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Zero Divisor Graph of Quotient Ring

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ABSTRACT

Recently, extensive research has been conducted on graphs derived from algebraic structures, including ring structures. One important example of a graph constructed from ring is the zero divisor graph. For a commutative ring R, the zero divisor graph is defined as a simple graph whose vertices are the non-zero zero divisors of R. Two distinct vertices are adjacent if only if their product equals to zero. In this paper, we analyze the properties of the zero divisor graph of quotient ring $\mathbb{Z}_p[x]/\langle x^5\rangle$, where p is a prime number. The methodology is to identifying all non-zero zero divisors to establish the vertices and their adjacency to construct the edges of the graph. We determine some graphs properties, including the order, size, adjacency matrix, degree, distance, diameter, girth, clique number, and chromatic number of the zero divisor graph of quotient ring $\mathbb{Z}_p[x]/\langle x^5\rangle$. This study gives new insights into the structural properties of zero divisor graphs and increase the understanding of the relatedness between algebraic structures and graph theory.

Keywords: commutative ring; quotient ring; graphs properties; zero divisor graph

INTRODUCTION

The concept of graphs originated in the 18th century with the analysis of the Königsberg bridge problem [1]. Despite being discovered long ago, graph theory continues to be an interesting topic until now, with applications in branches of mathematics and science, such as operations research [2], robot navigation network [3], coding theory [4], crystallography [5], transportation [6], pharmaceutical chemistry [7], biological network [8], quantum physics [9], and so on.

It is crucial to review some basic concepts of graph theory, as cited in [10]. Formally, a graph $G = (V(G), E(G))$ is a system consists of a finite non-empty set of vertices $V(G)$ and a finite set of edges $E(G)$, which are unordered pairs of vertices. An edge $(u, v) \in E(G)$ is commonly written as uv or vu and a vertices $u, v \in V(G)$ are adjacent if $uv \in E(G)$. The order of G is defined as number of vertices, $|V(G)|$, the size of G is defined as the number of edges, $|E(G)|$, the adjacency matrix of G, $A(G)$, is a matrix $[a_{ij}]$ where $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. The degree of a vertex $v \in V(G)$, $\deg(v)$, is defined as the number of vertices which are adjacent to v. The distance $d(u, v)$ is defined as the lenght of the shortest path between vertices u and v in G and the maximum distance between

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any two vertices in G is called the the diameter, diam(G). Additionally, the girth, $g(G)$, is defined as the length of the shortest cycle in G and the clique number, $\omega(G)$, is defined as the size of the maximum complete subgraph in G.

Another important concept in graph theory is vertex coloring, which is highly popular subject [10]. A vertex coloring is a function $c: V(G) \to \mathbb{N}$ such that $c(u) \neq c(v)$ for $uv \in E(G)$. If $c(u) \in \{1,2,3, \dots, k\}$ for each $u \in V(G)$, then function c is called a k-coloring. Graph G is called k-colorable if there exists a coloring of G from a set of k colors. The minimum positive integer k for which G is k-colorable is called the chromatic number of G, $\chi(G)$. For complete graph K_n, $\chi(K_n) = n$.

The development of graph theory has constructed a new types of graphs based on algebraic structures, such as ring. One type of graph constructed from a ring is the zero divisor graph. This study has attracted the attention of many researchers since 1988. In [11], Beck introduced the concept of the zero divisor graph on a commutative ring in connection with some coloring problems. He defined it as an undirected and simple graph (no multi-edges or loops), where the vertices represent elements of the ring and two distinc vertices a, b are adjacent if and only if $ab = 0$. In 1999 Anderson and Livingston [12] continued to investigate zero divisor graph and altered the definition concern in the set of vertices. They defined the vertex set of the zero divisor graph of a commutative ring R as including all non-zero zero divisors, with two distinct vertices being adjacent if the product of those vertices equals zero, similar to the previous definition. This refined zero divisor graph is denoted by $\Gamma(R)$. Additionally, they show that the zero divisor of quotient ring $\mathbb{Z}_p[x]/\langle x^2 \rangle$ isomorphic to K_{p-1}.

Furthermore, some researches about zero divisor graphs have been published, contributing to a deeper understanding of the graph-theoretic properties of algebraic structures. In 2005, Axtell et al. [13] explored the zero divisor graphs of the polynomial ring $R[x]$ and power series $R[\![x]\!]$. Following this, the relationship of the diameter between $\Gamma(R), \Gamma(R[x])$, and $\Gamma(R[\![x]\!])$ are investigated by Lucas in 2006 [14]. Further, its girth are investigated by Anderson and Mulay in 2007 [15]. In 2015, Haouaoui and Benhissi [16] focused on some properties of the zero divisor graph of $R[\![x]\!]$. In 2019, Pi et al. [17] explored the zero divisor graph of ring $\mathbb{Z}_n$ for $n = p^2, p^k, pq, pqr$, where $k \geq 3$ and distinct prime p, q, r. This research was continued in 2020 by Park et al. [18], who studied the zero divisor graph of polyniomial ring $\mathbb{Z}_n[x]$ and power series $\mathbb{Z}_n[\![x]\!]$ for the same cases n. In 2023, Johnson and Sankar investigated the zero divisor graph of quotient ring $\mathbb{Z}_p[x]/\langle x^4 \rangle$ with prime p [19]. Moreover, the research of zero divisor graphs can be seen in [20], [21], [22], and [23].

The research topic of the zero divisor graph of quotient rings is relatively new has the possibility to grow rapidly. While previous studies have examined zero divisor graphs in simpler rings, the analysis of the quotient ring $\mathbb{Z}_p[x]/\langle x^5 \rangle$ has not been thoroughly explored. Motivated by the work of Johnson and Sankar, this paper extends the analysis to the zero divisor graph of quotient ring $\mathbb{Z}_p[x]/\langle x^5 \rangle$, where p is a prime number. This study delves into a detailed examination of $\Gamma(\mathbb{Z}_p[x]/\langle x^5 \rangle)$, focusing on properties such as order, size, adjacency matrix, degree, distance, diameter, girth, clique number, and chromatic number.

METHODS

This study uses a literature-based approach to examine the properties of $\Gamma(\mathbb{Z}_p[x]/\langle x^5 \rangle)$ as discussed in recent research. To understand its structure, it is necessary to first review the formal definition of a zero divisor graph.

Definition 1 [24] Let $G = (V, E)$ be a graph and $(R, +, \cdot)$ be a commutative ring. The zero divisor graph of R, denoted as $\Gamma(R)$, is a simple graph with $V = Z^*(R)$, where $Z^*(R)$ is a set of non-zero zero divisors of R and $E = \{ab|ab = 0, \forall\ a, b \in\ Z^*(R)\}$.

Example 1 Let $\mathbb{Z}_3[x]/\langle x^4\rangle = \{kx^3 + lx^2 + mx + n|k, l, m, n \in \mathbb{Z}_3\}$ be a commutative ring. Then, all non-zero zero divisor of $\mathbb{Z}_3[x]/\langle x^4\rangle$ is

$V_1 = \{kx^3|\ k \in \mathbb{Z}_3\backslash\{0\}\} = \{x^3, 2x^3\}$

$V_2 = \{kx^3 + lx^2|\ k \in \mathbb{Z}_3, l \in \mathbb{Z}_3\backslash\{0\}\} = \{x^2, 2x^2, x^3 + x^2, x^3 + 2x^2, 2x^3 + x^2, 2x^3 + 2x^2\}$

$V_2 = \{kx^3 + lx^2 + mx|\ k, l \in \mathbb{Z}_3, m \in \mathbb{Z}_3\backslash\{0\}\} = \{x, 2x, x^2 + x, x^2 + 2x, 2x^2 + x,$
$= 2x^2 + 2x, x^3 + x, x^3 + 2x, x^3 + x^2 + x, x^3 + 2x^2 + x, x^3 + 2x^2 + 2x, 2x^3 + x,$
$2x^3 + 2x, 2x^3 + x^2 + x, 2x^3 + x^2 + 2x, 2x^3 + 2x^2 + x, 2x^3 + 2x^2 + 2x\}$.

Hence, we have $\Gamma(\mathbb{Z}_3[x]/\langle x^4\rangle)$ as shown in Figure 1.

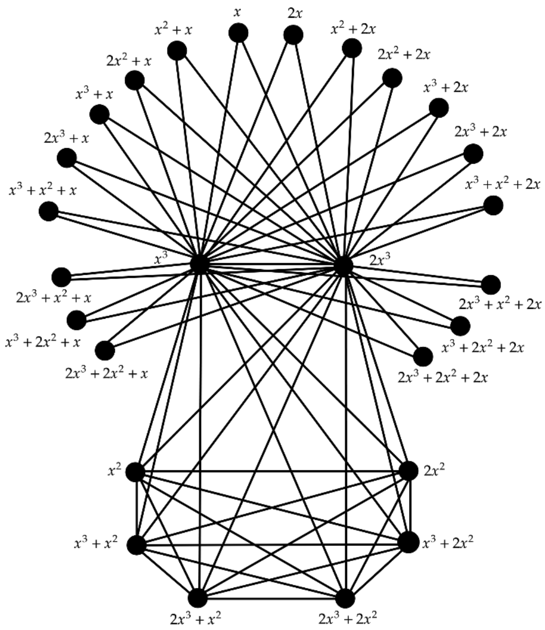

Figure 1. Zero divisor graph of ring $\mathbb{Z}_3[x]/\langle x^4\rangle$

By this foundational understanding, the research methodology involves the following steps:

i. Identity the set of vertices of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ by determining all non-zero zero divisors of $\mathbb{Z}_p[x]/\langle x^5\rangle$ and analyze the set of edges to find the order and size of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$.
ii. Construct the adjacency matrix of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ based on the adjacency between each vertex in $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$.
iii. Determine the degree of each vertex in $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ by counting the number of vertices that are adjacent to it.
iv. Determine the distance between any two vertices in $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ by calculating the lenght of the shortest path that connects them.
v. Consequently of step iv, identify the diameter of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ by finding the greatest distance between any two vertices in $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ and the girth by determining the lenght of the shortest cyle in $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$.
vi. Determine the clique number of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ by constructing the size of the maximum complete subgraph in $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$.
vii. Determine the chromatic number of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ by finding the minimum number of colors required to color the vertices such that no two adjacent vertices share the same color.

RESULTS AND DISCUSSION

Let $\mathbb{Z}_p[x]$, where p is a prime number, be a commutative ring and $\langle x^5 \rangle$ be the ideal of $\mathbb{Z}_p[x]$. The set of all cosets is

$$\mathbb{Z}_p[x]/\langle x^5 \rangle = \{rx^4 + sx^3 + tx^2 + ux + v + \langle x^5 \rangle | r, s, t, u, v \in \mathbb{Z}_p\}.$$

For convenience, element of $\mathbb{Z}_p[x]/\langle x^5 \rangle$ is commonly written as

$$\mathbb{Z}_p[x]/\langle x^5 \rangle = \{rx^4 + sx^3 + tx^2 + ux + v | r, s, t, u, v \in \mathbb{Z}_p\}.$$

We partition the all non-zero zero divisor of $\mathbb{Z}_p[x]/\langle x^5 \rangle$ into four as follows.

$$V_1 = \{rx^4 + sx^3 + tx^2 + ux \mid r, s, t \in \mathbb{Z}_p, u \in \mathbb{Z}_p \backslash \{0\}\},$$
$$V_2 = \{rx^4 + sx^3 + tx^2 \mid r, s \in \mathbb{Z}_p, t \in \mathbb{Z}_p \backslash \{0\}\},$$
$$V_3 = \{rx^4 + sx^3 \mid r \in \mathbb{Z}_p, s \in \mathbb{Z}_p \backslash \{0\}\},$$
$$V_4 = \{rx^4 \mid r \in \mathbb{Z}_p \backslash \{0\}\},$$

Next, we will discuss the order and size of $\Gamma(\mathbb{Z}_p[x]/\langle x^5 \rangle)$ as stated in **Lemma 1**.

Lemma 1 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5 \rangle)$. Order and size of G, respectively, is

$$|V(G)| = p^4 - 1 \text{ and } |E(G)| = \frac{1}{2}(4p^5 - 5p^4 - p^2 + 2)$$

Proof.

Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5 \rangle)$. From the partition set non-zero zero divisor of $\mathbb{Z}_p[x]/\langle x^5 \rangle$, it can be easily seen that the cardinality of V_1, V_2, V_3, and V_4 is

$$|V_1| = p^4 - p^3, \qquad |V_2| = p^3 - p^2, \qquad |V_3| = p^2 - p, \qquad |V_4| = p - 1.$$

Thus the vertex set of graph G, is $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, so order of G is

$$V(G) = |V_1| + |V_2| + |V_3| + |V_4| = p^4 - 1.$$

We also investigate the adjacency among various sets of vertices of G as follows.

1. For arbitrary $a_1, a_2 \in V_1$, where $a_1 \neq a_2$, $a_1 = r_i x^4 + s_i x^3 + t_i x^2 + u_i x$ and $a_2 = r_j x^4 + s_j x^3 + t_j x^2 + u_j x$, we have
$$a_1 a_2 = (r_i x^4 + s_i x^3 + t_i x^2 + u_i x)(r_j x^4 + s_j x^3 + t_j x^2 + u_j x)$$
$$= (t_i t_j + u_i s_j + s_i u_j)x^4 + (t_i u_j + u_i t_j)x^3 + u_i u_j x^2 \not\equiv 0 \bmod x^5.$$
This implies that every vertex in V_1 is not adjacent to every other vertex in V_1.
2. For arbitrary $a \in V_1, b \in V_2$ where $a = r_i x^4 + s_i x^3 + t_i x^2 + u_i x$ and $b = r_j x^4 + s_j x^3 + t_j x^2$, we have
$$ab = (r_i x^4 + s_i x^3 + t_i x^2 + u_i x)(r_j x^4 + s_j x^3 + t_j x^2)$$
$$= (t_i t_j + u_i s_j)x^4 + u_i t_j \not\equiv 0 \bmod x^5.$$
This implies that no vertex in V_1 is adjacent to any vertex in V_2.
3. For arbitrary $a \in V_1, c \in V_3$ where $a = r_i x^4 + s_i x^3 + t_i x^2 + u_i x$ and $c = r_j x^4 + s_j x^3$, we have
$$ac = (r_i x^4 + s_i x^3 + t_i x^2 + u_i x)(r_j x^4 + s_j x^3) = u_i s_j x^4 \not\equiv 0 \bmod x^5.$$
This implies that no vertex in V_1 is adjacent to any vertex in V_3.
4. For arbitrary $a \in V_1, d \in V_4$ where $a = r_i x^4 + s_i x^3 + t_i x^2 + u_i x$ and $d = r_j x^4$, we have
$$ad = (r_i x^4 + s_i x^3 + t_i x^2 + u_i x)(r_j x^4) \equiv 0 \bmod x^5.$$
This implies that every vertex in V_1 is adjacent to every vertex in V_4.
Let $E_1 = \{ad \in E(G) | a \in V_1, b \in V_2\}$, then
$$|E_1| = |V_1||V_2| = (p^4 - p^3)(p - 1) = p^5 - 2p^4 + p^3.$$
5. For arbitrary $b_1, b_2 \in V_2$ where $b_1 = r_i x^4 + s_i x^3 + t_i x^2$ and $b_2 = r_j x^4 + s_j x^3 + t_j x^2$, we have
$$b_1 b_2 = (r_i x^4 + s_i x^3 + t_i x^2)(r_j x^4 + s_j x^3 + t_j x^2) = t_i t_j x^4 \not\equiv 0 \bmod x^5.$$

This implies that every vertex in V_2 is not adjacent to every other vertex in V_2.

6. For arbitrary $b \in V_2, c \in V_3$ where $b = r_i x^4 + s_i x^3 + t_i x^2$ and $c = r_j x^4 + s_j x^3$, we have
$$bc = (r_i x^4 + s_i x^3 + t_i x^2)(r_j x^4 + s_j x^3) \equiv 0 \bmod x^5 .$$
This implies that every vertex in V_2 is adjacent to every vertex in V_3.
Let $E_2 = \{bc \in E(G) | b \in V_2, c \in V_3\}$, then
$$|E_2| = |V_2||V_3| = (p^3 - p^2)(p^2 - p) = p^5 - 2p^4 + p^3.$$
7. For arbitrary $b \in V_2, d \in V_4$ where $b = r_i x^4 + s_i x^3 + t_i x^2$ and $d = r_j x^4$, we have
$$bd = (r_i x^4 + s_i x^3 + t_i x^2)(r_j x^4) \equiv 0 \bmod x^5 .$$
This implies that every vertex in V_2 is adjacent to every vertex in V_4.
Let $E_3 = \{bd \in E(G) | b \in V_2, b \in V_4\}$, then
$$|E_3| = |V_2||V_4| = (p^3 - p^2)(p - 1) = p^4 - 2p^3 + p^2.$$
8. For arbitrary $c_1, c_2 \in V_3$ where $c_1 = r_i x^4 + s_i x^3$ and $c_2 = r_j x^4 + s_j x^3$, we have
$$c_1 c_2 = (r_i x^4 + s_i x^3)(r_j x^4 + s_j x^3) \equiv 0 \bmod x^5 .$$
This implies that every vertex in V_3 is adjacent to every other vertex in V_3.
Let $E_4 = \{c_1 c_2 \in E(G) | c_1, c_2 \in V_3\}$, then
$$|E_4| = \binom{|V_3|}{2} = \binom{p^2 - p}{2} = \frac{1}{2}(p^4 - 2p^3 + p).$$
9. For arbitrary $c \in V_3, d \in V_4$ where $c = r_i x^4 + s_i x^3$ and $d = r_j x^4$, we have
$$cd = (r_i x^4 + s_i x^3)(r_j x^4) \equiv 0 \bmod x^5 .$$
This implies that every vertex in V_3 is adjacent to every vertex in V_4.
Let $E_5 = \{cd \in E(G) | c \in V_3, b \in V_4\}$, then
$$|E_5| = |V_3||V_4| = (p^2 - p)(p - 1) = p^3 - 2p^2 + p.$$

10. For arbitrary $d_1, d_2 \in V_4$ where $d_1 = r_i x^4$ and $d_2 = r_j x^4$, we have
$$d_1 d_2 = (r_i x^4)(r_j x^4) \equiv 0 \bmod x^5 .$$
This implies that every vertex in V_4 is adjacent to every other vertex in V_4.
Let $E_6 = \{d_1 d_2 \in E(G) | d_1, d_2 \in V_4\}$, then
$$|E_6| = \binom{|V_4|}{2} = \binom{p - 1}{2} = \frac{1}{2}(p^2 - 3p + 2).$$

Therefore, size of G is
$$|E(G)| = |E_1| + |E_2| + |E_3| + |E_4| + |E_5| + |E_6| = \frac{1}{2}(4p^5 - 5p^4 - p^2 + 2).$$
■

Proposition 1 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5 \rangle)$. Adjacency matrix of G is

$$A(G) = \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \left[\begin{matrix} O_{p^4-p^3} \\ O_{(p^3-p^2)\times(p^4-p^3)} \\ O_{(p^2-p)\times(p^4-p^3)} \\ N_{(p-1)\times(p^4-p^3)} \end{matrix}\right. & \begin{matrix} O_{(p^4-p^3)\times(p^3-p^2)} \\ O_{(p^3-p^2)} \\ N_{(p^2-p)\times(p^3-p^2)} \\ N_{(p-1)\times(p^3-p^2)} \end{matrix} & \begin{matrix} O_{(p^4-p^3)\times(p^2-p)} \\ N_{(p^3-p^2)\times(p^2-p)} \\ N_{(p^2-p)} - I_{(p^2-p)} \\ N_{(p-1)\times(p^2-p)} \end{matrix} & \left.\begin{matrix} N_{(p^4-p^3)\times(p-1)} \\ N_{(p^3-p^2)\times(p-1)} \\ N_{(p^2-p)\times(p-1)} \\ N_{(p-1)} - I_{(p-1)} \end{matrix}\right] \end{matrix}$$

where O is a zero matrix, I is a identity matrix, and N is a matrix of all ones.

Proof.

Based on the proof of **Lemma 1**, it is obtained that:

i. Every vertex in V_1, is adjacent to every vertex in V_4.
ii. Every vertex in V_2, is adjacent to every vertex in V_3 and V_4.
iii. Every vertex in V_3, is adjacent to every vertex in V_2, V_3, and V_4. It implies that in V_3

established the complete subgraph K_{p^2-p}.

iv. Every vertex in V_4, is adjacent to every vertex in V_1, V_2, V_3, and V_4, thus it also established the complete subgraph K_{p-1} in V_4.

Therefore, graph G can be ilustrated as follows:

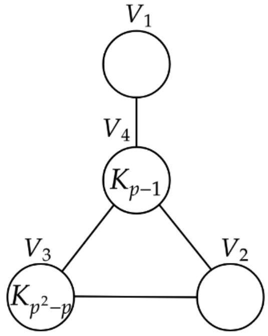

Figure 2. Illustrated of graph G

Then, we can write the adjacency matrix of graph G as follows.

$$A(G) = \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \left[\begin{matrix} O_{p^4-p^3} \\ O_{(p^3-p^2)\times(p^4-p^3)} \\ O_{(p^2-p)\times(p^4-p^3)} \\ N_{(p-1)\times(p^4-p^3)} \end{matrix}\right. & \begin{matrix} O_{(p^4-p^3)\times(p^3-p^2)} \\ O_{(p^3-p^2)} \\ N_{(p^2-p)\times(p^3-p^2)} \\ N_{(p-1)\times(p^3-p^2)} \end{matrix} & \begin{matrix} O_{(p^4-p^3)\times(p^2-p)} \\ N_{(p^3-p^2)\times(p^2-p)} \\ N_{(p^2-p)} - I_{(p^2-p)} \\ N_{(p-1)\times(p^2-p)} \end{matrix} & \left.\begin{matrix} N_{(p^4-p^3)\times(p-1)} \\ N_{(p^3-p^2)\times(p-1)} \\ N_{(p^2-p)\times(p-1)} \\ N_{(p-1)} - I_{(p-1)} \end{matrix}\right] \end{matrix}$$

where O is a zero matrix, I is a identity matrix, and N is a matrix of all ones.

■

Example 2 For $p = 2$, the vertex set of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ is given by

$V_1 = \{x^4 + x^3 + x^2 + x, x^4 + x^3 + x, x^4 + x^2 + x, x^4 + x, x^3 + x^2 + x, x^3 + x, x^2 + x, x\}$
$V_2 = \{x^4 + x^3 + x^2, x^4 + x^2, x^3 + x^2, x^2\}$,
$V_3 = \{x^4 + x^3, x^3\}$,
$V_4 = \{x^4\}$.

Therefore, we have a graph $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ in Figure 3.

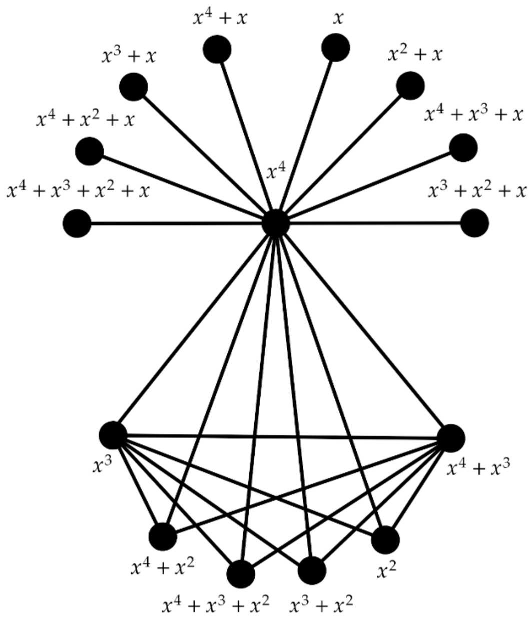

Figure 3. $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$

Additionally, the adjacency matrix of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ is

$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array}\begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} O_8 & O_{8\times 4} & O_{8\times 2} & N_{8\times 1} \\ O_{4\times 8} & O_4 & N_{4\times 2} & N_{4\times 1} \\ O_{2\times 8} & N_{2\times 4} & N_2 - I_2 & N_{2\times 1} \\ N_{1\times 8} & N_{1\times 4} & N_{1\times 2} & N_1 - I_1 \end{bmatrix} \end{array},$$

where O is a zero matrix, I is a identity matrix, and N is a matrix of all ones.

Lemma 3 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$. If $a \in V_1, b \in V_2, c \in V_3, d \in V_4$, with V_1, V_2, V_3, V_4 are partitions of vertex set of G, then the degree of each vertex is

$$\deg(a) = p - 1, \quad \deg(b) = p^2 - 1, \quad \deg(c) = p^3 - 2, \quad \deg(d) = p^4 - 2.$$

Proof.

Let $a \in V_1, b \in V_2, c \in V_3$, and $d \in V_4$ with V_1, V_2, V_3, V_4 are partitions of vertex set of G. Since $a \in V_1$ is adjacent with every vertex in V_4, then the vertex degree a is $\deg(a) = |V_4| = p - 1$. In the similarly way, we get

$$\deg(b) = |V_3| + |V_4| = p^2 - 1,$$
$$\deg(c) = |V_2| + |V_3| + |V_4| = p^3 - 2,$$
$$\deg(d) = |V_1| + |V_2| + |V_3| + |V_4| = p^4 - 2.$$

■

Lemma 4 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$. The distance between any adjacent vertices of graph G is 1, while the distance between any non-adjacent vertices is 2.

Proof.

Distance of each vertex of G is divided into two cases as follows:

(i) Distance between two adjacent vertices.
Let $k, l \in V(G)$. Since, vertex k adjacent to l, so $kl \in E(G)$. Therefore, $d(k, l) = 1$.

(ii) Distance between two non-adjacent vertices.
Let $k, l \in V(G)$. For the reason vertex k is not adjacent to l, so $kl \notin E(G)$. Then, $d(k, l) > 1$. Furthermore, we will identify every possible vertex that is not adjacent in $V(G)$. According to the proof of **Proposition 1**, the vertex $d \in V_4$ is adjacent to each vertex in V_1, V_2, and V_3. It implies that, for any non-adjacent vertices $a_1, a_2 \in V_1, a_1 \neq a_2$, we can construct the path $a_1 \sim d \sim a_2$. Therefore, $d(a_1, a_2) = 2$. Similarly way, we obtain:

$$d(b_1, b_2) = 2, \qquad b_1, b_2 \in V_2, b_1 \neq b_2,$$
$$d(a, b) = 2, \qquad a \in V_1, b \in V_2,$$
$$d(a, c) = 2, \qquad a \in V_1, c \in V_3.$$

■

As a consequence of **Lemma 4**, we also have the following corollary.

Corollary 1 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$. Diameter of G is $\mathrm{diam}(G) = 2$.

Proof.

According to **Lemma 4**, it is clear that the diameter of graph G is 2.

■

Corollary 2 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$. Girth of G is $g(G) = 3$.

Proof.

Let any vertices $b \in V_2, c \in V_3$, and $d \in V_4$. Based on Figure 2, we can construct a cycle from b to b and obtain the shortest cycle is $b \sim c \sim d \sim b$ with length 3. Thus, the girth of G is

$$g(G) = 3.$$

■

Based on the properties of the graph $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$, we can further explore its structure by determining the clique number and chromatic number of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$.

Lemma 5 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$. Clique number of G, is $\omega(G) = p^2$.

Proof.

From Figure 2, it can be seen that there is a complete subgraph K_{p-1} and K_{p^2-p}that can be constructed from the partition of vertices set V_3 and V_4, respectively. Because every vertex in V_3 is adjacent with every vertex in V_4, there exists a complete subgraph $K_{p-1+p^2-p} = K_{p^2-1}$. Furthermore, if we choose a vertex $b \in V_2$, then the clique number is increased by one. This is because b is adjacent with every other vertex in V_3 and V_4. On the other hand, if we choose two vertices $b_1, b_2 \in V_2$, then there are no complete subgraph exists because b_1 and b_2 are not adjacent. For that reason, we can only choose one vertex in V_2. Therefore, the clique number of G is

$$\omega(G) = p^2 - 1 + 1 = p^2.$$

■

Proposition 2 Let $G = \Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$. Chromatic number of G is $\chi(G) = p^2$.

Proof.

In order to prove that the chromatic number of G is p^2 or $\chi(G) = p^2$, we will show that there are p^2-coloring of G such that $\chi(G) \leq p^2$ and every coloring of G needs at least p^2color such that $\chi(G) \geq p^2$.

(i) Based on **Lemma 5**, clique number of G is p^2. Consequenly, the coloring of G is p^2-coloring, such that $\chi(G) \leq p^2$.

(ii) Suppose there exist $(p^2 - 1)$-coloring in G. In any coloring of G, the all vertex in V_3, and V_4 must be colored diferently as they induce a graph K_{p^2-1}. If we attemp to color G with only the $p^2 - 1$ colors, so every vertex in V_2 must have the one of colors in V_3 or V_4, however, this leads to a situation where adjacent vertices $b \in V_2$ and $d \in V_4$, or $b \in V_2$ and $c \in V_3$ have the same color, which is imposibble. This means that G cannot be colored in $(p^2 - 1)$-coloring which concludes a contradiction. As a result, we obtain the condition $\chi(G) \geq p^2$.

Based on the result above, we proved that $\chi(G) = p^2$.

■

Example 3 Let the graph $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ as drawn in Figure 3. Then we can state the following results:

i. $|V(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle))| = 15$.
ii. $|E(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle))| = 23$.
iii. For every vertex $a \in V_1, b \in V_2, c \in V_3, d \in V_4$, and $V_1, V_2, V_3, V_4 \in V(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle))$ the degree of each vertex is

$$\deg(a) = 1, \deg(b) = 3, \deg(c) = 6, \deg(d) = 18.$$

iv. For every vertex in $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$, the distance between any adjacent vertices of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ is 1, but the distance between any non-adjacent vertices is 2.
v. diam $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle) = 2$.
vi. $g\big(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)\big) = 3$.
vii. $\omega\big(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)\big) = 4$.
viii. $\chi\big(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)\big) = 4$.

We provide the example of a 4-colorable graph $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ as follows:

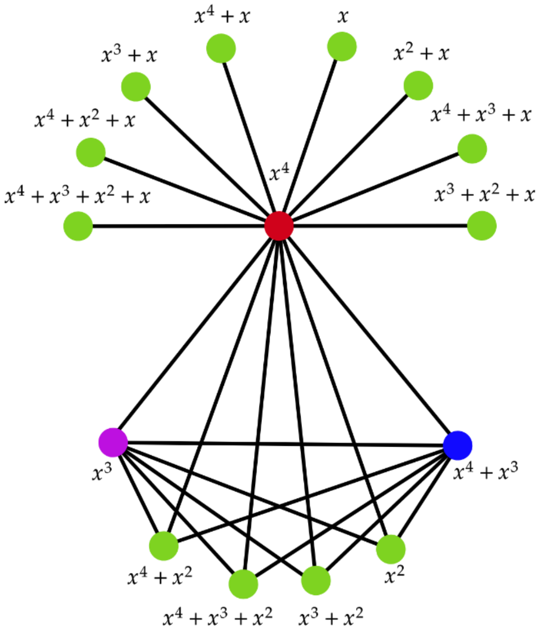

Figure 4. $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ with 4-colorable

CONCLUSIONS

This paper explored properties of zero divisor graph of quotient ring $\mathbb{Z}_p[x]/\langle x^5\rangle$, where p is a prime number, including the order, size, adjacency matrix, degree, distance, diameter, girth, clique number, and chromatic number. For further research, differenet propeties of $\Gamma(\mathbb{Z}_p[x]/\langle x^5\rangle)$ can be explored, such as chromatic index, domination number, matching, independence number, Ramsey numbers, planarity, Eulerian and Hamiltonian properties.

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