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A Left-Symmetric Structure on the Semi-Direct Sum Real Frobenius Lie Algebra of Dimension 8

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ABSTRACT

This paper relates quasi-associative algebras or Koszul algebras of matrix Lie algebras of finite dimension to finite dimensional Frobenius Lie algebras which is written as a semi direct sum. Let $\mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ be the Lie algebra of the semi-direct sum of the real vector space $\mathrm{M}_2(\mathbb{R})$ and the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ of the sets of all 2×2 real matrices. The research aims to give explicit formulas of left-symmetric algebra structures on $\mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$. In this paper, a Frobenius functional is constructed in order to show that the Lie algebra $\mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ is the real Frobenius Lie algebra of dimension 8. Moreover, a bilinear form corresponding to this Frobenius functional is symplectic. Then the obtained symplectic bilinear form induces the left-symmetric algebra structures on $\mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$. In other words, we show that the Lie algebra $\mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ is the left-symmetric algebra. Thus, we give the formulas of its left-symmetric algebra structure explicitely. The left-symmetric algebra structures for case of higher dimension of this Lie algebra type are still an open problem to be investigated.

Keywords: Left-symmetric algebra; Frobenius Lie algebra; Frobenius functional; Semi-direct sum; Symplectic form.

INTRODUCTION

Let $\mathbb{R}$ be the field of real numbers of characteristic zero and $\mathrm{M}_2(\mathbb{R})$ be the $\mathbb{R}$-vector space of all 2×2 real matrices with entries contained in $\mathbb{R}$. The space $\mathrm{M}_2(\mathbb{R})$ is considered as the $\mathbb{R}$-algebra structure given by usual addition and multiplication of matrices. In addition, The space $\mathrm{M}_2(\mathbb{R})$ is the Lie algebra equipped by the Lie brackets in the form of the matrix commutator $[x, y] := xy - yx$ for all matrices $x, y \in \mathrm{M}_2(\mathbb{R})$. We denote by $\mathfrak{gl}_2(\mathbb{R})$ the Lie algebra for $\mathrm{M}_2(\mathbb{R})$ with respect to the mentioned Lie bracket above. Moreover, we shall let $\mathfrak{h}_2 \coloneqq \mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ represent for the semi-direct sum Lie algebra of the vector space $\mathrm{M}_2(\mathbb{R})$ and the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ of dimension 8. Furthermore, the Lie algebra $\mathfrak{h}_2$ has the following matrix realization:

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$$\sigma := \sigma(U, X) = \begin{pmatrix} X & U \\ 0 & 0 \end{pmatrix} \in \mathfrak{h}_2 \subset \mathrm{M}_4(\mathbb{R}) \tag{1}$$

with $U \in \mathrm{M}_2(\mathbb{R})$ and $X \in \mathfrak{gl}_2(\mathbb{R})$. Let $\mathfrak{h}_2^*$ be a dual vector space of $\mathfrak{h}_2$. We also realize in the following matrix

$$\sigma^* := \sigma^*(\alpha, \beta) = \begin{pmatrix} \beta & 0 \\ \alpha & 0 \end{pmatrix} \in \mathfrak{h}_2^* \subset \mathrm{M}_4(\mathbb{R}) \tag{2}$$

with $\alpha \in \mathrm{M}_2(\mathbb{R})$ and $\beta \in \mathfrak{gl}_2(\mathbb{R})$. A value of a linear functional $\sigma^* \in \mathfrak{h}_2^*$ on a matrix $\sigma \in \mathfrak{h}_2$ is denoted by

$$\langle \sigma^*, \sigma \rangle = \mathrm{tr}(U\alpha) + \mathrm{tr}(X\beta) = \langle \alpha, U \rangle + \langle \beta, X \rangle \tag{3}$$

where tr is denoted the matrix trace. The notion of the Lie algebra $\mathfrak{h}_2$ comes originally from the Lie algebra $\mathfrak{h}_{n,p} := \mathrm{M}_{n,p}(\mathbb{K}) \rtimes \mathfrak{gl}_n(\mathbb{K})$ where $\mathrm{M}_{n,p}(\mathrm{K})$ is the real vector space of the set $n \times p$ matrices and $\mathfrak{gl}_n(\mathbb{K})$ is the Lie algebra of the set $n \times n$ matrices over a field $\mathbb{K}$ of characteristic 0. This Lie algebra was introduced by [1] in the context of coadjoint representations of the affine Lie group $\mathrm{G}_{n,p} := \mathrm{M}_{n,p}(\mathbb{K}) \rtimes \mathrm{GL}_n(\mathbb{K})$ where $\mathrm{GL}_n(\mathbb{K})$ is the Lie group of all invertible matrices of the Lie algebra $\mathfrak{gl}_n(\mathbb{K})$. In our case, we take $n = p = 2$ and $\mathbb{K} = \mathbb{R}$. We denote by $\mathrm{M}_2(\mathbb{R}) := \mathrm{M}_{2,2}(\mathbb{R})$ and we determine the Lie algebra $\mathfrak{h}_2$ of the Lie group G_2.

In addition, the case $p = 1$ is considered as the affine Lie algebra, denoted by $\mathfrak{aff}(n)$, of dimension $n(n+1)$. More precise, the realization of this affine Lie algebra $\mathfrak{aff}(n)$ is of the form $\mathrm{M}_{n,1}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ where $\mathrm{M}_{n,1}(\mathbb{R})$ is isomorphic to the space $\mathbb{R}^n$. Interestingly, the Lie algebra $\mathfrak{aff}(n)$ is Frobenius since it has a trivial stabilizer on a certain linear functional. In the other words, the Lie group $\mathrm{Aff}(n)$ of $\mathfrak{aff}(n)$ has an open coadjoint orbit [2]. Moreover, the Lie algebra $\mathfrak{aff}(n)$ is non-solvable. Regarding the notion of a Frobenius Lie algebra, this Lie algebra type was introduced by Ooms in the context to answer Professor Jacobson's question on properties of finite dimensional Lie algebras having an exact simple module over the universal enveloping algebras([3],[4]). The answer that the universal enveloping algebra has an exact simple module if its finite dimensional Lie algebra is Frobenius. Furthermore, if a Lie algebra $\mathfrak{g}$ is Frobenius then there exists a linear functional ψ defined on $\mathfrak{g}$ such that a bilinear alternating form is non-degenerate. Namely, it is a symplectic linear form and such the linear functional ψ is called a Frobenius functional [5].

The notion of the Frobenius functional gives some important implications. Among them, firstly, since the alternating bilinear form corresponding to the Frobenius functional is the symplectic linear form, then the dimension of Frobenius Lie algebra is always even. Secondly, the stabilizer of Frobenius Lie algebras corresponding to the Frobenius functional is trivial, then we have a fact that the Frobenius Lie algebra is not nilpotent [6]. Appearing in many different areas of Lie algebras, for instance, in the study of bounded homogeneous domain, simple hypersurface singularities, and solutions of Yang-Baxter equation [7], Frobenius Lie algebras bring great significance in many areas of Lie algebras.

In ([5],[8]), a notion of a left-symmetric algebra structure on a Lie algebra was introduced where the bilinear product was defined. It was proved that an n-dimensional Lie algebra $\mathfrak{g}$ has left-symmetric algebra structures if there exists a $\mathfrak{g}$-module K of dimension n such that the 1-cocycle space contains a bijective 1-cocycle. Indeed, not every type of Lie algebras has affine structure. These structures are important because it arise in many areas of mathematics such as convex homogeneous cones, affine manifolds, and vertex algebras. Associating to the Frobenius functional, the left-symmetric algebras can be induced by the symplectic structure corresponding to a Frobenius functional [9]. In the present work, we are interested in studying the Frobenius Lie algebra and left-symmetric algebra for the Lie algebra $\mathfrak{h}_2$ of the Lie group G_2.

In this paper, we calculate a Frobenius functional of $\mathfrak{h}_2$ corresponding to its Pfaffian and we prove that $\mathfrak{h}_2$ is 8-dimensional Frobenius Lie algebra. The obtained Frobenius functional of $\mathfrak{h}_2$ is applied to contruct a symplectic structure on $\mathfrak{h}_2$ which induces a left-symmetric algebra structure on $\mathfrak{h}_2$. The research aims to give explicit formulas of left-symmetric algebra structures on $\mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$. Furthermore, with respect to a basis $\mathfrak{h}_2$, we calculte the left-symmetric algebra structures explicitely. Let ε_{ij}, $1 \le i, j \le 4$ be element of $\mathfrak{h}_2 \subset \mathrm{M}_4(\mathbb{R})$ of the form in the equation (1).

We organize the paper as follows. Section 1 is explained the background of this research, state of art, the aim of research, statement of the main results, and some basic notions . Section 2 is devoted to research method. Section 3 discusses our main results to prove that $\mathfrak{h}_2 \coloneqq \mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ is 8-dimensional Frobenius Lie algebra and $\mathfrak{h}_2$ has left-symmectric structures. In this section, we also give explicit formulas for left-symmetric structures of $\mathfrak{h}_2$ and discussion for future research related to our main results. In the end of this paper, the conclusion is given.

METHODS

Our main object was considered from the notion of the affine Lie group $\mathrm{G}_{n,p} \coloneqq \mathrm{M}_{n,p}(\mathbb{K}) \rtimes \mathrm{GL}_n(\mathbb{K})$ which corresponds to the notion of coadjoint representations. Regarding this affine Lie group, particularly, we consider the special case for $n = p = 2$ and we obtained the Lie algebra $\mathfrak{h}_2 \coloneqq \mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ of the Lie group $\mathrm{G}_2 \coloneqq \mathrm{M}_2(\mathbb{R}) \rtimes \mathrm{GL}_2(\mathbb{R})$.

We studied some related papers corresponding to a Frobenius Lie algebra and we offered another alternative to show whether it was a Frobenius Lie algebra or not coresponding to its Pfaffian. Particularly on some Oom's work [4]. On the other hand, Burde introduced the notation of left-symmetric algebra structures on a Lie group and a Lie algebra [8]. We proved the existence of left-symmetric algebra structures on $\mathfrak{h}_2$ and we listed its structures explicitely. Our constructions based on symplectic form which induces its left-symmetric algebra structures.

In the next section, we shall briefly review some basic notions of Lie algebras, Frobenius Lie algebras and their properties, Pfaffian of a Frobenius Lie algebras, and left-symmetric structures on a Lie algebra. In Section Results and Discussion, we shall complete the proof of our main results.

1.1 Frobenius Lie algebra

We shall first introduce the notion of a Lie algebra as follows:

Definition 1[10]. *Let $\mathfrak{g}$ be a vector space. A Lie bracket on $\mathfrak{g}$ is a bilinear form on $\mathfrak{g} \times \mathfrak{g}$, usually denoted by $[\ ,\ ]$, which satisfies*:

1. $[x, x] = 0$ for $x \in \mathfrak{g}$,

2. Jacobi identity, that is

$$[x,[y,z]] = [[z,x],y] + [[x,y],z]. \tag{4}$$

A vector space $\mathfrak{g}$ *together with a Lie bracket* $[\ ,\ ]$ *is called a Lie algebra.*

Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{g}^*$ is dual vector space consisting of linear functionals on $\mathfrak{g}$. We denote by $\mathfrak{gl}(\mathfrak{g})$ the Lie algebra of endomorphism of $\mathfrak{g}$ to itself. Then a representation of the Lie algebra $\mathfrak{g}$ on himself is given by the following map

$$\mathrm{ad}:\ \mathfrak{g} \ni x \mapsto \mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g}). \tag{5}$$

Furthermore, the corresponding representation of the Lie algebra $\mathfrak{g}$ on the space $\mathfrak{g}^*$ is considered as the dual map to ad and is written in the following formula

$$\langle \mathrm{ad}^*(x)\sigma^*, y\rangle = \langle \sigma^*, -\mathrm{ad}(x)y\rangle = \langle \sigma^*, -[x,y]\rangle \tag{6}$$

where $x, y \in \mathfrak{g}$and $\sigma^* \in \mathfrak{g}^*$.

Let σ^* be an element of $\mathfrak{g}^*$ and B_{σ^*} be alternating bilinear form corresponding to σ^* which is defined by

$$B_{\sigma^*}: \mathfrak{g} \times \mathfrak{g} \ni (x,y) \mapsto \langle \sigma^*, [x,y]\rangle \coloneqq B_{\sigma^*}(x,y) \in \mathbb{R}. \tag{7}$$

The kernel of B_{σ^*} is given by the following formula

$$\begin{aligned}\mathrm{Ker}(B_{\sigma^*}) &= \{(x,y) \in \mathfrak{g} \times \mathfrak{g}\ \ ;\ \ \langle \sigma^*, [x,y]\rangle = 0\,\} \\ &= \{x \in \mathfrak{g}\ \ ;\ \ \langle \sigma^*, [x,y]\rangle = 0, \forall\, y \in \mathfrak{g}\,\} \\ &= \{x \in \mathfrak{g}\ \ ;\ \ \langle \mathrm{ad}^*(x)\sigma^*, y\rangle = 0, \forall\, y \in \mathfrak{g}\,\} \\ &= \{x \in \mathfrak{g}\ \ ;\ \ \mathrm{ad}^*(x)\sigma^* = 0\}\end{aligned}$$

The latter formula is nothing but a stabilizer of $\mathfrak{g}$ correponding to σ^* which is denoted by $\mathfrak{g}^{\sigma^*}$.

Definition 2[11]. *Let* $\mathfrak{g}$ *be a Lie algebra. The Lie algeba* $\mathfrak{g}$ *is said to be Frobenius if there exists a linear functional* $\sigma^* \in \mathfrak{g}^*$ *such that the alternating bilinear form in the equation* (7) *is non-degenerate. Such* σ^* *satisfying the eqution* (7) *is called a Frobenius functional.*

Indeed, the non-commutative Lie algebra of dimension 2 is always Frobenius. It is well known that affine Lie algebra $\mathfrak{aff}(1) \coloneqq \mathbb{R} \rtimes \mathbb{R}$ is 2-dimensional Frobenius Lie algebras. Those Frobenius Lie algebras form a large class of Lie algebras. For example, some parabolic and seaweed subalgebras of semi-simple Lie algebras, j-algebras, and Borel subalgebras of simple Lie algebras are Frobenius Lie algebras [7].

Now, let $T \coloneqq \{x_1, x_2, \dots, x_n\}$ be a basis for a Frobenius Lie algebra $\mathfrak{g}$ of dimension n with n is even. For $x_j, x_k \in T$, we denote by $M_{\mathfrak{g}}(\mathbb{R})$ an $n \times n$ matrix whose entries are defined by $M_{\mathfrak{g}}(\mathbb{R}) \coloneqq ([x_j, x_k])_{j,k=1}^{n}$. Moreover, for $\sigma^* \in \mathfrak{g}^*$ we define a matrix $M_{\mathfrak{g}}(\mathbb{R})(\sigma^*)$ stand for an $n \times n$ matrix whose entries are defined by

$$M_{\mathfrak{g}}(\mathbb{R})(\sigma^*) \coloneqq (\langle \sigma^*, [x_j, x_k]\rangle)_{j,k=1}^{n}. \tag{8}$$

We get the following theorem :

Theorem 1[4]. *Let* $\mathfrak{g}$ *be a Lie algebra of even dimensional with a basis* $T := \{x_1, x_2, \dots, x_n\}$. *The Lie algebra* $\mathfrak{g}$ *is Frobenius if one of the following equivalent conditions is satisfied:*

1. *The stabilizer* $\mathfrak{g}^{\sigma^*} = \{0\}$ *for some* $\sigma^* \in \mathfrak{g}^*$.
2. *The determinant of the matrix* $M_{\mathfrak{g}}(\mathbb{R})$ *is not equal to zero.*
3. *The determinant of the matrix* $M_{\mathfrak{g}}(\mathbb{R})(\sigma^*)$ *is not equal to zero for some Frobenius functionals* $\sigma^* \in \mathfrak{g}^*$.

The relation of alternating bilinear form and a linear functional on $\mathfrak{g}$ in the term of a Frobenius Lie algebra can be explained as follows:
In general, let ψ be alternating bilinear form in a Lie algebra $\mathfrak{g}$. If ψ is non-degenerate closed 2-form then $\mathfrak{g}$ is said to be a quasi-Frobenius Lie algebra. Furthermore, if there exists a linear functional $\sigma^* \in \mathfrak{g}^*$ such that

$$\psi(\alpha, \beta) = \mathrm{d}\sigma^*(\alpha, \beta) = -\langle \sigma^*, [\alpha, \beta] \rangle \quad (\alpha, \beta \in \mathfrak{g}) \tag{9}$$

with $\mathrm{d}\sigma^*$ denotes the differential of σ^*, then $\mathfrak{g}$ is also called a Frobenius Lie algebra [5]. Indeed, this statement is equivalent to definition of the Frobenius Lie algebra.

Example 1. The nice examples of low dimensional Frobenus Lie algebras are 4-dimensional Frobenius Lie algebras over a field with characteristic $\neq 2$ and 6-dimensional Frobenius Lie algebras over an algebraically closed field classified in [6]. In the other hand, the $(2n+1)$-dimensional heisenberg Lie algebra is not Frobenius Lie algebra.

Since we shall relate a Frobenius functional to Pfaffian, we need to recall the notion of Pfaffian for the square matrix $A := (A_{jk})_{j,k=1}^{2n}$ where A is an alternating matrix [12]. The formula of Pfaffian is given by

$$\mathrm{Pf}(A) := \frac{1}{2^n n!} \sum_{\tau \in S(2n)} \mathrm{sgn}(\tau) \prod_{k=1}^{n} A_{\tau(2i-1)\tau(2i)}. \tag{10}$$

The Pfaffian for the Lie algebra $\mathfrak{g}$ is defined as the Pfaffian of matrix $M_{\mathfrak{g}}(\mathbb{R})$.

Remark 1 [12]. *Let A be an even dimensional square-alternating matrix. Then* $\det(A) = \mathrm{Pf}(A)^2$.

Example 2. Let $\mathfrak{aff}(1)$ be 2-dimensional affine Lie algebra with non-zero bracket is $[a, b] = b$. We obtain $\mathrm{Pf}(\mathfrak{aff}(1)) = b$. Moreover, the 4-dimensional Frobenius Lie algebra Σ with non-zero brackets in the following formulas [6]

$$[d, a] = a, \qquad [c, b] = a$$
$$[d, b] = \frac{1}{2} b, \qquad [d, c] = \frac{1}{2} c$$

has Pfaffian of the form $\mathrm{Pf}(\Sigma) = a^2$ [13].

The notion of Pfaffian is very important in representation theory of Lie groups, for instance, in notion of square-integrable representation, the Duflo-Moore operator can be associated to Pfaffian [13]. In this paper, we shall relate Pfaffian to a Frobenius functional contruction.

1.2 Left-Symmetric Algebra Structure

The notion of left-symmetric algebra first have been arisen in the theory of Lie groups G endowed a left-invariant affine structure [14]. Let A, B, and C left-invariant vector fields of G that are the section of the map $\psi : TG \to G$ with TG is a tangent bundle of G. Let ∇ be a connection in TG with both curvature and torsion are zero, namely :

$$[A, B] = \nabla_{\mathrm{A}} B - \nabla_B A,$$
$$\nabla_{[\mathrm{A,B}]} C = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z. \tag{11}$$

Let a, b, and c be elements of a Lie algebra $\mathfrak{g}$ of G. Firstly, we define $ab := a * b = \nabla_{\mathrm{A}} B$. Let $\Delta(a, b, c)$ be associator for a, b, and c given by

$$\Delta(a, b, c) := (ab)c - a(bc). \tag{12}$$

In addition, a left-symmectric or a non-associative algebra structure on the Lie algebra $\mathfrak{g}$ is given by the following formula

$$\Delta(a, b, c) = \Delta(b, a, c). \tag{13}$$

We start with some basic definitions of left-symmetric algebra structure which shall be needed for the next section.

Definition 3[8]. *A non-associative algebra L is called a left-symmetric algebra (LSA) if a bilinear product defined by $L \times L \ni (a, b) \mapsto ab := a * b \in L$ satisfies the equation* (15).

A given Lie bracket by the following commutator

$$[a, b] = ab - ba \tag{14}$$

for all $a, b, c \in L$ defines a Lie algebra $\mathfrak{g}$ of L. In other words, the algebra L is a Lie algebra denoted by $\mathfrak{g} = \mathfrak{g}(L)$ with respect to the Lie bracket in the equation (14). We call a left-symmetric algebra by a Lie admissible Lie algebra or Vinberg algebra [8].

Example 3[5]. Let $\mathfrak{aff}(1)$ be a 2-dimensional affine lie algebra with basis $B := \{a, b\}$ and non-zero brackets $[a, b] = b$. The left-symmetric algebra structures on $\mathfrak{aff}(1)$ are given by

$$a^2 = -a,\ ab = 0,\ ba = -b,\ b^2 = 0. \tag{15}$$

Indeed, $\mathfrak{aff}(1)$ is LSA. Furthermore, we can see that the equation (15) satisfies the equation (14).

Remark 2[8]. A Lie algebra $\mathfrak{g}$ has structure affines if it satisfies the equations (13) and (14).

We mention here that not every Lie algebra has left-symmetric algebra structure. For examples, Filiform Lie algebras $\mathfrak{g}$ of dimension $10 \leq \dim \mathfrak{g} \leq 13$ do not have left-symmetric algebra structure ([8], [14]).

Let $\mathfrak{g}$ be a Frobenius Lie algebra whose Frobenius functional is σ^*. We define an alternating bilinear form B_{σ^*} as in the equation (7) whose a representation matrix is in the equation (8). For all $a, b, c \in \mathfrak{g}$, using a product $ab \coloneqq a * b$, then we have that the symplectic form B_{σ^*} induces a left-symmetric algebra structure([5], [9]) defined by

$$B_{\sigma^*}(ab, c) := -B_{\sigma^*}(b, [a, c]) = -\langle \sigma^*, [b, [a, c]] \rangle. \tag{16}$$

The non-degeneracy of B_{σ^*} and the Jacobi identity in the equation (4) guarantee that the equation (16) satisfies the equations (13) and (14). In other words, we find that the symplectic form B_{σ^*} induces a left-symplectic algebra structure. We also observe that the determinant of a representation matrix of B_{σ^*} is not equal to zero since $\mathfrak{g}$ is the Frobenius Lie algebra. Furthermore, we shall apply the equation (16) to find left-symmetric algebra structure explicitely.

RESULTS AND DISCUSSION

In this section, we shall prove our main result as we summarize as follows :

Proposition 1. *Let $S = \{\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{23}, \varepsilon_{24}\}$ be a basis for the Lie algebra $\mathfrak{h}_2$. Then $\mathfrak{h}_2$ is 8-dimensional Frobenius Lie algebra whose the Frobenius functional $\sigma_0^* \coloneqq \varepsilon_{14}^* + \varepsilon_{23}^* \in \mathfrak{h}_2^*$ is considered with respect to the Pfaffian of $\mathfrak{h}_2$, denoted by $\mathrm{Pf}(\mathfrak{h}_2)$. Its Pfaffian $\mathrm{Pf}(\mathfrak{h}_2)$ is written in the following form* :

$$\mathrm{Pf}(\mathfrak{h}_2) = (\varepsilon_{13}\varepsilon_{24} - \varepsilon_{14}\varepsilon_{23})^2 \in \mathrm{S}(\mathfrak{h}_2), \tag{17}$$

where $\mathrm{S}(\mathfrak{h}_2)$ is a symmetric algebra of degree 4. Furthermore, *the Lie algebra $\mathfrak{h}_2$ is left-symmetric algebra* whose *formulas are in the following products*

$\varepsilon_{11}^2 = -\varepsilon_{11}$	$\varepsilon_{11}\varepsilon_{12} = 0$	$\varepsilon_{11}\varepsilon_{21} = -\varepsilon_{21}$
$\varepsilon_{11}\varepsilon_{22} = 0$	$\varepsilon_{11}\varepsilon_{13} = \varepsilon_{13}$	$\varepsilon_{11}\varepsilon_{14} = 0$
$\varepsilon_{11}\varepsilon_{23} = 0$	$\varepsilon_{11}\varepsilon_{24} = -\varepsilon_{24}$	$\varepsilon_{12}\varepsilon_{21} = -\varepsilon_{22}$
$\varepsilon_{12}\varepsilon_{11} = -\varepsilon_{12}$	$\varepsilon_{12}^2 = 0$	$\varepsilon_{12}\varepsilon_{14} = -\varepsilon_{13}$
$\varepsilon_{12}\varepsilon_{22} = 0$	$\varepsilon_{12}\varepsilon_{13} = 0$	$\varepsilon_{21}^2 = 0$
$\varepsilon_{12}\varepsilon_{23} = \varepsilon_{13}$	$\varepsilon_{12}\varepsilon_{24} = \varepsilon_{14} - \varepsilon_{23}$	$\varepsilon_{21}\varepsilon_{24} = 0$
$\varepsilon_{21}\varepsilon_{11} = 0$	$\varepsilon_{21}\varepsilon_{12} = -\varepsilon_{11}$	$\varepsilon_{22}\varepsilon_{21} = 0$

$\varepsilon_{21}\varepsilon_{22} = -\varepsilon_{21}$	$\varepsilon_{21}\varepsilon_{13} = \varepsilon_{23} - \varepsilon_{14}$	$\varepsilon_{22}\varepsilon_{14} = 0$
$\varepsilon_{21}\varepsilon_{14} = \varepsilon_{24}$	$\varepsilon_{21}\varepsilon_{23} = -\varepsilon_{24}$	$\varepsilon_{21}\varepsilon_{24} = 0$
$\varepsilon_{22}\varepsilon_{11} = 0$	$\varepsilon_{22}\varepsilon_{12} = -\varepsilon_{12}$	$\varepsilon_{22}\varepsilon_{21} = 0$
$\varepsilon_{22}^2 = -\varepsilon_{22}$	$\varepsilon_{22}\varepsilon_{13} = -\varepsilon_{13}$	$\varepsilon_{13}\varepsilon_{21} = -\varepsilon_{14}$
$\varepsilon_{22}\varepsilon_{23} = 0$	$\varepsilon_{22}\varepsilon_{24} = \varepsilon_{24}$	$\varepsilon_{13}\varepsilon_{14} = 0$
$\varepsilon_{13}\varepsilon_{11} = 0$	$\varepsilon_{13}\varepsilon_{12} = 0$	$\varepsilon_{14}\varepsilon_{21} = 0$
$\varepsilon_{13}\varepsilon_{22} = -\varepsilon_{13}$	$\varepsilon_{13}^2 = 0$	$\varepsilon_{14}^2 = 0$
$\varepsilon_{13}\varepsilon_{23} = 0$	$\varepsilon_{13}\varepsilon_{24} = 0$	$\varepsilon_{23}\varepsilon_{21} = -\varepsilon_{24}$
$\varepsilon_{14}\varepsilon_{11} = -\varepsilon_{14}$	$\varepsilon_{14}\varepsilon_{12} = 0$	$\varepsilon_{23}\varepsilon_{14} = 0$
$\varepsilon_{14}\varepsilon_{22} = 0$	$\varepsilon_{14}\varepsilon_{13} = 0$	$\varepsilon_{24}\varepsilon_{21} = 0$
$\varepsilon_{14}\varepsilon_{23} = 0$	$\varepsilon_{14}\varepsilon_{24} = 0$	$\varepsilon_{24}\varepsilon_{14} = 0$
$\varepsilon_{23}\varepsilon_{11} = 0$	$\varepsilon_{23}\varepsilon_{12} = 0$	
$\varepsilon_{23}\varepsilon_{22} = -\varepsilon_{23}$	$\varepsilon_{23}\varepsilon_{13} = 0$	
$\varepsilon_{23}^2 = 0$	$\varepsilon_{23}\varepsilon_{24} = 0$	
$\varepsilon_{24}\varepsilon_{11} = -\varepsilon_{24}$	$\varepsilon_{24}\varepsilon_{12} = -\varepsilon_{23}$	
$\varepsilon_{24}\varepsilon_{22} = 0$	$\varepsilon_{24}\varepsilon_{13} = 0$	
$\varepsilon_{24}\varepsilon_{23} = 0$	$\varepsilon_{24}^2 = 0$	

where the products on $\mathfrak{h}_2$ is defined by

$$\mathfrak{h}_2 \times \mathfrak{h}_2 \ni (\varepsilon_{ij}, \varepsilon_{kl}) \mapsto \varepsilon_{ij}\varepsilon_{kl} \coloneqq \varepsilon_{ij} * \varepsilon_{kl} \in \mathfrak{h}_2 . \tag{18}$$

Proof. Let ε_{jk} be elements of $\mathfrak{h}_2 \subset \mathrm{M}_4(\mathbb{R})$ with $1 \leq j, k \leq 4$ corresponding to the form in the equation (1). We let the set $S = \{\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{23}, \varepsilon_{24}\}$ be a basis for $\mathfrak{h}_2$. The Lie algebra $\mathfrak{h}_2$ is endowed with the Lie brackets written as the matrix commutator with respect to the basis S as follows:

$$[\varepsilon_{jk}, \varepsilon_{il}] = \varepsilon_{jk}\varepsilon_{il} - \varepsilon_{il}\varepsilon_{jk} \tag{19}$$

where $1 \leq j, k, i, l \leq 4$. In addition, we give the non-zero brackets for $\mathfrak{h}_2$ in the following formulas :

$[\varepsilon_{12}, \varepsilon_{21}] = \varepsilon_{11} - \varepsilon_{22}$ $\quad$ $[\varepsilon_{11}, \varepsilon_{21}] = -\varepsilon_{21,}$

$[\varepsilon_{11}, \varepsilon_{12}] = \varepsilon_{12}$ $\qquad [\varepsilon_{21}, \varepsilon_{22}] = -\varepsilon_{21},$

$[\varepsilon_{12}, \varepsilon_{22}] = \varepsilon_{12}$ $\qquad [\varepsilon_{11}, \varepsilon_{13}] = \varepsilon_{13},$

$[\varepsilon_{11}, \varepsilon_{14}] = \varepsilon_{14}$ $\qquad [\varepsilon_{12}, \varepsilon_{23}] = \varepsilon_{13},$

$[\varepsilon_{12}, \varepsilon_{24}] = \varepsilon_{14}$ $\qquad [\varepsilon_{21}, \varepsilon_{13}] = \varepsilon_{23},$

$[\varepsilon_{21}, \varepsilon_{14}] = \varepsilon_{24}$ $\qquad [\varepsilon_{22}, \varepsilon_{23}] = \varepsilon_{23},$

$[\varepsilon_{22}, \varepsilon_{24}] = \varepsilon_{24}.$ $\qquad (20)$

Firstly, in order to show that $\mathfrak{h}_2$ is the Frobenius Lie algebra, we just compute the Pfaffian of the matrix $M_{\mathfrak{h}_2}(\mathbb{R}) \coloneqq ([\varepsilon_{kj}, \varepsilon_{il}])_{j,k,i,l=1}^{4}$ defined before with respect to the basis S. Since we have that the determinant for $M_{\mathfrak{h}_2}(\mathbb{R})$ in the following form

$$\det M_{\mathfrak{h}_2}(\mathbb{R}) = (\varepsilon_{13}\varepsilon_{24} - \varepsilon_{14}\varepsilon_{23})^4, \tag{21}$$

is not equal to zero, then $\mathfrak{h}_2$ is the Frobenius Lie algebra as desired. In the next step, using the Theorem 5, we can also see that $\mathfrak{h}_2$ is the Frobenius Lie algebra by constructing a Frobenius functional such that a stabilizer of $\mathfrak{h}_2$ ata that point is trivial. Secondly, to prove that $\mathfrak{h}_2$ is left-symmetric algebra, we should find a Frobenius functional corresponding the Pfaffian for $\mathfrak{h}_2$. We observe, using Remark 7, we have that the Pfaffian of $\mathfrak{h}_2$ can be written of the form

$$\mathrm{Pf}(\mathfrak{h}_2) = (\varepsilon_{13}\varepsilon_{24} - \varepsilon_{14}\varepsilon_{23})^2, \tag{22}$$

which is contained in the symmetric algebra $S(\mathfrak{h}_2)$ of degree 4. Since $\mathfrak{h}_2$ is Frobenius Lie algebra, the existence of some Frobenius functionals are guaranteed. From the equation (22), we first claim that a Frobenius functional is $\sigma_{mn}^* \coloneqq \varepsilon_{14}^* + \varepsilon_{23}^*$ which is contained in the dual space $\mathfrak{h}_2^*$ of vector space $\mathfrak{h}_2$. We recalll the value of a linear functional σ_{mn}^* on $\mathfrak{h}_2$ is defined by $\langle \sigma_{mn}^*, \varepsilon_{jk} \rangle = 1$ for $m = j, n = k$ and 0 otherwise. Let $M_{\mathfrak{h}_2}(\mathbb{R})(\sigma^*)$ be the matrix defined in the equation (10). Namely, we have the 8×8 matrix $M_{\mathfrak{h}_2}(\mathbb{R})(\sigma^*) \coloneqq (\langle \sigma_{mn}^*, [\varepsilon_{kj}, \varepsilon_{il}] \rangle)_{j,k,i,l=1}^{4}$ which can be seen in the following form

$$M_{\mathfrak{h}_2}(\mathbb{R})(\sigma_{mn}^*) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Since the determinant of matrix $M_{\mathfrak{h}_2}(\mathbb{R})(\sigma_{mn}^*)$ is equal to 1, then $\mathfrak{h}_2$ is again the Frobenius Lie algebra. In other words, we have shown that our claim is true. Thus, σ_{mn}^* is the Frobenius functional. Indeed, one can also check that the stabilizer of $\mathfrak{h}_2$ at this Frobenius functional σ_{mn}^* is trivial and this again show that σ_{mn}^* is the Frobenius functional.

Corresponding to the Frobenius functional σ_{mn}^*, we construct the symplectic linear form $B_{\sigma_{mn}^*}$ as defined in the equation (7). We shall show that $B_{\sigma_{mn}^*}$ induces left-symmetric algebra structure in the equations (13) and (14) with respect to the the product $\mathfrak{h}_2 \times \mathfrak{h}_2 \ni (a,b) \mapsto ab \coloneqq a * b \in \mathfrak{h}_2$. Let a, b, c, and α be elements of $\mathfrak{h}_2$. Firstly, we shall show that the equation (13). Namely, we shall show that

$$(ab)c - (ba)c = a(bc) - b(ac). \tag{24}$$

Let us observe that

$$\begin{aligned}
B_{\sigma_{mn}^*}\big((ba)c - (ab)c, \alpha\big) &= B_{\sigma_{mn}^*}\big((ba)c, \alpha\big) - B_{\sigma_{mn}^*}\big((ab)c, \alpha\big),\\
&= B_{\sigma_{mn}^*}\,(c, [ab, \alpha]) - B_{\sigma_{mn}^*}(c, [ba, \alpha]),\\
&= \langle \sigma_{mn}^*, \big[c, [ab, \alpha]\big]\rangle - \langle \sigma_{mn}^*, \big[c, [ba, \alpha]\big]\rangle,\\
&= \langle \sigma_{mn}^*, \big[c, [ab, \alpha]\big] - \big[c, [ba, \alpha]\big]\rangle,\\
&= \langle \sigma_{mn}^*, \big[c, [ab, \alpha] - [ba, \alpha]\big]\rangle,\\
&= \langle \sigma_{mn}^*, \big[c, [ab - ba, \alpha]\big]\rangle,\\
&= \langle \sigma_{mn}^*, \Big[c, \big[[a, b], \alpha\big]\Big]\rangle,\\
&= B_{\sigma_{mn}^*}\big(c, \big[[a, b], \alpha\big]\big).
\end{aligned}$$

Therefore, we get the nice formula as follows :

$$B_{\sigma_{mn}^*}\big((ba)c - (ab)c, \alpha\big) = B_{\sigma_{mn}^*}\big(c, \big[[a, b], \alpha\big]\big). \tag{25}$$

In the similar way, we have

$$B_{\sigma_{mn}^*}\big(b(ac) - a(bc), \alpha\big) = B_{\sigma_{mn}^*}\big(c, \big[[a, b], \alpha\big]\big). \tag{26}$$

By the equality of equations (25) and (26), then we have

$$B_{\sigma_{mn}^*}\big((ba)c - (ab)c + a(bc) - b(ac), \alpha\big) = 0. \tag{27}$$

But the non-degeneracy of the bilinear form $B_{\sigma_{mn}^*}$ guarantees that $(ba)c - (ab)c + a(bc) - b(ac) = 0$. In other words, the equation (13) holds for all $a, b, c \in \mathfrak{h}_2$. Therefore, the symplectic bilinear form $B_{\sigma_{mn}^*}$ corresponding to σ_{mn}^* induces the left-symmetric on the Frobenius Lie algebra $\mathfrak{h}_2$.

Secondly, we shall show that the equation (14) holds. Let us observe that

$$\begin{aligned}
&B_{\sigma_{mn}^*}(ab - ba - [a, b], \alpha)\\
&= B_{\sigma_{mn}^*}(ab, \alpha) - B_{\sigma_{mn}^*}(ba, \alpha) - B_{\sigma_{mn}^*}([a, b], \alpha),\\
&= B_{\sigma_{mn}^*}(a, [b, \alpha]) - B_{\sigma_{mn}^*}(b, [a, \alpha]) - B_{\sigma_{mn}^*}([a, b], \alpha),\\
&= B_{\sigma_{mn}^*}(a, [b, \alpha]) + B_{\sigma_{mn}^*}(b, [\alpha, a]) + B_{\sigma_{mn}^*}(\alpha, [a, b]),\\
&= \langle \sigma_{mn}^*, \big[a, [b, \alpha]\big]\rangle + \langle \sigma_{mn}^*, \big[b, [\alpha, a]\big]\rangle + \langle \sigma_{mn}^*, \big[\alpha, [a, b]\big]\rangle,\\
&= \langle \sigma_{mn}^*, \big[a, [b, \alpha]\big] + \big[b, [\alpha, a]\big] + \big[\alpha, [a, b]\big]\rangle,
\end{aligned}$$

$= \langle \sigma_{mn}^*, 0 \rangle = 0.$

Thus, we obtain

$$B_{\sigma_{mn}^*}(ab - ba - [a,b], \alpha) = 0. \quad (28)$$

By the same argument of non-degeneracy of $B_{\sigma_{mn}^*}$, we get $ab - ba - [a,b] = 0$. Thus, the equation (14) holds. We proved that, the alternating bilinear form $B_{\sigma_{mn}^*}$ induces the left-symmetric structure on $\mathfrak{h}_2$.

More specific, for $a, b \in \mathfrak{h}_2$, there exist scalars α_{kj} and β_{il} where $1 \le j,k,i,l \le 4$ such that

$$a \coloneqq \sum_{1\le j,k\le 4} \alpha_{kj}\varepsilon_{kj}\,, \; b \coloneqq \sum_{1\le i,l\le 4} \beta_{il}\varepsilon_{il}. \quad (29)$$

Moreover, since $ab \coloneqq a * b \in \mathfrak{h}_2$, then there also exists scalars φ_{st} where $1 \le s,t \le 4$ such that

$$ab \coloneqq \sum_{1\le s,t\le 4} \varphi_{st}\varepsilon_{st}. \quad (30)$$

We shall calculate scalar φ_{st} written as products of scalars α_{kj} and β_{il} where $1 \le j,k,i,l,s,t \le 4$ satisfying the equationss (13) and (14). Applying induced left-symmetric structure from the symplectic form $B_{\sigma_{mn}^*}$ corresponding to the Frobenius fuctional $\sigma_{mn}^* \coloneqq \varepsilon_{14}^* + \varepsilon_{23}^*$ then we get the following products

$$\varphi_{st} = \sum_{1\le j,k,i,l\le 4} \alpha_{kj}\beta_{il}. \quad (31)$$

Therefore, by choosing suitable scalars α_{kj} and β_{il}, we have the products $\varepsilon_{jk}\varepsilon_{il}$. Indeed, these product satisfies the equationss (13) and (14) because they are induced by simplectic form for $\mathfrak{h}_2$. Thus, $\mathfrak{h}_2$ is left-symmetric algebra.

We apply formulas in the equations (16), (29), (30), and (31) to find the explicit left-symmetric structures on $\mathfrak{h}_2$. We compute these left-symmetric structures with respect to the basis S as mentioned above. Then we obtain :

$$\begin{aligned} B_{\sigma_{mn}^*}(ab, \varepsilon_{kj}) &= -B_{\sigma_{mn}^*}(b, [a, \varepsilon_{kj}]), \\ &= -\langle \sigma_{mn}^*, \left[b, [a, \varepsilon_{kj}]\right] \rangle, \\ &= -\langle \sigma_{mn}^*, \left[\sum_{1\le i,l\le 4} \beta_{il}\varepsilon_{il}\,, \left[\sum_{1\le j,k\le 4} \alpha_{kj}\varepsilon_{kj}\,, \varepsilon_{kj}\right]\right] \rangle, \end{aligned} \quad (32)$$

where $\varepsilon_{kj} \in S \subset \mathfrak{h}_2$ for $1 \le j,k \le 4$. On the other hand, we have the following formulas :

$$\begin{aligned} B_{\sigma_{mn}^*}(ab, \varepsilon_{kj}) &= B_{\sigma_{mn}^*}\left(\sum_{1\le s,t\le 4} \varphi_{st}\varepsilon_{st}\,, \varepsilon_{kj} \right), \\ &= \langle \sigma_{mn}^*, \left[\sum_{1\le s,t\le 4} \varphi_{st}\varepsilon_{st}\,, \varepsilon_{kj} \right] \rangle, \end{aligned} \quad (33)$$

where $\varepsilon_{kj} \in S \subset \mathfrak{h}_2$ for $1 \le j,k \le 4$. Let $\varepsilon_{kj} = \varepsilon_{11}$, then the equation (32) can be computed with respect the Lie brackets in the equation (20) as follows :

$$B_{\sigma_{mn}^*}(ab,\varepsilon_{11}) = -\langle\sigma_{mn}^*,\left[b,\left[a,\varepsilon_{kj}\right]\right]\rangle,$$
$$= -\beta_8\alpha_2+\beta_5\alpha_3+\beta_3\alpha_5+\beta_1\alpha_6. \quad (34)$$

In the other hand, we obtain from the equation (33) that

$$B_{\sigma_{mn}^*}(ab,\varepsilon_{11}) = \langle\sigma_{mn}^*,\left[\sum_{1\le s,t\le 4}\varphi_{st}\varepsilon_{st},\varepsilon_{kj}\right]\rangle,$$
$$= -\varphi_{14}. \quad (35)$$

Therefore, we get

$$\varphi_{14} = \beta_8\alpha_2-\beta_5\alpha_3-\beta_3\alpha_5-\beta_1\alpha_6. \quad (36)$$

In the similar way, we obtain the following formulas using the equations (20), (32), and (33) simultaneously :

$\varphi_{11} = -(\beta_1\alpha_1+\beta_2\alpha_3),$

$\varphi_{12} = -(\beta_1\alpha_2+\beta_2\alpha_4),$

$\varphi_{21} = -(\beta_3\alpha_1+\beta_4\alpha_3),$

$\varphi_{22} = -(\beta_3\alpha_2+\beta_4\alpha_4),$

$\varphi_{13} = \beta_5\alpha_1-\beta_6\alpha_2+\beta_7\alpha_2-\beta_5\alpha_4-\beta_4\alpha_5-\beta_2\alpha_6,$

$\varphi_{23} = \beta_5\alpha_3-\beta_8\alpha_2-\beta_4\alpha_7-\beta_2\alpha_8,$

$\varphi_{24} = \beta_6\alpha_3-\beta_8\alpha_1-\beta_7\alpha_3+\beta_8\alpha_4-\beta_3\alpha_7-\beta_1\alpha_8.$ (34)

Therefore, the following formula

$$ab = \left(\sum_{1\le j,k\le 4}\alpha_{kj}\varepsilon_{kj}\right)\left(\sum_{1\le i,l\le 4}\beta_{il}\varepsilon_{il}\right) = \sum_{1\le s,t\le 4}\varphi_{st}\varepsilon_{st}$$

is determined by the equation (34). By choosing suitable α_{kj} and β_{il} then we have the left-symmetric structure as stated in Proposition 1. For example if we fix $\alpha_1 = 1$ and $\beta_1 = 1$, then we have the product $\varepsilon_{11} * \varepsilon_{11} = \varepsilon_{11}^2 = -\varepsilon_{11}$.

■

As discussion, there is a still open problem for a generalization of left-symmetric structure of $\mathfrak{h}_2$ to left-symmetric structure of $\mathfrak{h}_{n,p} := \mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$. The Lie algebra $\mathfrak{h}_{n,p}$ is Frobenius Lie algebra whenever p is factor of n [1]. We offered an alternative proof to show $\mathfrak{h}_{n,p} := \mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ is Frobenius for $n = p = 2$ as explained before. Since $\mathfrak{h}_{n,p}$ is Frobenius Lie algebra, then there exists a Frobenius Functional. Therefore, we can construct a symplectic linear form which induces left-symmetric structure for $\mathfrak{h}_{n,p}$. Thus, we consider the following conjecture:

Conjecture 1. *The Lie algebra* $\mathfrak{h}_{n,p} := \mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ *where* p *devides* n *has left-symmetric structures. These structures are induced by a symplectic form of* $\mathfrak{h}_{n,p}$.

One of the interesting problems is how to find a Frobenius functional for $\mathfrak{h}_{n,p}$ corresponding to the Pfaffian of $\mathfrak{h}_{n,p}$ in order to give explicit formulas for left-symmetric structure for $\mathfrak{h}_{n,p}$.

Furthermore, if the Conjecture 12 is true, then the next question is how about left-symmetric structures for general Frobenius Lie algebras. We also proved that that all 4-dimensional Frobenius Lie algebra are left-symmetric algebras. As mentioned before that not all Lie algebras have left-symmetric structure. But we guess that Lie algebras of Frobenius types in general have left-symmetric structures. In other words, in more general case we consider the following conjecture

Conjecture 2. *Let* $\mathfrak{g}$ *be a finite dimensional Frobenius Lie algebra. Then* $\mathfrak{g}$ *is left-symmetric algebra whose left-symmetric structures are induced by a symplectic form corresponding to Frobenius functional of* $\mathfrak{g}$.

It would be interesting to study the completeness of left-symmetric algebra of $\mathfrak{h}_{n,p}$ whenever a Frobenius Lie algebra is equal to its radical and in general, the completeness of any finite dimensional Frobenius Lie algebra.

CONCLUSIONS

We showed that $\mathfrak{h}_2 \coloneqq \mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ is the 8-dimensional Frobenius Lie algebra. Furthermore, we proved the existence of left-symmetric structures on the Frobenius Lie algebra $\mathfrak{h}_2$ and we listed the explicit formulas of left-symmetric structures. Therefore, $\mathfrak{h}_2$ is left-symmetric algebra. Our construction based on the symplectic form corresponding to a Frobenius functional of $\mathfrak{h}_2$ which induced the left-symmetric structures on $\mathfrak{h}_2$. Our result can motivate the left-symmetric structure for $\mathfrak{h}_{n,p} := \mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ and for general case of Frobenius Lie algebras.

For future research, we stated Conjecture 12 and Conjecture 13 which are still open problem to be investigated. This is very interesting if Conjecture 12 and Conjecture 13 are true because we can study the radical of $\mathfrak{h}_{n,p}$, denoted by $\mathrm{Rad}(\mathfrak{h}_{n,p})$, and if we have $\mathfrak{h}_{n,p} = \mathrm{Rad}(\mathfrak{h}_{n,p})$, then we come to the notion of a completeness of left-symmetric algebra of $\mathfrak{h}_{n,p}$. Moreover, left-symmetric algebras relate affine structures and it is interesting to investigate for the Lie groups case[15] .

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