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Strongly Summable Vector Valued Sequence Spaces Defined by 2 Modular

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ABSTRACT

Summability is an important concept in sequence spaces. One summability concept is strongly Cesaro summable. In this paper, we study a subset of the set of all vector-valued sequence in 2-modular space. Some facts that we investigated in this paper include linearity, the existence of modular and completeness with respect to these modular.

Keywords: Strongly; Summable; Sequence Spaces; 2-modular

INTRODUCTION

Summability is an important concept in sequence spaces. The familiar example of sequence spaces that using the summability concept is ℓ^p spaces. In [1], it is explained that Kutner discusses spaces of strongly Cesaro summable sequences, and furthermore, Maddox generalizes this concept. If ω denote the set of all infinite sequence of real/complex numbers, then the set

$$w = \left\{(x_k) \in \omega : \exists L, \ni \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}|x_k - L| = 0\right\},$$

denote the space of strongly Cesaro summable sequence [2] [3].

Let X be a real linear space of dimension $d \geq 2$. A 2-norm on X is a function $\|.,.\|: X \times X \to \mathbb{R}$, where for all $x, y, z \in X$, satisfy

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent
(ii) $\|x, y\| = \|y, x\|$
(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|, \alpha \in \mathbb{R}$
(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

The pair $(X, \|.,.\|)$ is then called a 2-normed space [4]. The concept is initially introduced by Gahler [5] in the middle of 1963. Furthermore, in 1989, Misiak generalized the 2-normed concept to be n-normed [6]. Since then, many kinds research on 2-normed (n-normed) spaces, include research on strongly Cesaro summable vector-valued sequences or the generalize in 2-normed (n-normed) spaces [7] [8] [9] [10] [11].

In 1950, Nakano developed modular function and it was generalized by Musielak and Orlicz [12] [13]. Modular is the generalization of the norm. Let Y be a real linear space, a functional $g: Y \to \mathbb{R}^*$ is said tobe modular if it satisfies the following conditions:

(i) $g(x) = 0$ if and if $x = 0$
(ii) $g(-x) = g(x)$
(iii) $g(\alpha x + \beta y) \leq g(x) + g(y)$, every $x, y \in Y, \alpha, \beta \geq 0, \alpha + \beta = 1$.

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The pair (Y, g) is then called a modular space. Following the 2-norm (n-norm) concept, K. Nourouzi and S. Shabanian in 2009 initially introduced the n-modular concept [14] [15]. Let X be a real linear space of dimension $d \geq 2$. A 2-modular on X is a function $\rho(.,.): X \times X \to \mathbb{R}^*$ where for all $x, y, z \in X$, satisfy

(i) $\rho(x, y) = 0$ if and only if x and y are linearly dependent
(ii) $\rho(x, y) = \rho(y, x)$
(iii) $\rho(-x, y) = \rho(x, y)$,
(iv) $\rho(\alpha x + \beta y, z) \leq \rho(x, z) + \rho(y, z)$, every $\alpha, \beta \geq 0, \alpha + \beta = 1$.

The pair $(X, \|.,.\|)$ is then called a 2-modular space. The 2-modular space, with ρ satisfies Δ_2-condition, if there exist $L > 0$, such that

$$\rho(2x, y) \leq L\rho(x, y),$$

for all $x, y \in X$. A sequence (x_k) in X is said to be 2-modular convergent to $x_0 \in X$ if

$$\lim_{k\to\infty} \rho(x_k - x_0, y) = 0, \forall y \in X.$$

It means that for every $\epsilon > 0$, there exists an $k_0 \in \mathbb{N}$, such that for any $k \in \mathbb{N}$, $k \geq k_0$, we have

$$\rho(x_k - x_0, y) < \epsilon, \forall y \in X.$$

Furthermore, a sequence (x_k) in X is called 2-modular Cauchy sequence if, for all $y \in X$, we have

$$\lim_{k,l\to\infty} \rho(x_k - x_l, y) = 0.$$

The standard example of a 2-modular space is $X = \mathbb{R}^2$, with 2-modular on $\mathbb{R}^2$ define by

$$\rho(\bar{x}, \bar{y}) = \sqrt{\left|\det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\right|},$$

Where $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in \mathbb{R}^2$. Clearly that ρ satisfies Δ_2-condition and the sequence $\left(\left(\frac{1}{n}, 0\right)\right)$ in $\mathbb{R}^2$ is 2-modular convergent to $(0,0) \in \mathbb{R}^2$.

This paper will be constructed t spaces of strongly Cesaro summable vector-valued sequences in 2-modular spaces based on the facts presented above.

METHODS

Let (X, ρ) be a 2-modular space, with ρ satisfies Δ_2-condition and the dimension of X greater than one. We define

$$X_\rho = \{x \in X: \rho(x, y) < \infty, \forall y \in X\}.$$

Because ρ satisfies Δ_2-condition, then there exists $K > 0$, such that for all $x, y \in X_\rho, z \in X$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}\rho(x + y, z) &= \rho\left(\frac{2x + 2y}{2}, z\right) \\ &\leq \rho(2x, z) + \rho(2y, z) \\ &\leq K\rho(x, z) + K\rho(y, z) \\ &< \infty\end{aligned}$$

Based on Archimedean property, there exists $n_0 \in \mathbb{N}$, such that $\alpha \leq 2^{n_0}$

$$\begin{aligned}\rho(\alpha x, z) &\leq \rho(2^{n_0} x, z) \\ &\leq K^{n_0}\rho(x, z) \\ &< \infty.\end{aligned}$$

Hence, we have that X_ρ is a subspace linear of X. Furthermore (X_ρ, ρ) is a 2-modular space too.

The notation $\omega(X_\rho)$ will donate as the set of all sequences in X_ρ

$$\omega(X_\rho) = \{(x_k): x_k \in X, k \in \mathbb{N}\} \quad (1)$$

where linear space operations are defined coordinatewise,

$$(x_k) + (y_k) = (x_k + y_k), \qquad \alpha(x_k) = (\alpha x_k)$$

for all $(x_k), (y_k) \in \omega(X_\rho)$ and $\alpha \in \mathbb{R}$.

The goal of this paper is that we want to extend the concept of strongly Cesaro summable to 2-modular spaces valued sequences, defined as

$$w_0^\rho(X_\rho) = \left\{(x_k) \in \omega(X_\rho): \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \rho(x_k, y) = 0, \forall y \in X_\rho\right\} \quad (2)$$

$$w^\rho(X_\rho) = \left\{(x_k) \in \omega(X_\rho): \exists x_0 \in X_\rho, \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \rho(x_k - x_0, y) = 0, \forall y \in X_\rho\right\} \quad (3)$$

Furthermore, we also studied the properties of $w_0^\rho(X_\rho)$ and $w^\rho(X_\rho)$.

RESULTS AND DISCUSSION

Henceforth, if not specified then X is a 2-modular space with 2-modular ρ, that satisfies the Δ_2-conditions.

First, we will prove that the mean Cesaro theorem applies to 2-modular space.

Theorem 1. Let sequence (x_k) in X_ρ 2-modular convergent to $x_0 \in X_\rho$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \rho(x_k - x_0, y) = 0, \forall y \in X_\rho$$

Proof. Since the sequence (x_k) in X_ρ 2-modular convergent to $x_0 \in X_\rho$, then for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$, such that for all $k \geq n_\epsilon$, we have

$$\rho(x_k - x_0, y) < \frac{\epsilon}{2},$$

for all $y \in X$. Note that, for all $n \geq n_\epsilon$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} \rho(x_k - x_0, y) &= \frac{1}{n}\sum_{k=1}^{n_\epsilon} \rho(x_k - x_0, y) + \frac{1}{n}\sum_{k=n_\epsilon+1}^{n} \rho(x_k - x_0, y) \\
&\leq \frac{1}{n}\sum_{k=1}^{n_\epsilon} \max_{1\leq k\leq n_\epsilon} \rho(x_k - x_0, y) + \frac{1}{n}\sum_{k=n_\epsilon+1}^{n} \max_{n_\epsilon+1\leq k\leq n} \rho(x_k - x_0, y) \\
&= \frac{\max_{1\leq k\leq n_\epsilon} \rho(x_k - x_0, y)}{n}\sum_{k=1}^{n_\epsilon} 1 + \frac{\max_{n_\epsilon+1\leq k\leq n} \rho(x_k - x_0, y)}{n}\sum_{k=n_\epsilon+1}^{n} 1 \\
&= \max_{1\leq k\leq n_\epsilon} \rho(x_k - x_0, y)\frac{n_\epsilon}{n} + \max_{n_\epsilon+1\leq k\leq n} \rho(x_k - x_0, y)\frac{n - n_\epsilon}{n} \\
&= \max_{1\leq k\leq n_\epsilon} \rho(x_k - x_0, y)\frac{n_\epsilon}{n} + \max_{n_\epsilon+1\leq k\leq n} \rho(x_k - x_0, y) \\
&= \max_{1\leq k\leq n_\epsilon} \rho(x_k - x_0, y)\frac{n_\epsilon}{n} + \frac{\epsilon}{2}.
\end{aligned}$$

By Archimedean property, there exists $n' \geq n_\epsilon$, such that for all $n \geq n'$, we have

$$\max_{1 \leq k \leq n_\epsilon} \rho(x_k - x_0, y) \frac{n_\epsilon}{n} < \frac{\epsilon}{2}.$$

Hence, for all $n \geq n'$, we have

$$\frac{1}{n} \sum_{k=1}^{n} \rho(x_k - x_0, y) < \epsilon.$$

In other words, the proof is complete. ■

Based on Theorem 1, we can say that for all 2-modular convergent sequence (x_k) in X_ρ is an element of $w^\rho(X_\rho)$.

Theorem 2. The set $w^\rho(X_\rho)$ is a linear subspace of $\omega(X_\rho)$.

Proof. Note that for all $(x_k), (y_k) \in w^\rho(X_\rho)$ and $\alpha \in \mathbb{R}$, there exsist $x_0, y_0 \in X_\rho$ so that for all $y \in X_\rho$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho(x_k - \mathrm{x}_0, y) = 0, \text{and } \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho(x_k - \mathrm{y}_0, y) = 0.$$

Therefore, ρ satisfy Δ_2-condition, then there exists $L > 0$ and $n_0 \in \mathbb{N}$ so that

$$\begin{aligned} 0 \leq \rho\big((x_k + y_k) - (x_0 + y_0), y\big) &= \rho\big((x_k - x_0) + (y_k - y_0), y\big) \\ &\leq \rho(2(x_k - x_0), y) + \rho(2(y_k - y_0), y) \\ &\leq L\rho\big((x_k - x_0), y\big) + L\rho\big((y_k - y_0), y\big) \end{aligned}$$

and

$$\begin{aligned} 0 \leq \rho(\alpha x_k - \alpha l, y) &= \rho(\alpha(x_k - l), y) \\ &\leq \rho(2^{n_0}(x_k - x_0), y) \\ &\leq L^{n_0} \rho(x_k - x_0, y). \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho((x_k + y_k) - (x_0 + y_0), y) = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho(\alpha x_k - \alpha x_0, y) = 0.$$

In other words $(x_k) + (y_k)$, $\alpha(x_k) \in w^\rho(X_\rho)$, and we proof that $w^\rho(X_\rho)$ is a subspace linear of $\omega(X_\rho)$.■

Theorem 3. If $(x_k) \in w^\rho(X_\rho)$, then for all $y \in X_\rho$, $\left(\frac{1}{n}\sum_{k=1}^{n} \rho(x_k, y)\right)$ is a bounded sequence of real numbers.

Proof. If $(x_k) \in w^\rho(X_\rho)$, then there exist $x_0 \in X_\rho$, such that for all $y \in X_\rho$, we have

$$\lim_{\mathrm{n}\to\infty} \frac{1}{\mathrm{n}} \sum_{\mathrm{k}=1}^{\mathrm{n}} \rho(\mathrm{x}_\mathrm{k} - x_0, \mathrm{y}) = 0.$$

Hence, there exist $n_0 \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, with $n \geq n_0$ we have

$$\frac{1}{\mathrm{n}}\sum_{\mathrm{k}=1}^{\mathrm{n}} \rho(\mathrm{x}_{\mathrm{k}} - x_0, \mathrm{y}) \leq 1.$$

Since ρ satisfies the Δ_2-conditions, there exist $L > 0$, for all $y \in X_\rho$, we have

$$\rho(x_k, y) = \rho\left(\frac{2(x_k - x_0)}{2} + \frac{2x_0}{2}, y\right) \leq L\rho(x_k - x_0, y) + L\rho(x_0, y).$$

It implies,

$$\frac{1}{n}\sum_{k=1}^{n} \rho(x_k, y) \leq \frac{L}{n}\sum_{k=1}^{n} \rho(x_k - x_0, y) + L\rho(x_0, y).$$

If we set

$$M = \sup\left\{\rho(x_1 - x_0, y), \frac{1}{2}\sum_{\mathrm{k}=1}^{2} \rho(\mathrm{x}_{\mathrm{k}} - x_0, \mathrm{y}), \cdots, \frac{1}{\mathrm{n}_0 - 1}\sum_{\mathrm{k}=1}^{\mathrm{n}_0 - 1} \rho(\mathrm{x}_1 - x_0, \mathrm{y}), 1\right\}$$

then it follows that we have $K = L\big(M + \rho(x_0, y)\big)$, such that

$$\frac{1}{n}\sum_{k=1}^{n} \rho(x_k, y) \leq K,$$

for all $n \in \mathbb{N}$. This implies that for all $y \in X_\rho$, $\left(\frac{1}{n}\sum_{k=1}^{n} \rho(x_k, y)\right)$ is a bounded sequence. ∎

Theorem 4. Function

$$g\big((x_k)\big) = \sup\left\{\frac{1}{n}\sum_{k=1}^{n} \rho(x_k, z), \forall z \in X_\rho\right\} \tag{5}$$

is a modular on $w^\rho\big(X_\rho\big)$.

Proof. If $(x_k) = \mathbf{0}$ is the zero sequence. Then it is clear that $g\big((x_k)\big) = 0$. Conversely, if $\big((x_k)\big) = 0$, then we have

$$\sup\left\{\frac{1}{n}\sum_{k=1}^{n} \rho(x_k, z), \forall z \in X_\rho\right\} = 0.$$

Hence, it implies for all $n \in \mathbb{N}$ and $z \in X_\rho$, we have

$$\frac{1}{n}\sum_{k=1}^{n} \rho(x_k, y_k) = 0 \Leftrightarrow \rho(x_k, z) = 0 \Leftrightarrow x_k = 0, \forall k \in \mathbb{N}.$$

Thus, it is evident that $(x_k) = \mathbf{0}$.

Since $\rho(-x, y) = \rho(x, y)$ applies, for all $x, y \in X_\rho$, consequently, it is clear that $g\big(-(x_k)\big) = g\big((x_k)\big)$. Finally, for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, the for all $(x_k), (y_k) \in w^\rho\big(X_\rho\big)$ we have,

$$\begin{aligned} g\big(\alpha(x_k) + \beta(y_k)\big) &= \sup\left\{\frac{1}{n}\sum_{k=1}^{n} \rho(\alpha x_k + \beta\, y_k, z), \forall z \in X_\rho\right\} \\ &= \sup\left\{\frac{1}{n}\sum_{k=1}^{n} \big(\rho(x_k, \mathrm{z}) + \rho(y_k, z)\big), \forall z \in X_\rho\right\} \end{aligned}$$

$$\leq \sup\left\{\frac{1}{n}\sum_{k=1}^{n}\rho(x_k,z)\,,\forall z\in X_\rho\right\}+\sup\left\{\frac{1}{n}\sum_{k=1}^{n}\rho(y_k,z\,)\,,\forall z\in X_\rho\right\}$$
$$= g\big((x_k)\big)+g\big((y_k)\big).$$
This completes the proof. ∎

Theorem 5. If X_ρ 2-modular complete, then $\big(w^\rho(X_\rho),g\big)$ is a modular complete.
Proof. Let $n\in\mathbb{N}$ and (x^i) be a 2-modular Cauchy sequence in $w^\rho(X_\rho)$, where $x^i=(x_k^i)$, for all $i\in\mathbb{N}$. Hence, for all $\epsilon>0$, there exists $n_0\in\mathbb{N}$, such that for all $i,j\in\mathbb{N}$, with $i,j\geq n_0$, we have
$$g(x^i-x^j)=\sup\left\{\frac{1}{n}\sum_{k=1}^{n}\rho\big(x_k^i-x_k^j,z\big)\,,\forall z\in X_\rho\right\}<\epsilon.$$
It implies that, for all $i,j\geq n_0$, we have
$$\frac{1}{n}\sum_{k=1}^{n}\rho\big(x_k^i-x_k^j,z\big)<\epsilon,\forall z\in X_\rho,$$
or
$$\sum_{k=1}^{n}\rho\big(x_k^i-x_k^j,z\big)<n\epsilon,\forall z\in X_\rho,$$
such that,
$$\rho\big(x_k^i-x_k^j,z\big)<n\epsilon,\forall z\in X_\rho.$$
Hence, for all $k\in\mathbb{N}$, (x_k^i) is a ρ-Cauchy sequence in X_ρ. Since X_ρ complete 2-modular, then (x_k^i) is 2-modular convergent in X_ρ, for all $k\in\mathbb{N}$. Therefore, for $k\in\mathbb{N}$, there exist $x_k\in X_\rho$, such that for all $z\in X_\rho$, we have
$$\lim_{i\to\infty}\rho\big(x_k^i-x_k,z\big)=0.$$
Since, for all $i,j\geq n_0$, we have
$$\frac{1}{n}\sum_{k=1}^{n}\rho\big(x_k^i-x_k,z\big)=\lim_{j\to\infty}\frac{1}{n}\sum_{k=1}^{n}\rho\big(x_k^i-x_k^j,z\big)<\epsilon,\forall z\in X_\rho,$$
then
$$g\left(\big(x_k^i\big)-(x_k)\right)=\sup\left(\frac{1}{n}\sum_{k=1}^{n}\rho\big(x_k^i-x_k,z\big)\right)<\epsilon,\text{for all } i\geq n_0,$$
such that
$$\rho\big(x_k^i-x_k,z\big)<n\epsilon,\text{for all } i\geq n_0$$
Therefore (x^i) modular convergent to (x_k), and $\big(x_k^i-x_k\big)\in w\big(X_\rho\big)$. Since $\big(x_k^i\big)\in w\big(X_\rho\big)$ and $w\big(X_\rho\big)$ is a linear spaces, so we have
$$(x_k)=\big(x_k^i\big)-\big(x_k^i-x_k\big)\in w\big(X_\rho\big).$$
This complete the proof that $\big(w^\rho(X_\rho),g\big)$ is a complete modular (ρ-complete). ∎

CONCLUSIONS

If (X,ρ) is a 2-modular space, with ρ satisfies Δ_2-condition, then we can construct $w^\rho\big(X_\rho\big)\subset w\big(X_\rho\big)$ is the space of strongly Cesaro summable vector-valued sequences in 2-modular $\big(X_\rho,\rho\big)$. It certainly can be shown that $w^\rho\big(X_\rho\big)$ is a linear space. Furthermore, if $(x_k)\in w^\rho\big(X_\rho\big)$, then we can prove that for all $y\in X_\rho$, $\left(\frac{1}{n}\sum_{k=1}^{n}\rho(x_k,y)\right)$ is a bounded sequence of real numbers. This fact provides a guarantee for us to be able to build a modular g on $w^{\rho}(X_{\rho})$. Finally, we proved that $(w^{\rho}(X_{\rho}), g)$ is modular complete, if (X_{ρ}, ρ) is a 2-modular complete.

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