A}_{n\ faktor}\ (n > 0).$$

Next, If $A$ is invertible, then the definition of squared of powered negative integers of $A$ is

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1} \dots A^{-1}}_{n\ faktor}.$$

**Theorem 1.1 (Andrilli,** [11]**)** If $A$ is a rectangle matrix, and if $r$ and $s$ are nonnegative integers, then

1. $A^r\ A^s = A^{r+s}$
2. $(A^r)^s = A^{rs} = (A^s)^t$

**Definition 1.2** [10] If $A$ is a rectangle matrix, then the *trace* of $A$ which is stated as $Tr(A)$, is defined as the total entries on main diagonal of $A$. *Trace* from $A$ cannot be defined when $A$ is not a rectangle matrix

$$Tr\ (A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}. \tag{1.1}$$

**Theorem 1.2** [12] If $A$ and $B$ are rectangle matrix in the same order and $c$ is $r$ scale, then apply:

a. $Tr(A) = Tr(A^T)$,
b. $Tr(cA) = c\ Tr(A)$, (1.2)
c. $Tr(A + B) = Tr(A) + Tr(B)$,
d. $Tr(AB) = Tr(BA)$.

**METHODS**

The method used in order to reach the aim of this paper is using literature study or conceptual foundation by following steps.

- Finding the general formula of power matrix $(A_n)^m$ with $m \in \mathbb{Z}^+$ and proof it using mathematical induction,
- Determining trace matrix $(A_n)^m$, notated by $Tr\ (A_n)^m$, finding the general formula and using mathematical induction, we proof the formula obtained.

**RESULTS AND DISCUSSION**

This research is going to discuss about positive integers squared trace of $m$ from special matrix of $n\ x\ n$ order with the entries of real numbers where each entry has the same value in a row, which is noted with matrix $A_n^m$. The research started by deciding the general form of matrix square of $A_n^m$ by calculating matrix square in order of 2 x 2 to order of 5 x 5 squared by $m$ positive integers. After the general matrix of $A_n^m$ is formed, then this research is continued by looking for $Tr(A_n^m)$.

**Special Matrix Order of $\boldsymbol{n \times n}$ $(\boldsymbol{n \geq 2})$ Squared by $\boldsymbol{m}$ Positive Integers**

This part is going to explain about squaring of special matrix order of $n \times n$ , $n \geq 2$ with the real number entries where each entry has the same value in a row, this matrix is formulated as follows

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_i & a_i & \cdots & a_i \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & \cdots & a_n \end{bmatrix},\ a_i \in \mathbb{R}\ ;\ \ i = 1,\ 2,\ \ldots\ ,\ n. \tag{2.1}$$

It is special matrix in order of $2 \times 2$ to $5 \times 5$ which is formulated as follows.

$$A_2 = \begin{bmatrix} a_1 & a_1 \\ a_2 & a_2 \end{bmatrix},\quad A_3 = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{bmatrix},\quad A_4 = \begin{bmatrix} a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 & a_4 \\ a_5 & a_5 & a_5 & a_5 & a_5 \end{bmatrix}$$

For $n = 2$, it is decided the matrix squaring values of $(A_2)^2$ to $(A_2)^{11}$ which are presented in Table 1 below.

**Table 1**. Special Matrix Squaring Values of $(A_2)^2$ to $(A_2)^{11}$

| No | Special Matrix Squaring of $\boldsymbol{A_2}$ | Matrix Squaring Values $\boldsymbol{A_2}$ |
|---|---|---|
| 1. | $(A_2)^2$ | $(a_1 + a_2)\ A_2$ |
| 2. | $(A_2)^3$ | $(a_1 + a_2)^2\ A_2$ |
| 3. | $(A_2)^4$ | $(a_1 + a_2)^3\ A_2$ |
| 4. | $(A_2)^5$ | $(a_1 + a_2)^4\ A_2$ |
| 5. | $(A_2)^6$ | $(a_1 + a_2)^5\ A_2$ |
| 6. | $(A_2)^7$ | $(a_1 + a_2)^6\ A_2$ |
| 7. | $(A_2)^8$ | $(a_1 + a_2)^7\ A_2$ |
| 8. | $(A_2)^9$ | $(a_1 + a_2)^8\ A_2$ |
| 9. | $(A_2)^{10}$ | $(a_1 + a_2)^9\ A_2$ |
| 10. | $(A_2)^{11}$ | $(a_1 + a_2)^{10}\ A_2$ |

After getting the values of special matrix squaring of $A_2$ which are in Table 1, then it can be predicted that the general form of the special matrix squaring based on its recursive pattern is $(A_2)^m = (a_1 + a_2)^{m-1} A_2$. According to the prediction, then the general form of matrix squaring of $A_2$ is presented in Theorem 2.1 below.

**Theorem 2.1** If given the special matrix of $A_2 = \begin{bmatrix} a_1 & a_1 \\ a_2 & a_2 \end{bmatrix}$; $a_1, a_2 \in \mathbb{R}$, then

$$(A_2)^m = (a_1 + a_2)^{m-1} A_2 \text{ with } m \in \mathbb{Z}^+. \tag{2.2}$$

**Proof:** Using mathematic induction.

For example $p(m) : (A_2)^m = (a_1 + a_2)^{m-1} A_2$

1. For $m = 1$ then

$$\begin{aligned} p\ (1) : (A_2)^1 &= (a_1 + a_2)^{1-1} A_2 \\ &= (a_1 + a_2)^0 A_2 \\ &= A_2 \end{aligned}$$

2. For $m = k$ then it is assumed that $p\ (k)$ is correct, which is

$$p\ (k) : (A_2)^k = (a_1 + a_2)^{k-1} A_2$$

Will be presented for $m = k + 1$ then $p\ (k + 1)$ is also correct, which is

$$p\ (k+1) : (A_2)^{k+1} = (a_1 + a_2)^k A_2. \tag{2.3}$$

Then,

$$\begin{aligned} (A_2)^{k+1} &= (A_2)^k (A_2) \\ &= (a_1 + a_2)^{k-1} A_2 A_2 \\ &= (a_1 + a_2)^{k-1} (A_2)^2 \\ &= (a_1 + a_2)^{k-1} (a_1 + a_2) A_2 \\ &= (a_1 + a_2)^{(k-1)+1} A_2 \\ &= (a_1 + a_2)^k A_2 \end{aligned}$$

By giving attention to the Equation (2.3) then $p\ (k + 1)$ is correct. Due to Step (1) and (2) are presented correctly, then Theorem 2.1 is proven. ■

For $n = 3$, it is decided the value of matrix squaring of $(A_3)^2$ to $(A_3)^{11}$ which are presented in Table 2 below.

**Table 2**. The Value of Special Matrix Squaring of $(A_3)^2$ to $(A_3)^{11}$

| No | Special Matrix Squaring of $A_3$ | The Value Matrix Squaring of $A_3$ |
|---|---|---|
| 1. | $(A_3)^2$ | $(a_1 + a_2 + a_3)A_3$ |
| 2. | $(A_3)^3$ | $(a_1 + a_2 + a_3)^2 A_3$ |
| 3. | $(A_3)^4$ | $(a_1 + a_2 + a_3)^3 A_3$ |
| 4. | $(A_3)^5$ | $(a_1 + a_2 + a_3)^4 A_3$ |
| 5. | $(A_3)^6$ | $(a_1 + a_2 + a_3)^5 A_3$ |
| 6. | $(A_3)^7$ | $(a_1 + a_2 + a_3)^6 A_3$ |
| 7. | $(A_3)^8$ | $(a_1 + a_2 + a_3)^7 A_3$ |
| 8. | $(A_3)^9$ | $(a_1 + a_2 + a_3)^8 A_3$ |
| 9. | $(A_3)^{10}$ | $(a_1 + a_2 + a_3)^9 A_3$ |
| 10. | $(A_3)^{11}$ | $(a_1 + a_2 + a_3)^{10} A_3$ |

After getting the values of the special matrix squaring of $A_3$ which is in Table 2, then it can be predicted the general form of the special matrix squaring is based on its recursive pattern which is $(A_3)^m = (a_1 + a_2 + a_3)^{m-1} A_3$. According to the prediction, then the general form of matrix squaring of $A_3$ is presented in Theorem 2.2 below.

**Theorem 2.2** If given the special matrix of $A_2 = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{bmatrix}$; $a_1, a_2, a_3 \in \mathbb{R}$, then

$$(A_3)^m = (a_1 + a_2 + a_3)^{m-1} A_3 \text{ with } m \in \mathbb{Z}^+. \tag{2.4}$$

**Proof:** Applying the same steps with Theorem 2.1, then this theorem is proved. ■

For $n = 4$, it is decided the value of matrix squaring of $(A_4)^2$ to $(A_4)^{11}$ which are presented in Table 3 below.

**Table 3**. The Value of Special Matrix Squaring of $(A_4)^2$ to $(A_4)^{11}$

| No | Special Matrix Squaring of $A_4$ | Matrix Squaring Value of$A_4$ |
|---|---|---|
| 1. | $(A_4)^2$ | $(a_1 + a_2 + a_3 + a_4) A_4$ |
| 2. | $(A_4)^3$ | $(a_1 + a_2 + a_3 + a_4)^2 A_4$ |
| 3. | $(A_4)^4$ | $(a_1 + a_2 + a_3 + a_4)^3 A_4$ |
| 4. | $(A_4)^5$ | $(a_1 + a_2 + a_3 + a_4)^4 A_4$ |
| 5. | $(A_4)^6$ | $(a_1 + a_2 + a_3 + a_4)^5 A_4$ |
| 6. | $(A_4)^7$ | $(a_1 + a_2 + a_3 + a_4)^6 A_4$ |
| 7. | $(A_4)^8$ | $(a_1 + a_2 + a_3 + a_4)^7 A_4$ |
| 8. | $(A_4)^9$ | $(a_1 + a_2 + a_3 + a_4)^8 A_4$ |
| 9. | $(A_4)^{10}$ | $(a_1 + a_2 + a_3 + a_4)^9 A_4$ |
| 10. | $(A_4)^{11}$ | $(a_1 + a_2 + a_3 + a_4)^{10} A_4$ |

After getting the values of special matrix squaring of $A_4$ which is in Table 3, then it can be predicted that the general form of the special matrix squaring is based on its recursive pattern which is $(A_4)^m = (a_1 + a_2 + a_3 + a_4)^{m-1} A_4$. According to the prediction, then the general form of matrix squaring of $A_4$ is presented in Theorem 2.3 below.

**Theorem 2.3** If given the special matrix of

$$A_4 = \begin{bmatrix} a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 \end{bmatrix}; \; a_1, a_2, a_3, a_4 \in \mathbb{R},$$

then

$$(A_4)^m = (a_1 + a_2 + a_3 + a_4)^{m-1} A_4 \text{ with } m \in \mathbb{Z}^+. \tag{2.5}$$

**Proof:** Adopting the proof in Theorem 2.1, then this theorem is proven as well. ■

For $n = 5$, it is decided the value of matrix squaring of $(A_5)^2$ to $(A_5)^{11}$ which is presented in the Table 4 below.

**Table 4**. The Value of Special Matrix Squaring of $(A_5)^2$to $(A_5)^{11}$

| No | Special Matrix Squaring of $A_5$ | Matrix Squaring Value of $A_5$ |
|---|---|---|
| 1. | $(A_5)^2$ | $(a_1 + a_2 + a_3 + a_4 + a_5)\ A_5$ |
| 2. | $(A_5)^3$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^2 A_5$ |
| 3. | $(A_5)^4$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^3 A_5$ |
| 4. | $(A_5)^5$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^4 A_5$ |
| 5. | $(A_5)^6$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^5 A_5$ |
| 6. | $(A_5)^7$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^6 A_5$ |
| 7. | $(A_5)^8$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^7 A_5$ |
| 8. | $(A_5)^9$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^8 A_5$ |
| 9. | $(A_5)^{10}$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^9 A_5$ |
| 10. | $(A_5)^{11}$ | $(a_1 + a_2 + a_3 + a_4 + a_5)^{10} A_5$ |

After getting the values of matrix squaring of $A_5$ which is in Table 3, then in can be predicted that the general form of the special matrix squaring is based on its recursive pattern which is $(A_5)^m = (a_1 + a_2 + a_3 + a_4 + a_5)^{m-1} A_5$. According to the prediction, then the general form of matrix squaring of $A_5$ is presented in Theorem 2.4 below.

**Theorem 2.4** If given the special matrix of $A_5 = \begin{bmatrix} a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 & a_4 \\ a_5 & a_5 & a_5 & a_5 & a_5 \end{bmatrix};$
$a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$ then

$$(A_5)^m = (a_1 + a_2 + a_3 + a_4 + a_5)^{m-1} A_5 \text{ with } m \in \mathbb{Z}^+. \tag{2.6}$$

**Proof:** It is clear from above theorems. ■

By giving attention to the recursive pattern of Equation (2.2), Equation (2.4), Equation (2.5) and Equation (2.6) which are

$$(A_2)^m = (a_1 + a_2)^{m-1} A_2$$
$$(A_3)^m = (a_1 + a_2 + a_3)^{m-1} A_3$$
$$(A_4)^m = (a_1 + a_2 + a_3 + a_4)^{m-1} A_4$$
$$(A_5)^m = (a_1 + a_2 + a_3 + a_4 + a_5)^{m-1} A_5.$$

It can be predicted that the general form of the special matrix squaring in order of $n \times n$, $n \geq 2$ is equal to the Equation (2.1), which is

$$\begin{aligned}(A_n)^m &= (a_1 + a_2 + \dots + a_n)^{m-1} A_n \\ &= \left(\sum_{i=1}^{n} a_i\right)^{m-1} A_n\end{aligned}$$

According to the prediction, then the general form of special matrix squaring in order of $n \times n$ , $n \geq 2$ is equal to Equation (2.1) is presented in the Theorem 2.5 below.

**Theorem 2.5** If given the special matrix in order $n \times n$ , $n \geq 2$ which is

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_i & a_i & \cdots & a_i \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & \cdots & a_n \end{bmatrix} ; \; a_i \in \mathbb{R} , \; i = 1, 2, \dots , n$$

then

$$(A_n)^m = (\textstyle\sum_{i=1}^{n} a_i)^{m-1} A_n \text{ , } dengan\ m \in \mathbb{Z}^+.$$

**Proof:** Again, by using mathematic induction, for example $p\ (m) : (A_n)^m = (\sum_{i=1}^{n} a_i)^{m-1} A_n$ , with $m \in \mathbb{Z}^+$

1. For $m = 1$ then

$$\begin{aligned} p\ (1) : (A_n)^1 &= \left(\sum_{i=1}^{n} a_i\right)^{1-1} A_n \\ &= \left(\sum_{i=1}^{n} a_i\right)^{0} A_n \\ &= A_n \end{aligned}$$

2. For $m = k$ is assumed that $p\ (k)$ is correct, which is
   $p\ (k) : (A_n)^k = (\sum_{i=1}^{n} a_i)^{k-1} A_n$, with $m \in \mathbb{Z}^+$.

   Will be presented for $m = k + 1$ then $p\ (k + 1)$ is also correct, which is

$$\begin{aligned} p\ (k + 1) : (A_n)^{k+1} &= \left(\sum_{i=1}^{n} a_i\right)^{(k+1)-1} A_n \\ &= (\textstyle\sum_{i=1}^{n} a_i)^{k} A_n \qquad (2.7) \end{aligned}$$

   The proof is below

$$(A_n)^{k+1} = (A_n)^k \ (A_n)$$

$$= \left(\sum_{i=1}^{n} a_i\right)^{k-1} A_n A_n$$

$$= \left(\sum_{i=1}^{n} a_i\right)^{k-1} \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_i & a_i & \cdots & a_i \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & \cdots & a_n \end{bmatrix} \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_i & a_i & \cdots & a_i \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & \cdots & a_n \end{bmatrix}$$

$$= (\textstyle\sum_{i=1}^{n} a_i)^{k-1} \begin{bmatrix} a_1^2 + a_1a_2 + \cdots + a_1a_i + \cdots + a_1a_n & a_1^2 + a_1a_2 + \cdots + a_1a_i + \cdots + a_1a_n & \cdots & a_1^2 + a_1a_2 + \cdots + a_1a_i + \cdots + a_1a_n \\ a_1a_2 + a_2^2 + \cdots + a_2a_i + \cdots + a_2a_n & a_1a_2 + a_2^2 + \cdots + a_2a_i + \cdots + a_2a_n & \cdots & a_1a_2 + a_2^2 + \cdots + a_2a_i + \cdots + a_2a_n \\ \vdots & \vdots & & \vdots \\ a_1a_i + a_2a_i + \cdots + a_i^2 + \cdots + a_ia_n & a_1a_i + a_2a_i + \cdots + a_i^2 + \cdots + a_ia_n & \cdots & a_1a_i + a_2a_i + \cdots + a_i^2 + \cdots + a_ia_n \\ \vdots & \vdots & \vdots & \vdots \\ a_1a_n + a_2a_n + \cdots + a_ia_n + \cdots + a_n^2 & a_1a_n + a_2a_n + \cdots + a_ia_n + \cdots + a_n^2 & \cdots & a_1a_n + a_2a_n + \cdots + a_ia_n + \cdots + a_n^2 \end{bmatrix}$$

$$= \left(\sum_{i=1}^{n} a_i\right)^{k-1} \begin{bmatrix} (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_1 & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_1 & \cdots & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_1 \\ (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_2 & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_2 & \cdots & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_2 \\ \vdots & \vdots & \vdots & \vdots \\ (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_i & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_i & \cdots & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_i \\ \vdots & \vdots & \vdots & \vdots \\ (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_n & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_n & \cdots & (a_1 + a_2 + \cdots + a_i + \cdots + a_n)\, a_n \end{bmatrix}$$

$$= (\textstyle\sum_{i=1}^{n} a_i)^{k-1} (a_1 + a_2 + \cdots + a_i + \cdots + a_n) \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_i & a_i & \cdots & a_i \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & \cdots & a_n \end{bmatrix}$$

$$= (\textstyle\sum_{i=1}^{n} a_i)^{k-1} (\sum_{i=1}^{n} a_i)\, A_n$$
$$= (\textstyle\sum_{i=1}^{n} a_i)^{(k-1)+1} A_n$$
$$= (\textstyle\sum_{i=1}^{n} a_i)^{k} A_n$$

By giving attention to the Equation (2.7) then $p\ (k + 1)$ is correct. Due to step (1) and (2) are presented correctly, then the Theorem 2.5 is proven. ■

## *Trace* of Special Matrix in Order $n \times n\ (n \geq 2)$ Squared by Positive Integers

In this part is going to be given the trace of special matrix of $A_2^m, A_3^m, A_4^m$, and $A_5^m$ which are contained in Theorem 3.1 to Theorem 3.4 as follows.

**Theorem 3.1** If it is given the special matrix of $A_2 = \begin{bmatrix} a_1 & a_1 \\ a_2 & a_2 \end{bmatrix}$; $a_1, a_2 \in \mathbb{R}$ then

$$Tr\ (A_2)^m = (a_1 + a_2)^m, \text{ with } m \in \mathbb{Z}^+. \tag{3.1}$$

**Proof.** The Proof of Theorem uses direct proof. Because of the known matrix of $A_2$ then $Tr(A_2) = a_1 + a_2$. According to Theorem 2.1, is got Equation (2.2) which is $(A_2)^m = (a_1 + a_2)^{m-1} A_2$. By using the Definition 1.2 and Theorem 1.2 (b), it is formulated

$$\begin{aligned} Tr\ (A_2)^m &= Tr((a_1 + a_2)^{m-1} A_2) \\ &= (a_1 + a_2)^{m-1} Tr(A_2) \\ &= (a_1 + a_2)^{m-1} (a_1 + a_2) \\ &= (a_1 + a_2)^{(m-1)+1} \end{aligned}$$

$$= (a_1 + a_2)^m.$$

According to the proof, then Theorem 3.1 is proven. ■

**Theorem 3.2** If it is given special matrix of $A_3 = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{bmatrix}$; $a_1, a_2, a_3 \in \mathbb{R}$ then

$$Tr\ (A_3)^m = (a_1 + a_2 + a_3)^m, \text{ with } m \in \mathbb{Z}^+. \quad (3.2)$$

**Proof.** It is clear from above theorem. ■

**Theorem 3.3** If it is given the special matrix of

$$A_4 = \begin{bmatrix} a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 \end{bmatrix};\ a_1,\ a_2,\ a_3,\ a_4\ \in\ \mathbb{R},$$

then

$$Tr\ (A_4)^m = (a_1 + a_2 + a_3 + a_4)^m, \text{ with } m \in \mathbb{Z}^+. \quad (3.3)$$

**Proof.** The proof is clear. ■

**Theorem 3.4** If given the special matrix of $A_5 = \begin{bmatrix} a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 & a_4 \\ a_5 & a_5 & a_5 & a_5 & a_5 \end{bmatrix}$; $a_1,\ a_2,\ a_3, a_4$ and $a_5 \in \mathbb{R}$ then

$$Tr\ (A_5)^m = (a_1 + a_2 + a_3 + a_4 + a_5)^m, \text{ with } m \in \mathbb{Z}^+. \quad (3.4)$$

**Proof.** Clearly proven by following Theorem 3.1. ■

By giving attention to the recursive pattern on Equation (3.1), Equation (3.2), Equation (3.3) and Equation (3.4) which are

$$\begin{aligned} Tr\ (A_2)^m &= (a_1 + a_2)^m \\ Tr\ (A_3)^m &= (a_1 + a_2 + a_3)^m \\ Tr\ (A_4)^m &= (a_1 + a_2 + a_3 + a_4)^m \\ Tr\ (A_5)^m &= (a_1 + a_2 + a_3 + a_4 + a_5)^m. \end{aligned}$$

It can be predicted that the general form of the *trace* of special matrix in order $n\ x\ n, n \geq 2$ is equal to Equation (2.1) squared by positive integer (*nonnegative integer*) which is

$$Tr\ (A_n)^m = (a_1 + a_2 + \dots + a_n)^m = \left(\sum_{i=1}^{n} a_i\right)^m.$$

According to the prediction, then the general form of *trace* of special matrix in order $n\ x\ n\ , n \geq 2$ is presented in Theorem 3.5 below.

**Theorem 3.5** If given special matrix in order $n\ x\ n$ , $n \geq 2$ which is

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_i & a_i & \cdots & a_i \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & \cdots & a_n \end{bmatrix}, \quad a_i \in \mathbb{R}\ ; \quad i = 1,\ 2,\ \dots\ ,\ n.$$

then

$$Tr\ (A_n)^m = \left(\sum_{i=1}^{n} a_i\right)^m, \text{ with } m \in \mathbb{Z}^+.$$

**Proof:** This theorem will be proven by direct proof. Because matrix $A_n$ is known, then $Tr\ (A_n) = \sum_{i=1}^{n} a_i$. From Theorem 2.5, obtained $(A_n)^m = (\sum_{i=1}^{n} a_i)^{m-1} A_n$. So that by using Definition 1.2 and Theorem 1.2 (b) obtained

$$\begin{aligned} Tr(A_n)^m &= Tr\left(\left(\sum_{i=1}^{n} a_i\right)^{m-1} A_n\right) \\ &= \left(\sum_{i=1}^{n} a_i\right)^{m-1} Tr\ (A_n) \\ &= \left(\sum_{i=1}^{n} a_i\right)^{m-1} \left(\sum_{i=1}^{n} a_i\right) \\ &= \left(\sum_{i=1}^{n} a_i\right)^{m}. \end{aligned}$$

Based on the evidence, then Theorem 3.5 is Proven. ■

**The Application of Matrix $A_n^m$ and $Tr(A_n^m)$ in Examples**

The following is given the example of question related to Theorem 2.5 and Theorem 3.5 as follows.

**Example 1.** Consider Matrix $A_4$ as follows

$$A_4 = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 12 & 12 & 12 & 12 \\ 25 & 25 & 25 & 25 \\ 10 & 10 & 10 & 10 \end{bmatrix}$$

Determine $(A_4)^{80}$ and $Tr(A_4)^{80}$.

**Solution:**

By giving attention to matrix $A_4$, value of $a_1 = 3, a_2 = 12, a_3 = 25$, and $a_4 = 10$. Based on Theorem 2.5 obtained

$$\begin{aligned} (A_4)^{80} &= \left(\sum_{i=1}^{4} a_i\right)^{80-1} A_4 \\ &= (a_1 + a_2 + a_3 + a_4)^{79} A_4 \\ &= (3 + 12 + 25 + 10)^{79} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 12 & 12 & 12 & 12 \\ 25 & 25 & 25 & 25 \\ 10 & 10 & 10 & 10 \end{bmatrix} \end{aligned}$$

$$= (50)^{79}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 12 & 12 & 12 & 12 \\ 25 & 25 & 25 & 25 \\ 10 & 10 & 10 & 10 \end{bmatrix}$$

Based on Theorem 3.5 obtained

$$\begin{aligned} Tr(A_4)^{80} &= \left(\sum_{i=1}^{4} a_i\right)^{80} \\ &= (a_1 + a_2 + a_3 + a_4)^{80} \\ &= (3 + 12 + 25 + 10)^{80} \\ &= (50)^{80} \end{aligned}$$

**Example 2.** Given matrix $A_5$ as follows

$$A_5 = \begin{bmatrix} 8 & 8 & 8 & 8 & 8 \\ 5/16 & 5/16 & 5/16 & 5/16 & 5/16 \\ -12 & -12 & -12 & -12 & -12 \\ 2/3 & 2/3 & 2/3 & 2/3 & 2/3 \\ -5/12 & -5/12 & -5/12 & -5/12 & -5/12 \end{bmatrix}$$

Determine $(A_5)^{27}$ and $Tr(A_5)^{27}$.

**Solution :**

By giving attention to matrix $A_5$, value of $a_1 = 8, a_2 = 5/16, a_3 = -12, a_4 = 2/3$ and $a_5 = -5/12$. Based on Theorem 2.5 obtained

$$(A_5)^{27} = \left(\sum_{i=1}^{5} a_i\right)^{27-1} A_5$$

$$= (a_1 + a_2 + a_3 + a_4 + a_5)^{26} A_5$$

$$= (8 + (5/16) + (-12) + (2/3) + (-5/12))^{26} \begin{bmatrix} 8 & 8 & 8 & 8 & 8 \\ 5/16 & 5/16 & 5/16 & 5/16 & 5/16 \\ -12 & -12 & -12 & -12 & -12 \\ 2/3 & 2/3 & 2/3 & 2/3 & 2/3 \\ -5/12 & -5/12 & -5/12 & -5/12 & -5/12 \end{bmatrix}$$

$$= (-3\frac{7}{16})^{26} \begin{bmatrix} 8 & 8 & 8 & 8 & 8 \\ 5/16 & 5/16 & 5/16 & 5/16 & 5/16 \\ -12 & -12 & -12 & -12 & -12 \\ 2/3 & 2/3 & 2/3 & 2/3 & 2/3 \\ -5/12 & -5/12 & -5/12 & -5/12 & -5/12 \end{bmatrix}$$

According to Theorem 3.5 it is formulated

$$\begin{aligned} Tr(A_5)^{27} &= \left(\sum_{i=1}^{5} a_i\right)^{27} \\ &= (a_1 + a_2 + a_3 + a_4 + a_5)^{27} \\ &= (8 + (5/16) + (-12) + (2/3) + (-5/12))^{27} \\ &= (-3\frac{7}{16})^{27}. \end{aligned}$$

## CONCLUSIONS

Based on elaboration and discussion in previous part, several conclusions can be drawn as follows.

1. The general form of integer of a special matrix form in order $n \times n$ , $n \geq 2$ in Equation (2.1) is as follows.
$$(A_n)^m = \left(\sum_{i=1}^{n} a_i\right)^{m-1} A_n, \text{ with } m \in \mathbb{Z}^+.$$
2. General form of *trace* in a special matrix form in order $n \times n$ , $n \geq 2$ in Equation (2.1) is as follows.
$$Tr\ (A_n)^m = (\textstyle\sum_{i=1}^{n} a_i)^m, \text{ with } m \in \mathbb{Z}^+.$$

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