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The Properties of Intuitionistic Anti Fuzzy Module t-norm and t-conorm

Ongky Denny Wijaya, Abdul Rouf Alghofari, Noor Hidayat, Mohamad Muslikh*

Mathematics Department Brawijaya University, Malang, Indonesia

*Corresponding Author
Email: mslk@ub.ac.id*
ongkydenny@gmail.com, abdul_rouf@ub.ac.id, noorh@ub.ac.id

ABSTRACT

Zadeh have introduced fuzzy set in 1965 and Atanassov have introduced intuitionistic fuzzy set in 1986 in theirs paper. Now, many of researcher connecting intuitionistic fuzzy set with algebra theory. We interested to combine some concepts in references over intuitionistic fuzzy set, module of a ring, t-norm, t-conorm, and intuitionistic anti fuzzy. In this paper, we discuss about intuitionistic anti fuzzy module t-norm and t-conorm (IAFMTC) and their properties with respect to module homomorphism, maps, pre-image, and anti-image from intuitionistic fuzzy sets. We have investigated and prove all general properties of IAFMTC and properties related to module homomorphism, maps, pre-image, and anti-image.

Keywords: Intuitionistic Fuzzy Set; t-norm; t-conorm; Module

INTRODUCTION

In 1965, Zadeh [1] introduced new concept about fuzzy set. This concept highlighting the membership status of an indeterminate or fuzzy set. In this concept, membership status is defined as a function whose value is in the interval [0, 1]. In 1986, Atanassov [2] introduced the notion of intuitionistic fuzzy sets as generalization of fuzzy sets, i.e. highlight membership function and non membership function which the value sum of both functions lies in [0,1].

Today, much of the research in the field of algebra was utilised with intuitionistic fuzzy sets on their works. As on the article which written by Isaac and John [3] they introduce intuitionistic fuzzy module and their properties. Next, Rahman and Saikia [4] was introduced a concept of t-norms with respect to the Intuitionistic Fuzzy Submodule, then he describes some properties. In other side, Rasuli [5] was introduced the concepts and properties of intuitionistic fuzzy vector space with respect to t-norm and t-conorm. In contrast to the results above, Sharma in his article [6] introduces a submodule of the intuitionistic anti-fuzzy module. Sharma also discusses the concept of intuitionistic anti-fuzzy modules and the properties it gives rise to. In [7] Sharma was discussed about several properties in maps, pre-image, and anti-image in intuitionistic fuzzy module. A function called t-norm is studied on statistical metric [8] and both of t-norm and its dual, i.e. t-conorm is studied on probabilistic metric spaces [9]. Before many research in

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intuitionistic fuzzy sets related with algebra, there are also many research in fuzzy sets related with algebra. Some of them are fuzzy modules over a t-norm by Rasuli in [10] and anti-fuzzy submodule of a module by Sharma in [11]. Some properties about homomorphism in group algebraic structure related by intuitionistic fuzzy was discussed by Sharma in [12].

Motivated by the results of [3-12] in this article, we introduced some relatively new concepts on intuitionistic anti fuzzy module with respect to t-norm and t-conorm (IAFMTC) as opposite of intuitionistic fuzzy submodule related by t-norm and t-conorm. This concept combines some concepts from the results mentioned above as the gap of previous research. We investigate general properties on IAFMTC and properties with respect to module homomorphism, maps, pre-image, and anti-image from intuitionistic fuzzy sets. We give also some examples to illustrate the main results in this article.

METHODS

We make literature review from [3-12]. Then we construct a new structure named intuitionistic anti fuzzy module t-norm and t-conorm by modifying definition intuitionistic anti fuzzy module (IAFM) from [6] or intuitionistic fuzzy module w.r.t t-norm and t-conorm (IFMTC) from [4]. We have modified definition of IAFM by change minimum value with t-norm and maximum value using t-conorm. Also, we have modified definition of IFMTC by change "less than or equal" sign with "greater than or equal" sign.

RESULTS AND DISCUSSION

In this paper let R denotes commutative ring with unity 1. Definition of ring see [13] or [14]. Now, this is definition of module over R.

Definition 1. [14]
Let R be a ring with identity 1. An abelian group M said to be a module over R (denoted by R-module) if the map

$$\cdot : R \times M \to M$$
$$(r, m) \mapsto \cdot\, (r, m) = rm$$

satisfying the four conditions:

(1) $r(m_1 + m_2) = rm_1 + rm_2$,

(2) $(r_1 + r_2)m = r_1 m + r_2 m$,

(3) $(r_1 r_2)m = r_1(r_2 m)$,

(4) $1m = m$,

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$.

For the future, module over a ring R will be denoted by R-module. For a submodule of a module have simple criteria as defined as below.

Definition 2. [14]
Let M be a R-module and N be a non empty subset of M. N is said to be submodule of M if two conditions below is satisfied:

(1) N is a subgroup of M, i.e. for all $a, b \in N$ then $a - b \in N$.

(2) For all $n \in N$ and $r \in R$, $rn \in N$.

The homomorphism maps on the module not much different on the other algebraic structures. For instance, homomorphism in vector spaces (we called it by linear maps, see [15]) is relatively similar with homomorphism maps in module. Now, definition of homomorphism maps on the module given in Definition 3.

Definition 3. [14]
Let M and N be a R-module and $f: M \to N$ is a map. f is said to be R-module homomorphism, if for all $a, b \in M$ and $r \in R$ satisfy two conditions as follows.
(1) $f(a+b) = f(a) + f(b)$,
(2) $f(ra) = rf(a)$.

Homomorphism f is said an epimorphism if f surjective, monomorphism if f injective, isomorphism if f bijective. If domain and codomain of a homomorphism f are equal, then we said f endomorphism. We called f automorphisma, if f bijective and endomorphism.

Zadeh have introduced fuzzy sets as follows.

Definition 4. [1]
Let X be non empty set. A fuzzy set A in X is characterized by a membership function μ_A which associates each point in X with a real number in the interval $[0,1]$. In other words, we say the fuzzy sets is a set $A = \{(x, \mu_A(x)) | x \in X\}$ which $\mu_A: X \to [0,1]$. The value of μ_A at x, i.e. $\mu_A(x)$ representing the degree of membership of x in A.

Atanassov [2] on 1986 have introduced the concept of intuitionistic fuzzy set as follows. This concept is considered as generalization of fuzzy sets.

Definition 5. [2]
Let X be a non empty set. An intuitionistic fuzzy set (IFS) A in X is defined as an set of the form $A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$ which $\mu_A(x)$ and $\nu_A(x)$ are the functions defined by $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$, which $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$. The functions $\mu_A(x)$ and $\nu_A(x)$ define the degree of membership and degree of non-membership respectively, for every $x \in X$.

The definition of triangular norm was introduced on [8] in 1942. In the beginning, triangular norm used in the study of probabilistic metric spaces. In [9], triangular norm and triangular conorm was studied more at probabilistic metric spaces. In the following, triangular norm and triangular conorm are defined as follows.

Definition 6. [4, 5]
Let T be a function which defined as $T: [0,1] \times [0,1] \to [0,1]$. T is said to be a triangular norm (denoted by t-norm) if for all $x, y, z \in [0,1]$, four axioms are hold:
(1) Neutral element, i.e. $T(x, 1) = x$,
(2) Monotonicity, i.e. if $y \leq z$ then $T(x, y) \leq T(x, z)$,
(3) Commutativity, i.e. $T(x, y) = T(y, x)$,
(4) Associativity, i.e. $T(x, T(y, z)) = T(T(x, y), z)$.

Definition 7. [4, 5]
Let C be a function which defined as $C: [0,1] \times [0,1] \rightarrow [0,1]$. T is said to be a triangular conorm (denoted by t-conorm) if for all $x, y, z \in [0,1]$, four axioms are hold:
(1) Neutral element, i.e. $C(x, 0) = x$,
(2) Monotonicity, i.e. if $y \leq z$ then $C(x, y) \leq C(x, z)$,
(3) Commutativity, i.e. $C(x, y) = C(y, x)$,
(4) Associativity, i.e. $C\big(x, C(y, z)\big) = C(C(x, y), z)$.

Example 1. [5]
The examples of t-norm and t-conorm is given as follows.

Table 1. The examples of t-norm and t-conorm

Name	t-norm	t-conorm
Standard intersection/ standard union	$T_m(x, y) = \min\{x, y\}$	$C_m(x, y) = \max\{x, y\}$
Bounded sum	$T_b(x, y) = \max\{0, x + y - 1\}$	$C_b(x, y) = \min\{1, x + y\}$
Algebraic product/ Algebraic sum	$T_p(x, y) = xy$	$C_p(x, y) = x + y - xy$
Drastic	$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$	$C_D(x, y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise} \end{cases}$
Nilpotent minimum/ Nilpotent maximum	$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$	$C_{nM}(x, y) = \begin{cases} \max\{x, y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise} \end{cases}$
Hamacher product/ Einstein sum	$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \dfrac{xy}{x + y - xy} & \text{otherwise} \end{cases}$	$C_{H_2}(x, y) = \dfrac{x + y}{1 + xy}$

Definition 8. [5]
Let T be a t-norm and C be a t-conorm. T is said to be idempotent t-norm and C is said to be idempotent t-conorm if for all $x \in [0,1]$ satisfy $T(x, x) = x$ and $C(x, x) = x$ respectively.

In [6], there are several corollaries of t-norm and t-conorm as follows.

Corollary 1. [5]
Let T be a t-norm, then for all $x \in [0,1]$,
(1) $T(x, 0) = 0$,
(2) $T(0,0) = 0$.

Corollary 2. [5]
Let C be a t-conorm, then for all $x \in [0,1]$,
(1) $C(x,1) = 1$,
(2) $C(0,0) = 0$.

Definition 9. [7]
Let X and Y are non empty sets, $f: X \to Y$ a maps, $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy sets on X and Y respectively.
(1) For all $y \in Y$, intuitionistic fuzzy sets

$$f(A) = \left\{ \left(y, \mu_{f(A)}(y), \mu_{\nu(A)}(y) \right) \middle| y \in Y \right\}$$

which

$$\mu_{f(A)}(y) = \begin{cases} \max\limits_{x \in f^{-1}(y)} \mu_A(x) & f^{-1}(y) \neq 0 \\ 0 & f^{-1}(y) = 0 \end{cases}$$

and

$$\nu_{f(A)}(y) = \begin{cases} \min\limits_{x \in f^{-1}(y)} \nu_A(x) & f^{-1}(y) \neq 0 \\ 1 & f^{-1}(y) = 0 \end{cases},$$

is called by image of A by f.
(2) For all $x \in X$, intuitionistic fuzzy sets

$$f^{-1}(B)(x) = \left(\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) \right)$$

which $\mu_{f^{-1}(B)}(x) = \mu_B\big(f(x)\big)$ and $\nu_{f^{-1}(B)}(x) = \nu_B\big(f(x)\big)$, is called by pre-image A by f.
(3) For all $y \in Y$, intuitionistic fuzzy sets

$$\hat{f}(A) = \left\{ \left(y, \mu_{\hat{f}(A)}(y), \nu_{\hat{f}(A)}(y) \right) \middle| y \in Y \right\}$$

which

$$\mu_{\hat{f}(A)}(y) = \begin{cases} \min\limits_{x \in f^{-1}(y)} \mu_A(x) & f^{-1}(y) \neq 0 \\ 1 & f^{-1}(y) = 0 \end{cases}$$

and

$$\nu_{\hat{f}(A)}(y) = \begin{cases} \max\limits_{x \in f^{-1}(y)} \nu_A(x) & f^{-1}(y) \neq 0 \\ 0 & f^{-1}(y) = 0 \end{cases},$$

is called by anti-image of A by f.

Intuitionistic fuzzy module has discussed before in [3] by Isaac and John on 2011. Let M be a R-module and $A = (\mu_A, \nu_A)$ is intuitionistic fuzzy set on M, then A is called by intuitionistic fuzzy module if satisfy six axioms, i.e. (1) $\mu_A(0) = 1$, (2) $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$, (3) $\mu_A(ax) \geq \mu_A(x)$, (4) $\nu_A(0) = 0$, (5) $\nu_A(x + y) \leq \max\{\nu_A(x), \nu_A(y)\}$, (6) $\nu_A(ax) \leq \nu_A(x)$, for all $x \in M$ and $a \in R$.

Intuitionistic anti fuzzy module (abbreviated by IAFM) was discussed by Sharma in [6]. Intuitionistic anti fuzzy module obtained by modifying $\mu_A(0) = 1$ by $\mu_A(0) = 0$, min by max on second and fifth axiom, sign $\geq$ by $\leq$ and vice versa.

In this section, we introduce intuitionistic anti fuzzy module t-norm and t-conorm (abbreviated by IAFMTC), the example, some corollaries, and the properties related by module homomorphism, image, pre-image, and anti-image.

Definition 10.
Let M be R-module and $A = (\mu_A, \nu_A)$ be intuitionistic fuzzy set on M. A is called by intuitionistic anti fuzzy module t-norm and t-conorm (denoted by $A \in IAFMTC(M)$) if six axioms below are holds:
(1) $\mu_A(0) = 0$,
(2) $\mu_A(x + y) \leq C\big(\mu_A(x), \mu_A(y)\big)$,
(3) $\mu_A(ax) \leq \mu_A(x)$,
(4) $\nu_A(0) = 1$,
(5) $\nu_A(x + y) \geq T\big(\nu_A(x), \nu_A(y)\big)$,
(6) $\nu_A(ax) \geq \nu_A(x)$,

for all $x, y \in M$ and $a \in R$.

In following, we give example of intuitionistic anti fuzzy module t-norm and t-conorm.

Example 2.
Let Y is a set $Y = \{1,2\}$. The power set of Y is $P(Y) = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}\big\}$. It is easy to check $P(Y)$ form a ring under the operations $\oplus$ (symmetric difference) and $\cap$ (intersection). Now, also easy to check that $P(Y)$ is a $P(Y)$-module. If $A = (\mu_A, \nu_A)$ is intuitionistic fuzzy set on $P(Y)$ which

$$\mu_A : P(Y) \to [0,1]$$
$$B \mapsto \mu_A(B) = \begin{cases} 0 & B = \emptyset \\ 0.4 & B = \{1\} \text{ or } \{2\} \\ 0.5 & B = \{1,2\} \end{cases}$$

and

$$\nu_A : P(Y) \to [0,1]$$
$$B \mapsto \nu_A(B) = \begin{cases} 1 & B = \emptyset \\ 0.3 & B = \{1\} \text{ or } \{2\} \\ 0.25 & B = \{1,2\} \end{cases}$$

then A is intuitionistic anti fuzzy module:
(a) bounded sum t-norm and bounded sum t-conorm.
(b) algebraic product t-norm and algebraic sum t-conorm.

Proof.
To prove it, we must check all conditions on Definition 10.

■

Corollary 3.
If M is R-module and $A = (\mu_A, \nu_A) \in IAFMTC(M)$ then $\mu_A(-x) \leq \mu_A(x)$ and $\nu_A(-x) \geq \nu_A(x)$, for all $x \in M$.

Proof.
Since M is R-module then for all $x \in M$ is satisfy $-x \in M$. Based on axiom on Definition 10, we have $\mu_A\big(a(-x)\big) = \mu_A\big((-a)x\big) \leq \mu_A(x)$ and $\nu_A\big(a(-x)\big) = \nu_A\big((-a)x\big) \geq \nu_A(x)$. Now, choose a as multiplication identity on R, i.e. $a = 1$. We have $\mu_A(-x) \leq \mu_A(x)$ and $\nu_A(-x) \geq \nu_A(x)$.

■

Corollary 4.
If M is R-module and $A = (\mu_A, \nu_A) \in IAFMTC(M)$ then $A(x) = A(-x)$, for all $x \in M$.

Proof.
We prove based on axiom on Definition 10. Consider that $\mu_A(-x) \leq \mu_A(x) = \mu_A\big(-(-x)\big) \leq \mu_A(-x)$ and $\nu_A(-x) \geq \nu_A(x) = \nu_A\big(-(-x)\big) \geq \nu_A(-x)$. This imply $\mu_A(-x) = \mu_A(x)$ dan $\nu_A(-x) = \nu_A(x)$. Now, we have $A(-x) = \big(\mu_A(-x), \nu_A(-x)\big) = \big(\mu_A(x), \nu_A(x)\big) = A(x)$.

■

Next, we give lemma about idempotent t-norm and t-conorm.

Lemma 1.
T is a idempotent t-norm if and only if $T(x,y) = \min\{x,y\}$ for all $x,y \in [0,1]$.

Proof.
(⇒) Take any $x,y \in [0,1]$. If $x \leq y$ then $T(x,y) \leq T(x,1) = x = T(x,x) \leq T(x,y)$ such that $T(x,y) = x$. If $x \geq y$ then $T(x,y) \geq T(y,y) = y = T(1,y) \geq T(x,y)$ such that $T(x,y) = y$. Therefore, we have $T(x,y) = \min\{x,y\}$.
(⇐) Given that $T(x,y) = \min\{x,y\}$ for all $x,y \in [0,1]$. Let $x = y$, we have $T(x,x) = \min\{x,x\} = x$. Therefore, $\min\{x,y\}$ is idempotent t-norm.

■

Lemma 2.
C is a idempotent t-norm if and only if $C(x,y) = \max\{x,y\}$ for all $x,y \in [0,1]$.

Proof.
(⇒) Take any $x,y \in [0,1]$. If $x \geq y$ then $C(x,y) \geq C(x,0) = x = C(x,x) \leq C(x,y)$ such that $C(x,y) = x$. If $x \leq y$ then $C(x,y) \leq C(y,y) = y = C(0,y) \leq C(x,y)$ such that $C(x,y) = y$. Therefore, we have $C(x,y) = \max\{x,y\}$.
(⇐) Given that $C(x,y) = \max\{x,y\}$ for all $x,y \in [0,1]$. Let $x = y$, we have $T(x,x) = \max\{x,x\} = x$. Therefore, $\max\{x,y\}$ is idempotent t-conorm.

■

Corollary 5.
If M be a R-module, $A = (\mu_A, \nu_A) \in IAFMTC(M)$, T and C are idempotent t-norm and t-conorm respectively then $A = (\mu_A, \nu_A) \in IAFM(M)$. (IAFM denote intuitionistic anti fuzzy module in [6]).

Proof.
According to Lemma 1 and 2, if T and C are idempotent t-norm and t-conorm respectively then $T(x,y) = \min\{x,y\}$ and $C(x,y) = \max\{x,y\}$ for all $x,y \in [0,1]$. Therefore, $A = (\mu_A, \nu_A) \in IAFM(M)$.

■

Remark 1.
Every intuitionistic anti fuzzy module with respect to idempotent t-norm and idempotent t-conorm is intuitionistic anti fuzzy module.

The next main result is about some properties with respect to intuitionistic anti fuzzy module t-norm and t-conorm.

Theorem 1.
If M be a R-module, a is a unit on R, and $A = (\mu_A, \nu_A) \in IAFMTC(M)$ then $\mu_A(ax) = \mu_A(x)$ and $\nu_A(ax) = \nu_A(x)$ for all $x \in M$.

Proof.
Take any $x \in M$ and let a is a unit on R. Since a is a unit, then there exist $b \in R$ such that $ab = ba = 1$. Consider that $\mu_A(ax) \leq \mu_A(x) = \mu_A(1x) = \mu_A\big((ab)x\big) = \mu_A\big((ba)x\big) = \mu_A\big(b(ax)\big) \leq \mu_A(ax)$ and $\nu_A(ax) \geq \nu_A(x) = \nu_A(1x) = \nu_A\big((ab)x\big) = \nu_A\big((ba)x\big) = \nu_A\big(b(ax)\big) \geq \nu_A(ax)$. Therefore, $\mu_A(ax) = \mu_A(x)$ and $\nu_A(ax) = \nu_A(x)$ for all $x \in M$.

■

Theorem 2.
Let M be a R-module, N is a submodule of M, and $A = (\mu_A, \nu_A)$ with degree of membership μ_A and degree of non membership ν_A which defined as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases} \text{ and } \nu_A(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases}.$$

If $A' = \big(\mu_{A^c}, \nu_A\big)$ is intuitionistic fuzzy set, where $\mu_{A^c}: M \to \{0,1\}$ is defined by

$$\mu_{A^c}(x) = \begin{cases} 1 & \text{if } x \notin N \\ 0 & \text{if } x \in N \end{cases}$$

then $A' \in IAFMTC(M)$.

Proof.
Take any $x, y \in M$ and $a \in R$. The proof is divided into three cases as follows.

(1) First case. If $x, y \in N$ then $x + y \in N$ and $ax \in N$ such that
 (a) $\mu_{A^c}(0) = 0$.
 (b) $\mu_{A^c}(x + y) = 0 \leq 0 = C(0,0) = C\left(\mu_{A^c}(x), \mu_{A^c}(y)\right)$.
 (c) $\mu_{A^c}(ax) = 0 \leq 0 = \mu_{A^c}(x)$.
 (d) $\nu_A(0) = 1$.
 (e) $\nu_A(x + y) = 1 \geq 1 = T(1,1) = T\big(\nu_A(x), \nu_A(y)\big)$.
 (f) $\nu_A(ax) = 1 \geq 1 = \nu_A(x)$.

(2) Second case. If $x \notin N$ and $y \in N$ then $x + y \notin N$ and $ax \notin N$ such that
 (a) $\mu_{A^c}(0) = 0$.
 (b) $\mu_{A^c}(x + y) = 1 \leq 1 = C(1,0) = C\left(\mu_{A^c}(x), \mu_{A^c}(y)\right)$.
 (c) $\mu_{A^c}(ax) = 1 \leq 1 = \mu_{A^c}(x)$.
 (d) $\nu_A(0) = 1$.
 (e) $\nu_A(x + y) = 0 \geq 0 = T(0,1) = T\big(\nu_A(x), \nu_A(y)\big)$.
 (f) $\nu_A(ax) = 0 \geq 0 = \nu_A(x)$.

(3) Third case. $x, y \notin N$ then $x + y \notin N$ or $x + y \in N$ and $ax \notin N$ such that
 (a) $\mu_{A^c}(0) = 0$.
 (b) $\mu_{A^c}(x + y) \leq 1 = C(1,1) = C\left(\mu_{A^c}(x), \mu_{A^c}(y)\right)$.
 (c) $\mu_{A^c}(ax) = 1 \leq 1 = \mu_{A^c}(x)$.
 (d) $\nu_A(0) = 1$.
 (e) $\nu_A(x + y) \geq 0 = T(0,0) = T\big(\nu_A(x), \nu_A(y)\big)$.
 (f) $\nu_A(ax) = 0 \geq 0 = \nu_A(x)$.

Therefore, based on three cases proof above, we conclude that $A' \in IAFMTC(M)$.

■

Theorem 3.
Let M be a R-module, N is a non empty set of M, and $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set which degree of membership $\mu_A: N \to [0,1]$ and degree of non membership $\nu_A: N \to [0,1]$ defined as

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \in N \\ c_1 & \text{if } x \notin N \end{cases} \text{ and } \nu_A(x) = \begin{cases} 1 & \text{if } x \in N \\ c_2 & \text{if } x \notin N' \end{cases}$$

for $0 \le c_1 \le 1$, $0 \le c_2 \le 1$, and , $0 \le c_1 + c_2 \le 1$. $A = (\mu_A, \nu_A) \in IAFMTC(M)$ if and only if N is a submodule of M.

Proof.
($\Rightarrow$) Take any $x, y \in M$ and $a \in R$ which means $\mu_A(x) = \mu_A(y) = 0$ and $\nu_A(x) = \nu_A(y) = 1$. Based on given $\mu_A(x)$ and $\nu_A(x)$, we have $\mu_A(0) = 0$ and $\nu_A(0) = 1$ imply $0 \in N$. Now, consider that

$$\mu_A(x - y) \le C\big(\mu_A(x), \mu_A(-y)\big) = C\big(\mu_A(x), \mu_A(y)\big) = C(0,0) = 0$$

and

$$\nu_A(x - y) \ge T\big(\nu_A(x), \nu_A(-y)\big) = T\big(\nu_A(x), \nu_A(y)\big) = T(1,1) = 1,$$

which mean $x - y \in N$. Consider that $\mu_A(ax) \le \mu_A(x) = 0$ and $\nu_A(ax) \ge \nu_A(x) = 1$, which mean $ax \in N$. Based on Definition 2, N is a submodule of M.
($\Leftarrow$) The proof of $A = (\mu_A, \nu_A) \in IAFMTC(M)$ directly from Definition 10 as follows. Take any $x, y \in M$ and $a \in R$.
(1) Obviously, $0 \in N$ imply $\mu_A(0) = 0$.
(2) The proof is divided into three cases as follows.
  (a) If $x \in N$ and $y \in N$ then $x + y \in N$ and $\mu_A(x) = \mu_A(y) = 0$. Thus,
  $$\mu_A(x + y) = 0 \le 0 = C(0,0) = C\big(\mu_A(x), \mu_A(y)\big).$$
  (b) If $x \in N$ and $y \notin N$ then $x + y \notin N$, $\mu_A(x) = 0$, dan $\mu_A(y) = c_1$. Thus,
  $$\mu_A(x + y) = c_1 \le c_1 = C(0, c_1) = C\big(\mu_A(x), \mu_A(y)\big).$$
  (c) If $x \notin N$ and $y \notin N$ then $x + y \in N$ or $x + y \notin N$, $\mu_A(x) = c_1$, and $\mu_A(y) = c_1$. If $x + y \in N$ then
  $$\mu_A(x + y) = 0 = C(0,0) \le C(c_1, c_1) = C\big(\mu_A(x), \mu_A(y)\big).$$
  If $x + y \notin N$ then
  $$\mu_A(x + y) = c_1 = C(c_1, 0) \le C(c_1, c_1) = C\big(\mu_A(x), \mu_A(y)\big).$$
(3) The proof is divided into two cases as follows.
  (a) If $x \in N$ then $\mu_A(x) = 0$ and $ax \in N$. Thus,
  $$\mu_A(ax) = 0 \le 0 = \mu_A(x).$$
  (b) If $x \notin N$ then $\mu_A(x) = c_1$ and $ax \in N$ or $ax \notin N$. If $ax \in N$ then
  $$\mu_A(ax) = 0 \le c_1 = \mu_A(x).$$
  If $ax \notin N$ then
  $$\mu_A(ax) = c_1 \le c_1 = \mu_A(x).$$
(4) Obviously, $0 \in N$ imply $\nu_A(0) = 1$.
(5) The proof is divided into three cases as follows.
  (a) If $x \in N$ and $y \in N$ then $x + y \in N$ and $\nu_A(x) = \nu_A(y) = 1$. Thus,
  $$\nu_A(x + y) = 1 \ge 1 = T(1,1) = T\big(\nu_A(x), \nu_A(y)\big).$$
  (b) If $x \in N$ and $y \notin N$ then $x + y \notin N$, $\nu_A(x) = 1$, dan $\nu_A(y) = c_2$. Thus,
  $$\nu_A(x + y) = c_2 \ge c_2 = T(1, c_2) = T\big(\nu_A(x), \nu_A(y)\big).$$

(c) If $x \notin N$ and $y \notin N$ then $x+y \in N$ or $x+y \notin N$, $\nu_A(x)=c_2$, and $\nu_A(y)=c_2$. If $x+y \in N$ then

$$\nu_A(x+y)=1=T(1,1)\geq T(c_2,c_2)=T\big(\nu_A(x),\nu_A(y)\big).$$

If $x+y \notin N$ then

$$\nu_A(x+y)=c_2=T(c_2,1)\geq T(c_2,c_2)=T\big(\nu_A(x),\nu_A(y)\big).$$

(6) The proof is divided into two cases as follows.

(a) If $x \in N$ then $\nu_A(x)=1$ and $ax \in N$. Thus,

$$\nu_A(ax)=1\geq 1=\nu_A(x).$$

(b) If $x \notin N$ then $\nu_A(x)=c_2$ and $ax \in N$ or $ax \notin N$. If $ax \in N$ then

$$\nu_A(ax)=1\geq c_2=\nu_A(x).$$

If $ax \notin N$ then

$$\nu_A(ax)=c_2\geq c_2=\nu_A(x).$$

Therefore, $A=(\mu_A,\nu_A)\in IAFMTC(M)$.

■

Theorem 4.
If M be a R-module and $A=(\mu_A,\nu_A)\in IAFMTC(M)$ then the set

$$N=\{x\in M|A(x)=(0,1)\}$$

is a submodule of M.

Proof.
Take any $x,y\in N$ and $a\in R$. This means $\mu_A(x)=\mu_A(y)=0$ and $\nu_A(x)=\nu_A(y)=1$. Now, we want to show N is a submodule of M as follows.

(1) Consider that

$$\mu_A(x-y)\leq C\big(\mu_A(x),\mu_A(-y)\big)\leq C\big(\mu_A(x),\mu_A(y)\big)=C(0,0)=0,$$
$$\nu_A(x-y)\geq T\big(\nu_A(x),\nu_A(-y)\big)\geq T\big(\nu_A(x),\nu_A(y)\big)=T(1,1)=1.$$

Now, we have $\mu_A(x-y)\leq 0$ and $\nu_A(x-y)\geq 1$ which means $\mu_A(x-y)=0$ and $\nu_A(x-y)=1$. Thus, $A(x-y)=\big(\mu_A(x-y),\nu_A(x-y)\big)=(0,1)$ which imply $x-y\in N$.

(2) Consider that $\mu_A(ax)\leq\mu_A(x)=0$ and $\nu_A(ax)\geq\nu_A(x)=1$ which $\mu_A(ax)=0$ and $\nu_A(ax)=1$ must hold. Thus, $A(ax)=\big(\mu_A(ax),\nu_A(ax)\big)=(0,1)$, which implies $ax\in N$.

Therefore, N is a submodule of M.

■

Theorem 5.
Let M be a R-module and $A=(\mu_A,\nu_A)\in IAFMTC(M)$. If $A(x-y)=(0,1)$ then A(x) = $A(y)$ for all $x,y\in M$.

Proof.
Take any $x,y\in M$. Given $\mu_A(x-y)=0$ and $\nu_A(x-y)=1$. We will show $\mu_A(x)=\mu_A(y)$ and $\nu_A(x)=\nu_A(y)$ directly from axiom on the Definition 10. Consider that

$$\mu_A(x)=\mu_A(x-y+y)\leq C\big(\mu_A(x-y),\mu_A(y)\big)=C\big(0,\mu_A(y)\big)=\mu_A(y)$$

and

$$\mu_A(y) = \mu_A(x - x + y) = \mu_A\big(x - (x - y)\big) \le C\left(\mu_A(x), \mu_A\big(-(x - y)\big)\right)$$
$$= C\big(\mu_A(x), \mu_A(x - y)\big) = C(\mu_A(x), 0) = \mu_A(x).$$

Thus, we have $\mu_A(x) \le \mu_A(y)$ and $\mu_A(y) \le \mu_A(x)$ or equivalently $\mu_A(x) = \mu_A(y)$. Now, consider that

$$\nu_A(x) = \nu_A(x - y + y) \ge T\big(\nu_A(x - y), \nu_A(y)\big) = T\big(1, \nu_A(y)\big) = \nu_A(y)$$

and

$$\nu_A(y) = \nu_A(x - x + y) = \nu_A\big(x - (x - y)\big) \ge T\left(\nu_A(x), \nu_A\big(-(x - y)\big)\right)$$
$$= T\big(\nu_A(x), \nu_A(x - y)\big) = T(\nu_A(x), 1) = \nu_A(x).$$

Thus, we have $\nu_A(x) \le \nu_A(y)$ and $\nu_A(y) \le \nu_A(x)$ or equivalently $\nu_A(x) = \nu_A(y)$. Therefore,

$$A(x) = \big(\mu_A(x), \nu_A(x)\big) = \big(\mu_A(y), \nu_A(y)\big) = A(y).$$

■

In the next theorem, we discuss about properties of intuitionistic anti fuzzy module t-norm and t-conorm with respect to module homomorphism, image, pre-image, and anti-image.

Theorem 6.
Let M and N be a R-module, $\alpha: M \to N$ is R-module homomorphism, and $B = (\mu_B, \nu_B)$ is intuitionistic fuzzy set on N. If $B = (\mu_B, \nu_B) \in IAFMTC(N)$ then $\alpha^{-1}(B) = \big(\mu_{\alpha^{-1}(B)}, \nu_{\alpha^{-1}(B)}\big) \in IAFMTC(M)$.

Proof.
Let identity element of addition on M and N is denoted by 0_M and 0_N respectively. Given $B = (\mu_B, \nu_B) \in IAFMTC(N)$. We will prove $\alpha^{-1}(B) \in IAFMTC(M)$. Take any $x_1, x_2 \in M$ and $a \in R$. Consider that

(1) $\mu_{\alpha^{-1}(B)}(0_M) = \mu_B\big(\alpha(0_M)\big) = \mu_B(0_N) = 0.$

(2) $\mu_{\alpha^{-1}(B)}(x_1 + x_2) = \mu_B\big(\alpha(x_1 + x_2)\big) = \mu_B\big(\alpha(x_1) + \alpha(x_2)\big) \le$
$C\left(\mu_B\big(\alpha(x_1)\big), \mu_B\big(\alpha(x_2)\big)\right) = C\left(\mu_{\alpha^{-1}(B)}(x_1), \mu_{\alpha^{-1}(B)}(x_2)\right).$

(3) $\mu_{\alpha^{-1}(B)}(ax_1) = \mu_B\big(\alpha(ax_1)\big) = \mu_B\big(a\alpha(x_1)\big) \le \mu_B\big(\alpha(x_1)\big) = \mu_{\alpha^{-1}(B)}(x_1).$

(4) $\nu_{\alpha^{-1}(B)}(0_M) = \nu_B\big(\alpha(0_M)\big) = \nu_B(0_N) = 1.$

(5) $\nu_{\alpha^{-1}(B)}(x_1 + x_2) = \nu_B\big(\alpha(x_1 + x_2)\big) = \nu_B\big(\alpha(x_1) + \alpha(x_2)\big) \ge$
$T\left(\nu_B\big(\alpha(x_1)\big), \nu_B\big(\alpha(x_2)\big)\right) = T\left(\nu_{\alpha^{-1}(B)}(x_1), \nu_{\alpha^{-1}(B)}(x_2)\right).$

(6) $\nu_{\alpha^{-1}(B)}(ax_1) = \nu_B\big(\alpha(ax_1)\big) = \nu_B\big(a\alpha(x_1)\big) \ge \nu_B\big(\alpha(x_1)\big) = \nu_{\alpha^{-1}(B)}(x_1).$

Therefore, $\alpha^{-1}(B) = \big(\mu_{\alpha^{-1}(B)}, \nu_{\alpha^{-1}(B)}\big) \in IAFMTC(M)$.

■

Theorem 7.
Let M and N be a R-module, $\alpha: M \to N$ is R-module epimorphism, and $A = (\mu_A, \nu_A)$ is intuitionistic fuzzy set on M. If $A = (\mu_A, \nu_A) \in IAFMTC(M)$ then $\hat{\alpha}(A) = \big(\mu_{\hat{\alpha}(A)}, \nu_{\hat{\alpha}(A)}\big) \in IAFMTC(N)$.

Proof.

Take any $y_1, y_2 \in N$ and $a \in R$. Given α is R-module epimorphism, i.e. surjective homomorphism, obviously $\alpha^{-1}(y_1 + y_2) \neq \emptyset$. Let $y_1 = \alpha(x_1)$ and $y_2 = \alpha(x_2)$, for all $x_1, x_2 \in M$. Consider that

(1) $\mu_{\widehat{\alpha}(A)}(0_N) = \min_{x\in\alpha^{-1}(0_N)} \mu_A(x) = \mu_A(0_M) = 0.$

(2) $\mu_{\widehat{\alpha}(A)}(y_1 + y_2) = \mu_{\widehat{\alpha}(A)}\big(\alpha(x_1) + \alpha(x_2)\big) = \mu_{\widehat{\alpha}(A)}\big(\alpha(x_1 + x_2)\big) = \min_{x\in\alpha^{-1}(\alpha(x_1+x_2))} \mu_A(x) = \mu_A(x_1 + x_2) \leq C\big(\mu_A(x_1), \mu_A(x_2)\big) = C\left(\min_{x\in\alpha^{-1}(\alpha(x_1))} \mu_A(x), \min_{x\in\alpha^{-1}(\alpha(x_2))} \mu_A(x)\right) = C\left(\mu_{\widehat{\alpha}(A)}\big(\alpha(x_1)\big), \mu_{\widehat{\alpha}(A)}\big(\alpha(x_2)\big)\right) = C\left(\mu_{\widehat{\alpha}(A)}(y_1), \mu_{\widehat{\alpha}(A)}(y_2)\right).$

(3) $\mu_{\widehat{\alpha}(A)}(ay_1) = \mu_{\widehat{\alpha}(A)}\big(a\alpha(x_1)\big) = \mu_{\widehat{\alpha}(A)}\big(\alpha(ax_1)\big) = \min_{x\in\alpha^{-1}(\alpha(ax_1))} \mu_A(x) = \mu_A(ax_1) \leq \mu_A(x_1) = \min_{x\in\alpha^{-1}(\alpha(x_1))} \mu_A(x) = \mu_{\widehat{\alpha}(A)}\big(\alpha(x_1)\big) = \mu_{\widehat{\alpha}(A)}(y_1).$

(4) $\nu_{\widehat{\alpha}(A)}(0_N) = \max_{x\in\alpha^{-1}(0_N)} \nu_A(x) = \nu_A(0_M) = 1.$

(5) $\nu_{\widehat{\alpha}(A)}(y_1 + y_2) = \nu_{\widehat{\alpha}(A)}\big(\alpha(x_1) + \alpha(x_2)\big) = \nu_{\widehat{\alpha}(A)}\big(\alpha(x_1 + x_2)\big) = \max_{x\in\alpha^{-1}(\alpha(x_1+x_2))} \nu_A(x) = \nu_A(x_1 + x_2) \geq T\big(\nu_A(x_1), \nu_A(x_2)\big) = T\left(\max_{x\in\alpha^{-1}(\alpha(x_1))} \nu_A(x), \max_{x\in\alpha^{-1}(\alpha(x_2))} \nu_A(x)\right) = T\left(\nu_{\widehat{\alpha}(A)}\big(\alpha(x_1)\big), \nu_{\widehat{\alpha}(A)}\big(\alpha(x_2)\big)\right) = T\left(\nu_{\widehat{\alpha}(A)}(y_1), \nu_{\widehat{\alpha}(A)}(y_2)\right).$

(6) $\nu_{\widehat{\alpha}(A)}(ay_1) = \nu_{\widehat{\alpha}(A)}\big(a\alpha(x_1)\big) = \nu_{\widehat{\alpha}(A)}\big(\alpha(ax_1)\big) = \max_{x\in\alpha^{-1}(\alpha(ax_1))} \nu_A(x) = \nu_A(ax_1) \geq \nu_A(x_1) = \max_{x\in\alpha^{-1}(\alpha(x_1))} \nu_A(x) = \nu_{\widehat{\alpha}(A)}\big(\alpha(x_1)\big) = \nu_{\widehat{\alpha}(A)}(y_1).$

Therefore, $\hat{\alpha}(A) = \big(\mu_{\widehat{\alpha}(A)}, \nu_{\widehat{\alpha}(A)}\big) \in IAFMTC(N).$

■

CONCLUSIONS

We have define the intuitionistic anti fuzzy module t-norm and t-conorm and investigate their general properties and properties with respect to module homomorphism, maps, pre-image, and anti-image from intuitionistic fuzzy sets. We give some example, corollary, and the properties. Several properties which proved are IAFMTC with respect to idempotent t-norm and t-conorm, IAFMTC with respect to characteristic function, submodule of a module IAFMTC, module homomorphism, maps, pre-image, and anti-image from intuitionistic fuzzy sets.

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