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A Note on Generalized Strongly p-Convex Functions of Higher Order

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ABSTRACT

Generalized strongly p-convex functions of higher order is a new concept of convex functions which introduced by Saleem *et al.* in 2020. The Schur type inequality for generalized strongly p-convex functions of higher order also studied by them. This paper aims to revise Schur type inequality for generalized strongly p-convex functions of higher order in their paper. In order to revise it, we show that the contradiction was true. This paper showed that Schur type inequality for generalized strongly p-convex functions of higher order previously is not valid and we give the correct Schur type inequality for generalized strongly p-convex functions of higher order.

Keywords: Schur type inequality; p-Convex functions; Strongly convex of higher order

INTRODUCTION

Convexity is a basic notion in many branches of applied mathematics. Convexity is an important thing on Functional analysis, Geometry, Mathematical programming, Probability, and Statistics. On Functional analysis, convexity has intended to ensure existence and uniqueness of solutions of problems of Calculus of variations and optimal control. On Mathematical programming, convexity has intended to ensure convergence of optimization algorithms [1].

Convexity appears in ancient Greek Geometry. Archimedes (ca. 250 BC) used convexity on study of the area and arch length. Archimedes has been the first person who gave a definition of convexity, similar to the geometric definition which used till today, a set is said to be convex if it contains all line segments between each of its points [1].

Some geometric properties of convex sets and functions have studied before 1960 by great mathematicians Hermann Minkowski and Werner Fenchel. In 1891, Minkowski proved that, in Euclidean space $\mathbb{R}^n$, every compact convex set with center at the origin and volume greater than 2^n contains at least one point with integer coordinates different from the origin [2]. Afterwards, in 1951, Werner Fenchel's monograph stimulated the development of convexity theory.

Several researchers have been considered for classical convexity such that some of these new concepts are based on an extension of the domain of convex functions or

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sets to a generalized form[3-9]. Some examples of these new concepts are quasi-convexity [10], exponential convexity [11, 12], logarithmical convexity [13], h-convexity [14, 15], and p-convexity [3, 4, 16, 17]. p-Convex funtions and their properties was introduced by Zhang and Wan [16] in 2007. In 2018 Maden et al. [17] discussed strongly p-convex funtions and Hermite-Hadamard inequality for it. In 2020, Saleem *et al.* [3, 4] discussed about generalized p-convex funtions and generalized strongly p-convex funtions of higher order.

In [3] discussed definition and properties of generalized strongly p-convex functions of higher order also some of type inequalities which are Hermite-Hadamard, Fejér, and Schur. In Schur type inequality for generalized strongly p-convex funtions of higher order in [3] indirectly mentioned that $\phi\left(x_3^p - x_2^p\right) \leq \left(x_3^p - x_1^p\right)\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)$ for any $x_3^p - x_2^p, x_3^p - x_1^p \in (0,1)$ and $x_3^p - x_2^p < x_3^p - x_1^p$ with $q > 0$ and $\phi(t) = t^q(1-t) + t(1-t)^q$. So, this paper revises the correction of Schur type inequality in [3].

METHODS

In this section, we discussed about some definitions which are implemented with generalized strongly p-convex functions of higher order. We also give examples of strongly p-convex funtions of higher order and its generalized.

Definition 1. (See [1]) **(Convex Set)**

Let $X \subset \mathbb{R}^n$. X is called to be convex if

$$tx + (1-t)y \in X, \tag{1}$$

For any $x, y \in X$ and $t \in [0,1]$.

Definition 2. (See [3]) **(p-Convex Set)**

Let $X \subset \mathbb{R}^n$. X is called to be p-convex if

$$[tx^p + (1-t)y^p]^{\frac{1}{p}} \in X, \tag{2}$$

for any $x, y \in X$ and $t \in [0,1]$.

Definition 3. (See [18]) **(q-Convex Uniform)**

Let X be a Banach space and real number $q \geq 2$. Defined $\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{f+g}{2}\right\|; f, g \in X, \|f\| \leq 1, \|g\| \leq 1, \|f-g\| \geq \epsilon\right\}$. X is called to be p-convex uniform if there exists a constant $c > 0$ such that

$$\delta_X(\epsilon) \geq c\epsilon^q,$$

for $0 < \epsilon \leq 2$.

Lemma 1. (See [19])

Let X be a q-convex uniform with $q \geq 2$, then there exists a constant $\mu > 0$ such that

$$\|tx + (1-t)y\|^q \leq t\|x\|^q + (1-t)\|y\|^q - \mu\phi(t)\|x-y\|^q$$

for every $x, y \in X$ and $t \in (0,1)$, where

$$\phi(t) = t^q(1-t) + t(1-t)^q.$$

Lemma 1 used to prove an example of strongly p-convex funtions of higher order and generalized strongly p-convex funtions of higer order. It has been proved in [19].

Definition 4. (See [3]) **(Strongly $\boldsymbol{p}$-Convex Functions of Higher Order)**

Let X be a p-convex set. A function $f: X \to \mathbb{R}$ is called to be strongly p-convex functions of higher order if for any $x, y \in X$ and $t \in [0,1]$, then

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y) - \mu\phi(t)\|x^p - y^p\|^q, \tag{3}$$

with $\mu \geq 0, q > 0$, and $\phi(t) = t^q(1-t) + t(1-t)^q$.

Example:
Let $q \geq 2$ and $p > 1$. Defined $X = \{x \in \mathbb{R}; |x|^p \in l^q\}$ and $\psi: X \to \mathbb{R}$, where $\psi(x) = \||x|^p\|_q^q$, then ψ is strongly p–convex functions of higher order.

Definition 5. (See [3]) **(Generalized Strongly $\boldsymbol{p}$-Convex Funtions of Higher Order)**

Let X be a p-convex set. A function $f: X \to \mathbb{R}$ is called to be strongly p-convex functions of higher order with respect to $\eta: A \times A \to B$, with $A, B \subseteq \mathbb{R}$ if for any $x, y \in X$ and $t \in [0,1]$, then

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq f(y) + \eta\big(f(x), f(y)\big) - \mu\phi(t)\|x^p - y^p\|^q, \tag{4}$$

with $\mu \geq 0, q > 0$, and $\phi(t) = t^q(1-t) + t(1-t)^q$.

Example:
Function ψ in example of strongly p-convex functions of higher order with respect to $\eta(x, y) = x - y$ is generalized strongly p-convex functions of higher order with rescpect to η.

RESULTS AND DISCUSSION

In this section, we discussed about revised Schur type inequality for generalized strongly p-convex functions of higher order. We showed that $\phi\big(x_3^p - x_2^p\big) \leq \big(x_3^p - x_1^p\big)\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)$ is not valid for any $x_3^p - x_2^p, x_3^p - x_1^p \in (0,1)$ and $x_3^p - x_2^p < x_3^p - x_1^p$ with $q > 0$ and $\phi(t) = t^q(1-t) + t(1-t)^q$.

Theorem 1. (Schur type inequality)

Let X is p-convex set and $A, B \subset \mathbb{R}$. Defined a generalized strongly p-convex function of higher order with respect to $\eta(\cdot,\cdot): A \times A \to B$ with $\mu \geq 0$ and $p > 0$, then for every $x_1, x_2, x_3 \in X$ such that $x_1 < x_2 < x_3$ and $x_3^p - x_1^p, x_3^p - x_2^p, x_2^p - x_1^p \in (0,1)$, then the following inequality hold

$$\begin{aligned} f(x_3)\big(x_3^p - x_1^p\big) - f(x_2)\big(x_3^p - x_1^p\big) + \big(x_3^p - x_2^p\big)\eta\big(f(x_1), f(x_3)\big) \\ - \big(x_3^p - x_1^p\big)\mu\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)\left\|x_3^p - x_1^p\right\|^q \geq 0. \end{aligned} \tag{5}$$

Proof:
Let $x_1, x_2, x_3 \in X$ with some criterion which are $x_1 < x_2 < x_3$ and $x_3^p - x_1^p, x_3^p - x_2^p, x_2^p - x_1^p \in (0,1)$. Based on those criteria, then we have

$$0 < x_3^p - x_2^p < x_3^p - x_1^p \text{ and } 0 < x_2^p - x_1^p < x_3^p - x_1^p. \tag{6}$$

Based on (6), then we have

$$\frac{x_3^p - x_2^p}{x_3^p - x_1^p}, \frac{x_2^p - x_1^p}{x_3^p - x_1^p} \in (0,1). \tag{7}$$

It's clear that

$$\frac{x_3^p - x_1^p}{x_3^p - x_1^p} = 1 \Leftrightarrow \frac{x_3^p - x_1^p - x_2^p + x_2^p}{x_3^p - x_1^p} = 1 \Leftrightarrow \frac{\left(x_3^p - x_2^p\right) + \left(x_2^p - x_1^p\right)}{x_3^p - x_1^p} = 1$$
$$\Leftrightarrow \frac{x_3^p - x_2^p}{x_3^p - x_1^p} + \frac{x_2^p - x_1^p}{x_3^p - x_1^p} = 1. \tag{8}$$

Choose $t = \frac{x_3^p - x_2^p}{x_3^p - x_1^p}$, so from (8) can get

$$t + \frac{x_2^p - x_1^p}{x_3^p - x_1^p} = 1 \Leftrightarrow \frac{x_2^p - x_1^p}{x_3^p - x_1^p} = 1 - t \Leftrightarrow x_2^p - x_1^p = (1-t)\left(x_3^p - x_1^p\right) \Leftrightarrow x_2^p - x_1^p$$
$$= (1-t)x_3^p - (1-t)x_1^p \Leftrightarrow x_2^p - x_1^p = (1-t)x_3^p - x_1^p + tx_1^p \Leftrightarrow x_2^p$$
$$= tx_1^p + (1-t)x_3^p \Leftrightarrow x_2 = \left(tx_1^p + (1-t)x_3^p\right)^{\frac{1}{p}}. \tag{9}$$

From (9), we can write

$$f(x_2) = f\left(\left[tx_1^p + (1-t)x_3^p\right]^{\frac{1}{p}}\right). \tag{10}$$

After that, because f is generalized strongly p-convex functions of higher order, then from (10) and $t = \frac{x_3^p - x_2^p}{x_3^p - x_1^p}$ we can get

$$f(x_2) \le f(x_3) + \left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)\eta\left(f(x_1), f(x_3)\right) - \mu\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)\left\|x_3^p - x_1^p\right\|^q \tag{11}$$

If all segments on (11) times by $x_3^p - x_1^p$, then we have

$$\begin{aligned}
&f(x_2)\left(x_3^p - x_1^p\right)\\
&\le f(x_3)\left(x_3^p - x_1^p\right) + \left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)\eta\left(f(x_1), f(x_3)\right)\left(x_3^p - x_1^p\right)\\
&\quad - \mu\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)\left\|x_3^p - x_1^p\right\|^q\left(x_3^p - x_1^p\right)\\
&\Leftrightarrow f(x_3)\left(x_3^p - x_1^p\right) - f(x_2)\left(x_3^p - x_1^p\right) + \left(x_3^p - x_2^p\right)\eta\left(f(x_1), f(x_3)\right)\\
&\quad - \left(x_3^p - x_1^p\right)\mu\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)\left\|x_3^p - x_1^p\right\|^q \ge 0,
\end{aligned}$$

this completes the proof of theorem 1.

To show that $\phi\left(x_3^p - x_2^p\right) \le \left(x_3^p - x_1^p\right)\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)$ is not valid for any $x_3^p - x_2^p, x_3^p - x_1^p \in (0,1)$ and $x_3^p - x_2^p < x_3^p - x_1^p$ with $q > 0$ and $\phi(t) = t^q(1-t) + t(1-t)^q$, first we take $q = 2$, then we have

$$\phi(t) = t^2(1-t) + t(1-t)^2 = t^2 - t^3 + t(1 - 2t + t^2) = t^2 - t^3 + t - 2t^2 + t^3$$
$$= t - t^2 = t(1-t). \tag{12}$$

After that, we have to take $x = x_3^p - x_1^p = \frac{1}{2}$ and $y = x_3^p - x_2^p = \frac{1}{4}$, then its clear that $x_3^p - x_2^p = \frac{1}{4} < \frac{1}{2} = x_3^p - x_1^p$. So, we can get

$$\phi\left(\frac{1}{4}\right) = \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{3}{16},$$

$$\frac{1}{2}\phi\left(\frac{1/4}{1/2}\right)=\frac{1}{2}\phi\left(\frac{1}{2}\right)=\frac{1}{2}\left[\frac{1}{2}\left(1-\frac{1}{2}\right)\right]=\frac{1}{8}=\frac{2}{16}.$$

And $\phi\left(\frac{1}{4}\right)=\frac{3}{16}>\frac{2}{16}=\frac{1}{2}\phi\left(\frac{1/4}{1/2}\right)$. So, there exists $x_3^p-x_2^p=\frac{1}{4}, x_3^p-x_1^p=\frac{1}{2}$ and $x_3^p-x_2^p=\frac{1}{4}<x_3^p-x_1^p=\frac{1}{2}$ with $q=2$ and $\phi(t)=t^q(1-t)+t(1-t)^q$ but $\left(x_3^p-x_2^p\right)>\left(x_3^p-x_1^p\right)\phi\left(\frac{x_3^p-x_2^p}{x_3^p-x_1^p}\right)$. In other words, $\phi\left(x_3^p-x_2^p\right)>\left(x_3^p-x_1^p\right)\phi\left(\frac{x_3^p-x_2^p}{x_3^p-x_1^p}\right)$ is not valid for $x_3^p-x_2^p, x_3^p-x_1^p\in(0,1)$ and $x_3^p-x_2^p<x_3^p-x_1^p$ with $q>0$ and $\phi(t)=t^q(1-t)+t(1-t)^q$.

Remarks:

If ϕ satisfies $x\phi(y)\geq\phi(xy)$ for any $x,y\in(0,1)$ and $q>0$, then (5) can be written as

$$f(x_3)\left(x_3^p-x_1^p\right)-f(x_2)\left(x_3^p-x_1^p\right)+\left(x_3^p-x_2^p\right)\eta\left(f(x_1),f(x_3)\right)$$
$$-\mu\phi\left(x_3^p-x_2^p\right)\left\|x_3^p-x_1^p\right\|^q\geq 0.$$

CONCLUSION

Schur type inequality for generalized strongly p-convex functions of higher order on [3] has a correction.

$$\left(x_3^p-x_1^p\right)\mu\phi\left(\frac{x_3^p-x_2^p}{x_3^p-x_1^p}\right)\geq\phi\left(x_3^p-x_1^p\right),$$

is not valid for any $x_3^p-x_2^p, x_3^p-x_1^p\in(0,1)$ and $x_3^p-x_2^p<x_3^p-x_1^p$ with $p,q>0$, $t\in[0,1]$, and $\phi(t)=t^q(1-t)+t(1-t)^q$. So, (5) is the correct Schur type inequality for generalized strongly p-convex funtions of higher order.

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