



## Levi Decomposition of Frobenius Lie Algebra of Dimension 6

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### ABSTRACT

In this paper, we study notion of the Frobenius Lie algebra  $M_{2,1}(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$  of dimension 6. The finite dimensional Lie algebra can be expressed in terms of decomposition between Levi subalgebra and the radical (maximal solvable ideal). This form of decomposition is called Levi decomposition. Our main object is further denoted by  $\text{aff}(2) := M_{2,1}(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ . The work aims to obtain Levi decomposition of Frobenius Lie algebra  $\text{aff}(2)$  of dimension 6. To obtained Levi subalgebra and the radical, we apply literature reviews about Lie algebra and decomposition Levi in Dagli result. The main result of this paper is Frobenius Lie algebra  $\text{aff}(2)$  can be decomposition be semisimple Levi subalgebra  $\mathfrak{h}$  of dimension 4 and radical solvable  $\text{Rad}(\mathfrak{g})$  of dimension 2. Thus, the Levi decomposition form of the Frobenius Lie algebra is given.

**Keywords:** frobenius lie algebra; levi decomposition; lie algebra; radical

### INTRODUCTION

A vector space over a field that is equipped by Lie brackets which is neither commutative nor associative is called Lie algebra [1]. Any finite dimensional Lie algebra can be expressed as semidirect sum between Levi subalgebra (Lie subalgebra) and its radical (maximal solvable ideal) and this form is called a Levi decomposition [1]. We denote a finite dimensional Lie algebra by  $\mathfrak{g}$ . On the other hand, for finite dimensional case, the Lie algebra  $\mathfrak{g}$  can be written in Levi decomposition form which is given in the following form

$$\mathfrak{g} = \mathfrak{h} \ltimes \text{Rad}(\mathfrak{g}) \quad (1)$$

where  $\mathfrak{h}$  is a Levi subalgebra of  $\mathfrak{g}$  and  $\text{Rad}(\mathfrak{g})$  is a radical or solvable maximal ideal of  $\mathfrak{g}$ . Let  $S = \{e_1, e_2, \dots, e_n\}$  be a basis of  $\mathfrak{g}$  and we define  $C(\mathfrak{g}) = (C(\mathfrak{g})_{i,j})$  be a matrix whose Lie brackets entries of  $\mathfrak{g}$  are given by

$$C(\mathfrak{g})_{i,j} := [e_i, e_j]_{\mathfrak{g}}, 1 \leq i, j \leq n. \quad (2)$$

This matrix  $C(\mathfrak{g}) \in \text{Mat}(n \times n, S(\mathfrak{g}))$  is called a structure matrix of  $\mathfrak{g}$  where  $S(\mathfrak{g})$  denotes as symmetric algebra of  $\mathfrak{g}$  [2]. The notion of Lie algebras has been widely studied. One of which is the investigation of Lie algebra with dimension 8 which can be carried out by Levi's decomposition [3]. Rais introduced the Lie algebra notion  $M_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$  where  $M_{n,p}(\mathbb{R})$  is a vector space of matrices of size  $n \times p$  with real number entries and  $\mathfrak{gl}_n(\mathbb{R})$  is the Lie algebra of a vector space of matrices of size  $n \times n$  equipped with Lie brackets [4]. Furthermore, we can see the notions of Lie algebra in [5] and [6].

Let  $\mathfrak{g}$  be a Lie algebra with  $\mathfrak{g}^*$  is a dual vector space of  $\mathfrak{g}$  where  $\mathfrak{g}^*$  consisting of real valued all linear functional on  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  is said to be a Frobenius Lie algebra if there exists a linear functional  $\varphi \in \mathfrak{g}^*$  so that the skew-symmetric bilinear form  $B_\varphi(x, y) := \varphi([x, y])$  is non degenerate. Many studies of Frobenius Lie algebras have been carried out over the years. For instance, the properties of principal elements on Frobenius Lie algebra one of them is Frobenius Lie algebra cannot be unimodular [7]. An example of Frobenius Lie algebra is the affine Lie algebra  $\text{aff}(2)$  can be seen in the classification of Frobenius Lie algebra with dimension less than or equal to 6 [8]. Kurniadi have constructed Frobenius Lie algebra with dimension less than or equal to 6 from non-commutative nilpotent Lie algebra with dimension less than or equal to 4 [9]. Other example of Frobenius Lie algebra, notation Lie algebra  $M_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$  where  $n = p = 3$  is Frobenius Lie algebra of dimension 18 [10]. Moreover, the Lie algebra  $M_3(\mathbb{R}) \rtimes \mathfrak{gl}_3(\mathbb{R})$  has quasi-associative algebra structure [11]. The Frobenius Lie algebra  $M_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$  is the left-symmetric algebra [12]. It has been proven that the affine Lie algebra that is denoted by  $\text{aff}(n) := \mathbb{R}^n \rtimes \mathfrak{gl}_n(\mathbb{R})$  is Frobenius Lie algebra where  $\mathbb{R}^n$  is another form of  $M_{n,1}(\mathbb{R})$  [13]. Readers can study more about Frobenius Lie algebra in the following articles: [14], [15], [16], and [17].

In this paper, we study about decompose Frobenius Lie algebra for special case  $n = 2$  of the affine Lie algebra  $\text{aff}(n)$ . The notion  $M_{2,1}(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$  can be written in simpler formulas as  $\mathbb{R}^2 \rtimes \mathfrak{gl}_2(\mathbb{R})$  and we can denote it by  $\text{aff}(2)$  which is known as the affine Lie algebra. In the nice formula, the affine Lie algebra  $\text{aff}(2) := \mathbb{R}^2 \rtimes \mathfrak{gl}_2(\mathbb{R})$  can be expressed in the form of a matrix

$$\text{aff}(2) := \left\{ \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}; X \in \mathfrak{gl}_2(\mathbb{R}), Y \in \mathbb{R}^2 \right\} \subseteq \mathfrak{gl}_3(\mathbb{R}) \quad (3)$$

where  $\mathfrak{gl}_3(\mathbb{R})$  is  $3 \times 3$  real matrix. The purpose of this research is to give decompose this Lie algebra into Levi subalgebra and radical.

## METHODS

We used literature study for the research method, especially the study of Frobenius Lie algebra  $\text{aff}(2)$  and about Levi decomposition of Lie algebra in [18]. First, we given an affine Lie algebra  $\text{aff}(2)$ . We proved the affine Lie algebra  $\text{aff}(2)$  not solvable. Then, it is proved that Lie algebra  $\text{aff}(2)$  can be decomposed into its subalgebra and radical.

Before going into the discussion, we would like to introduce the theoretical foundations used in this study as follows:

**Definition 1** [19] Let  $\mathfrak{g}$  be a vector space and a bilinear form

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto [x, y] \in \mathfrak{g}.$$

The bilinear form  $[\cdot, \cdot]$  is called a Lie bracket for  $\mathfrak{g}$  if the following conditions are satisfied:

1.  $[x, y] = -[y, x]; \forall x, y \in \mathfrak{g}$
2.  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0; \forall x, y, z \in \mathfrak{g}.$

The vector space  $\mathfrak{g}$  equipped by Lie brackets is called Lie algebra.

**Definition 2** [19] A linear subspace  $\mathfrak{h}$  of  $\mathfrak{g}$  is called a Lie sub-algebra if  $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ , we denote by  $\mathfrak{h} < \mathfrak{g}$ . If we have  $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ , we call  $\mathfrak{h}$  as an ideal of  $\mathfrak{g}$  and then write  $\mathfrak{h} \trianglelefteq \mathfrak{g}$ .

**Definition 3** [19] Let  $\mathfrak{g}$  be a Lie algebra. The derived series of  $\mathfrak{g}$  is defined by

$$D^0(\mathfrak{g}) = \mathfrak{g} \text{ and } D^n(\mathfrak{g}) = [D^{n-1}(\mathfrak{g}), D^{n-1}(\mathfrak{g})] \forall n \in \mathbb{N} \quad (4)$$

The Lie algebra  $\mathfrak{g}$  is said to be solvable, if there exists an  $n \in \mathbb{N}$  with  $D^n(\mathfrak{g}) = \{0\}$ .

**Theorem 1** Let  $\mathfrak{g}$  be Lie algebra then,

- i. If  $\mathfrak{g}$  is solvable then the subalgebras and homomorphic images of  $\mathfrak{g}$  are solvable.
- ii. If  $\mathfrak{h}$  is a solvable ideal of  $\mathfrak{g}$  and  $\mathfrak{g}/\mathfrak{h}$  is solvable, then  $\mathfrak{g}$  is solvable
- iii. If  $\mathfrak{h}$  and  $\mathfrak{i}$  are solvable ideals of  $\mathfrak{g}$  then  $\mathfrak{h} + \mathfrak{i}$  is also a solvable ideal of  $\mathfrak{g}$ .

This theorem shows that the sum of all solvable ideals of a Lie algebra is a solvable ideal. So, in every finite-dimensional Lie algebra  $\mathfrak{g}$ , there exists a maximal solvable ideal. This ideal is called the radical of  $\mathfrak{g}$  and denoted by  $Rad(\mathfrak{g})$ .

**Theorem 2** [18] *Let  $V$  be vector space over a field and let  $\mathfrak{g}$  be a subalgebra of  $gl(V)$ , the  $\mathfrak{g}$  is solvable if  $Tr(xy) = 0$  for all  $x \in \mathfrak{g}$  and  $y \in [\mathfrak{g}, \mathfrak{g}]$ .*

**Theorem 3** [18] *Let  $\mathfrak{g}$  be Lie algebra over a field  $\mathbb{F}$ , then*

$$Rad(\mathfrak{g}) = \{x \in \mathfrak{g} \mid Tr(ad x \cdot ad y) = 0\} \tag{5}$$

for all  $y \in [\mathfrak{g}, \mathfrak{g}]$ .

**Definition 4** [19] *Let  $\mathfrak{g}$  be a Lie algebra. If its radical is trivial i.e  $Rad(\mathfrak{g}) = \{0\}$  then  $\mathfrak{g}$  is called semisimple. The Lie algebra  $\mathfrak{g}$  is said to be simple if it is not abelian and if it contains no ideal other than  $\mathfrak{g}$  and  $\{0\}$ .*

**Definition 5** [1] *Let  $V$  be a space vector. A linear map  $\rho: V \rightarrow V$  is said to be endomorphism on  $V$  if the following condition satisfied:*

1.  $\rho(x + y) = \rho(x) + \rho(y)$
2.  $\rho(xy) = (\rho(x))y = x(\rho(y))$

for all  $x, y \in V$ . The set of all endomorphism on  $V$  is denoted by  $End(V)$ . Furthermore, the endomorphism  $End(V)$  equipped by Lie bracket  $[x, y] = xy - yx$  for all  $x, y \in End(V)$  is Lie algebra and it is called a general linear algebra, we denoted by  $gl(V)$ .

**Definition 6** [19] *Let  $\mathfrak{g}$  be a Lie algebra and  $x \in \mathfrak{g}$ . The map  $ad: \mathfrak{g} \rightarrow \mathfrak{g}$  defined by*

$$ad x : \mathfrak{g} \in y \mapsto ad x(y) = [x, y] \in \mathfrak{g} \tag{6}$$

is a derivation. The map  $ad: \mathfrak{g} \rightarrow gl(\mathfrak{g})$  is called an adjoint representation.

Let a representation of Lie algebra  $\mathfrak{g}$  in the dual vector space  $\mathfrak{g}^*$  is denoted by  $ad^*$  whose value on  $\mathfrak{g}$  is defined by

$$\langle ad^*(x)\varphi, y \rangle = \langle \varphi, ad^*(-x)y \rangle = \langle \varphi, [y, x] \rangle \tag{7}$$

for  $\varphi \in \mathfrak{g}^*$ , for all  $x, y \in \mathfrak{g}$ .

A stabilizer of Lie algebra  $\mathfrak{g}$  at the point  $\varphi \in \mathfrak{g}^*$  is given in the following form:

$$\mathfrak{g}^\varphi = \{x \in \mathfrak{g} \mid ad^*(x)\varphi = 0\} \tag{8}$$

**Definition 7** [20] *Let  $\mathfrak{g}$  be a Lie algebra whose  $\mathfrak{g}^*$  be a dual vector space of  $\mathfrak{g}$ . A Lie algebra  $\mathfrak{g}$  is said to be Frobenius Lie algebra if there exist linear functional  $\varphi \in \mathfrak{g}^*$  such that the stabilizer of  $\mathfrak{g}$  on  $\varphi$  is equal to 0.*

Furthermore, we review briefly some basic notations needed in Levi decomposition. We explain Levi's theorem which states that a finite dimensional Lie algebra can be expressed as the semidirect sum of the Levi subalgebra and the radical.

**Theorem 4** [18] *Let  $\mathfrak{g}$  be a Lie algebra and let  $\mathfrak{g}$  be not solvable, then  $\mathfrak{g}/Rad(\mathfrak{g})$  is a semisimple Lie subalgebra.*

**Theorem 5** [18] *Let  $\mathfrak{g}$  be a finite dimensional Lie algebra. If  $\mathfrak{g}$  is not solvable, then there is a semisimple subalgebra  $\mathfrak{s}$  of  $\mathfrak{g}$  such that*

$$\mathfrak{g} = \mathfrak{s} \oplus Rad(\mathfrak{g}). \tag{9}$$

In this decomposition,  $\mathfrak{s} \cong \mathfrak{g}/Rad(\mathfrak{g})$  and we have commutation relations as follows

$$[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}, [\mathfrak{s}, Rad(\mathfrak{g})] \subseteq Rad(\mathfrak{g}), [Rad(\mathfrak{g}), Rad(\mathfrak{g})] \subseteq Rad(\mathfrak{g}). \tag{10}$$

The example of Levi decomposition can be seen in the work of [18], one of all example its Levi decomposition as follows

**Example 1** [18] *Let  $\mathfrak{g}$  be a Lie algebra spanned by*

$$\left\{ x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad (11)$$

where Lie bracket non-zero is

$$\begin{aligned} [x_1, x_2] &= 4x_1 - 4x_3 - 2x_4, & [x_1, x_3] &= x_1 - x_2 - x_4, \\ [x_2, x_3] &= -2x_1 + 2x_3 + 2x_4, & [x_2, x_4] &= -4x_1 + 4x_3 + 2x_4, \\ [x_3, x_4] &= -x_1 + x_2 + x_4. \end{aligned}$$

The Lie algebra  $\mathfrak{g}$  can be express in  $\mathfrak{g} = \text{Rad}(\mathfrak{g}) \rtimes \mathfrak{h}$  where radical  $\text{Rad}(\mathfrak{g}) = \text{span} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$  and Levi subalgebra  $\mathfrak{h}$  is spanned by  $\left\{ z_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ .

## RESULTS AND DISCUSSION

In this section, let  $\text{aff}(2)$  be the affine Lie algebra and let  $\text{aff}(2)$  be realized in the following matrix form

$$\text{aff}(2) = \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d, x, y \in \mathbb{R} \right\} \subseteq \mathfrak{gl}_3(\mathbb{R}), \quad (12)$$

with the standard basis for  $\text{aff}(2)$ , we have

$$S = \left\{ x_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right. \\ \left. x_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \quad (13)$$

The Lie brackets for the affine Lie algebra  $\text{aff}(2)$  is defined by  $[a, b] = ab - ba, \forall a, b \in \text{aff}(2)$  such that the non-zero Lie brackets for the affine Lie algebra  $\text{aff}(2)$  as follows

$$\begin{aligned} [x_1, x_2] &= x_2, & [x_1, x_3] &= -x_3, & [x_1, x_5] &= x_5, \\ [x_2, x_3] &= x_1 - x_4, & [x_2, x_4] &= x_2, & [x_2, x_6] &= x_5, \\ [x_3, x_4] &= -x_3, & [x_3, x_5] &= x_6, & [x_4, x_6] &= x_6. \end{aligned} \quad (14)$$

**Theorem 5** [8] *Let  $\text{aff}(2)$  be a Lie algebra of dimension 6 with basis in the equation (13). Let  $\text{aff}(2)^*$  be its dual vector space of  $\text{aff}(2)$ . Then there exist a linear functional  $\varphi = x_2^* + x_6^* \in \text{aff}(2)^*$  such that  $\mathfrak{g}^\varphi = \{0\}$ . Therefore, the affine Lie algebra  $\text{aff}(2)$  is Frobenius.*

In this section of the discussion is our main result, we will prove the Proposition 1 and the Proposition 2 as follows.

**Proposition 1.** *The affine Lie algebra  $\text{aff}(2)$  is not solvable.*

**Proof.** We have that

$$D^1(\text{aff}(2)) = [D(\text{aff}(2)), D(\text{aff}(2))] = \text{span}\{x_1 - x_4, x_2, x_3, x_5, x_6\}.$$

Next, we compute  $D^2(\text{aff}(2))$  also obtained

$$D^2(\text{aff}(2)) = [D^1(\text{aff}(2)), D^1(\text{aff}(2))] = \text{span}\{x_1 - x_4, x_2, x_3, x_5, x_6\}.$$

Therefore, there not exist  $n > 0$  that causes  $D^n(\text{aff}(2)) = \{0\}$ . Thus, the affine Lie algebra  $\text{aff}(2)$  is not solvable. ■

**Proposition 2.** *Let  $\text{aff}(2)$  be Frobenius affine Lie algebra whose basis  $S = \{x_i\}_{i=1}^6$  where the non-zero brackets for  $\text{aff}(2)$  in the equation (14). The affine Lie algebra  $\text{aff}(2)$  is not*

solvable then there exist  $\mathfrak{h} = \text{span}\{x_1, x_2 + x_5 + x_6, x_3 + x_5 + x_6, x_4\}$  is the semisimple Lie subalgebra of  $\text{aff}(2)$  and  $\text{Rad}(\text{aff}(2)) = \text{span}\{x_5, x_6\}$  is the radical of  $\text{aff}(2)$  such that

$$\text{aff}(2) = \text{span}\{x_5, x_6\} \rtimes \text{span}\{x_1, x_2 + x_5 + x_6, x_3 + x_5 + x_6, x_4\}. \quad (15)$$

**Proof.** Firstly, we have the structure matrix of  $\text{aff}(2)$  is

$$C(\text{aff}(2)) = \begin{pmatrix} 0 & x_2 & -x_3 & 0 & x_5 & 0 \\ -x_2 & 0 & x_1 - x_4 & x_2 & 0 & x_5 \\ x_3 & x_4 - x_1 & 0 & -x_3 & x_6 & 0 \\ 0 & -x_2 & x_3 & 0 & 0 & x_6 \\ -x_5 & 0 & -x_6 & 0 & 0 & 0 \\ 0 & -x_5 & 0 & -x_6 & 0 & 0 \end{pmatrix}. \quad (16)$$

Then, we find the maximal linearly independent set in the structure matrix such that basis

$$B = \{y_1 = x_1 - x_4, y_2 = x_2, y_3 = x_3, y_4 = x_5, y_5 = x_6\}$$

of the product space  $[\text{aff}(2), \text{aff}(2)]$ . Next, calculate  $\text{ad } x_i$  and  $\text{ad } y_j$  for  $1 \leq i \leq 6, 1 \leq j \leq 5$ , we get

$$\begin{aligned} \text{ad } x_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ad } x_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \text{ad } x_3 &= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ad } x_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\ \text{ad } x_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \text{ad } x_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \\ \text{ad } y_1 &= \text{ad } x_1 - x_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned} \quad (17)$$

Furthermore, compute the radical of  $\text{aff}(2)$  where  $x = \sum_{i=1}^6 \alpha_i x_i \in \text{Rad}(\text{aff}(2))$ , then find value  $\alpha_i$  using equations (17), we have

$$\sum_{i=1}^6 \alpha_i \text{Tr}(\text{ad } x_i \cdot \text{ad } y_j) = 0; 1 < j \leq 5$$

$$\alpha_1 \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (18)$$

Next, we solve linear equations (18) such we find that

$$\alpha_1 - \alpha_4 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_5 = s, \alpha_6 = t,$$

and with  $\alpha_1 = 0$ , then we get

$$x = \sum_{i=1}^6 \alpha_i x_i = 0x_1 + 0x_2 + 0x_3 + 0x_4 + sx_5 + tx_6 = sx_5 + tx_6.$$

Therefore, we obtain the radical of  $\text{aff}(2)$  is

$$\text{Rad}(\text{aff}(2)) = \text{span}\{x_5, x_6\} = \text{span}\{r_1, r_2\}. \tag{19}$$

Next, we find basis Levi subalgebra of  $\text{aff}(2)$ . In this cases,  $\text{Rad}(\text{aff}(2))$  is abelian because  $[r_i, r_j] = 0$  for all  $1 \leq i, j \leq 2$ . Complement on  $\text{aff}(2)$  respect to  $\text{Rad}(\text{aff}(2))$  spanned by  $\{x_1, x_2, x_3, x_4\}$ . The quotient algebra  $\text{aff}(2)/\text{Rad}(\text{aff}(2))$  is spanned by  $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$  and we have its brackets as follows

$$[\bar{x}_1, \bar{x}_2] = \bar{x}_2, [\bar{x}_1, \bar{x}_3] = -\bar{x}_3, [\bar{x}_2, \bar{x}_3] = \bar{x}_1 - \bar{x}_4, [\bar{x}_2, \bar{x}_4] = \bar{x}_2, [\bar{x}_3, \bar{x}_4] = -\bar{x}_3. \tag{20}$$

We set Levi subalgebra spanned by

$$\begin{aligned} z_1 &= x_1 + \alpha_1 r_1 + \alpha_2 r_2, \\ z_2 &= x_2 + \beta_1 r_1 + \beta_2 r_2, \\ z_3 &= x_3 + \gamma_1 r_1 + \gamma_2 r_2, \\ z_4 &= x_4 + \delta_1 r_1 + \delta_2 r_2. \end{aligned} \tag{21}$$

Next, we calculate to determine the four unknown  $\alpha, \beta, \gamma, \delta$  such that  $z_1, z_2, z_3, z_4$  span a semisimple Lie algebra that is isomorphic to  $\text{aff}(2)/\text{Rad}(\text{aff}(2))$ . Since  $z_1, z_2, z_3, z_4$  have the same commutation relations as  $\bar{x}_i, 1 \leq i \leq 4$  written in equations (20), we then get

$$[z_1, z_2] = z_2, [z_1, z_3] = -z_3, [z_2, z_3] = z_1 - z_4, [z_2, z_4] = z_2, [z_3, z_4] = -z_3. \tag{22}$$

We substitution the equation (22) onto (23) such that equation can be written as

$$[x_1 + \alpha_1 r_1 + \alpha_2 r_2, x_2 + \beta_1 r_1 + \beta_2 r_2] = x_2 + \beta_1 r_1 + \beta_2 r_2, \tag{23}$$

$$[x_1 + \alpha_1 r_1 + \alpha_2 r_2, x_3 + \gamma_1 r_1 + \gamma_2 r_2] = -(x_3 + \gamma_1 r_1 + \gamma_2 r_2), \tag{24}$$

$$[x_2 + \sum_{j=1}^2 \beta_j r_j, x_3 + \sum_{j=1}^2 \gamma_j r_j] = (x_1 + \sum_{j=1}^2 \alpha_j r_j) - (x_4 + \sum_{j=1}^2 \delta_j r_j) \tag{25}$$

$$[x_2 + \beta_1 r_1 + \beta_2 r_2, x_4 + \delta_1 r_1 + \delta_2 r_2] = x_2 + \beta_1 r_1 + \beta_2 r_2, \tag{26}$$

$$[x_3 + \gamma_1 r_1 + \gamma_2 r_2, x_4 + \delta_1 r_1 + \delta_2 r_2] = -(x_3 + \gamma_1 r_1 + \gamma_2 r_2). \tag{27}$$

Then, we apply the equations (23), (24), (25), (26), and (27) to compute  $\alpha_i, \beta_i, \gamma_i, \delta_i, 1 \leq i \leq 2$ . From equations (23) and (26) obtained that  $\beta_1 = \beta_2 = 1$ . From equations (24) and (27) obtained that  $\gamma_1 = \gamma_2 = 1$ . For equations (25), we obtained that  $\alpha_1 - \delta_1 = \alpha_2 - \delta_2 = 0$  and let  $\alpha_i = 0$ , such that we have

$$z_1 = x_1, z_2 = x_2 + r_1 + r_2, z_3 = x_3 + r_1 + r_2, z_4 = x_4.$$

Thus, the Levi subalgebra spanned by

$$\left\{ z_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{28}$$

■

## CONCLUSIONS

It has been proven in Proposition 2 that the affine Lie algebra  $\text{aff}(2)$  can be decomposed into its subalgebra and radicals which written in the equations (15). From our result of this paper, other research can study about decomposition of the general formula affine Lie algebra  $\text{aff}(n)$  of dimension  $n(n + 1)$ . For future research, the decomposition process can be expanded from the decomposition result of  $\text{aff}(n)$  in its radical and Levi subalgebra form such that we can find structure Frobenius Lie algebra  $\text{aff}(n)$  of its decomposition.

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