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C-Type Ops Transformation

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ABSTRACT

The scope of the ops transformation is limited to the McLaurin series only. While there are still many cases in mathematical modeling which are modeled in the form of a more general series. The purpose of this study is to generalize the ops transformation into a more general form that can be used for any Taylor series. This study uses a literature study method, namely by reviewing the ops transformation and then observing aspects that can be generalized. Next is to construct a more general definition of ops transformation which is referred to as c-type ops transformation. At the end of this research, the ops transformation will be applied to solve ordinary differential equations with variable coefficients very briefly.

Keywords: C-Type ops transformation; ops transformation; power series

INTRODUCTION

Taylor series is accessible to all students and it is a useful mathematical tool to nonlinear equations [1]. Power series is an essensial method for solving many problems in mathematics such as algebra and differential equations [2]. Many cases also appear in algebra which is involving power series such as solving polynomial homotopies equations [3]. As we know on algebraic geometry topics we talk about rings, ring extensions and ideals which recently appears as series forms. For example, is on commutative ring topics. If R be a commutative ring with identity. Let R[x] and R[[x]] be the collection of polynomials and, respectively, of power series with coefficients in R. Their multiplications are from a class of sequences $\lambda = \{\lambda_n\}$ of positive integers which is related one-one correspondence to its power series [4].

Another branch of mathematics which also involved by power series recently is differential equations[5]. The problem of finding formal power series solutions of differential equations has a long history and it has been extensively studied in literature. Using power series method however, is a more systematic way and standard basic method for approximating the solutions of such differential equations analytically and thus studying the method is of greater importance [6]. Ordinary Power series has also appear as solutions for fractional differential equations as told by Angstmann and Henry in their publication namely Generalized series expansions involving integer powers and fractional powers in the independent variable have recently been shown to provide solutions to certain linear fractional order differential equations [7]. This incredible discovery is related to I. Area and J. Nieto who discover the solution for fractional logistics equation which is appeared as power series form [8].

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This research discovers new theory of ordinary power series expecially the alternating method to analyze ordinary power series by considering ordinary power series as a transformation which called ops transformation. This transformation will simplify the counting process of some equations involving sigma notation. Reducing the use of sigma algebra, alternating it by algebra of linear transformation.

The aim of this research is to generalize the concept of ops transformation for power series of form (1). We will name it as c-type ops transformation. The letter c is indicated the center of the power series.

METHODS

We will proceed this reseach through the following procedures. First we will define the more general form of ops transforamtion. After that we will find some basic results of our new definition. We will use such results to extend the theory of ops transformation. Next we will make sure that our previous definition of ops transformation to become special case for our new definition. Next we will find some theorems regarding our new transformation properties. We also will make sure that our new definition is able to be applied much wider than our previous one.

RESULTS AND DISCUSSION

First we shall define the c-Type of ops transformation. Recall that power series centered at c is defined to be the real valued function of the form [9]

$$\sum_{n=0}^{\infty} a_n (x-c)^n. \qquad (1)$$

The series has a value depending on what value of x we choose. Some value of x will result the series tend to infinity. Some other will result the series converges[10]. The set of x which result (1) converges is called convergence interval. The term "interval" makes sense because such set always forms an interval. The half-length of such interval is called convergence radius.

Some smooth function f(x) at point c is able to be approximated by power series which on some value of x_0 lies on its convergence interval centered at c, the result will be same, i.e f(x_0) = $\sum_{n=0}^{\infty} a_n(x_0 - c)^n$. Such functions are called "real analytic" functions. The method of resulting such series was given by Taylor which is called Taylor series as below [11]

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n. \qquad (2)$$

Special case of Taylor series is when the value of c = 0, the series is called McLaurin series. Lazwardi (2021) has already able to reformulate such series into more simple form called ops transformation. The ops transformation is defined as below

$$Ops(\{a_n\})(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Therefore, for some McLaurin series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \qquad (4)$$

its enough to write the series simply as $Ops\left\{\frac{f^{(n)}(0)}{n!}\right\}$. Simplification of the form will make calculations and manipulations easier [12]. There are some properties regarding ops transformation as following:

Theorem 1. (shifting-entry) For each $\{a_n\}$ sequence, we have

$$Ops\{0, a_0, a_1, \dots\} = xOps\{a_n\}. \tag{5}$$

Theorem 2. For each $\{a_n\}$ sequence, we have

$$Ops(\{a_n\}) - a_0 = xOps(\{a_{n+1}\}). \tag{6}$$

Beside two above theorems, ops transformation inherits linearity properties as well as sigma notations.

Theorem 3. For each $\{a_n\}, \{b_n\}$ sequences and any real numbers α, β , we have

$$Ops(\alpha\{a_n\} + \beta\{b_n\}) = \alpha Ops\{a_n\} + \beta Ops\{b_n\}. \tag{7}$$

The last theorem notices us that we can view ops transformation as linear transformation which mapping from the space of all real sequences to real numbers on its convergence radius. We shall use this necessary fact to simplify several calculations.

Besides that Lazwardi was able to prove the formula regarding product of two ops transformations as following.

Theorem 4. For each $\{a_n\}, \{b_n\}$ sequences, we have

$$Ops\{a_n\}Ops\{b_n\} = Ops\left\{\sum_{k=0}^{n} a_k b_{n-k}\right\}. \tag{8}$$

This is just similiar with the product two power series but wihout involving double sigma notation. Another important result of previous research is we can use the fact that the power series is always able to differentiate n-times, to construct the rule of differentiation for ops transformation.
Talking about differentiation of ops transformation meas we have to state the symbol for its derivative. We use $D_x Ops\{a_n\}$ to notate the derivative of ops transformation on its radius convergence. Therefore we have the following theorems.

Theorem 5. For each $\{a_n\}$ sequence, we have

$$D_x Ops(\{a_n\}) = Ops(\{(n+1)a_{n+1}\}). \tag{9}$$

Here is some nice modification formula
Theorem 6. For each $\{a_n\}$ sequence, we have

$$xD_x Ops(\{a_n\}) = Ops(\{na_n\}). \tag{10}$$

If we pay more attention to the (1). There are some difference between (1) and (3) i.e the value of c will be varied and able to consider it as a variable. Therefore we have at least 4 variable involved in calculations of (1) which is more complicated than (3) expecially special type of ordinary power series which called Taylor series. As told by Salwa in her research that many infinite series form are recently appear in sequence spaces expecially on $\beta - dual$ sequence spaces which is defined as infinite series form [13] One of popular application from Taylor series is the iterative method of the differential transform methor has already been used for a while, by the "Traditional" Taylor series method users which have even better developed the method.

Suppose that we have power series of form (1) centered at c. We define
Definition 1. Let c be a real number and suppose power series $\sum a_n(x-c)^n$ has positive convergence radius near c. Define the c-type ops transformation as following

$$Ops^c\{a_n\}(x) = \sum_{n=0}^{\infty} a_n\,(x-c)^n. \tag{11}$$

It looks similiar to the previous form with additional superscript c. Note that the additional superscript c on *Ops* roles as index depending on value c on the right side. For some reason, we shall keep c to become upper index because we shall use lower index with another use on the next research.

Recall that one of the most suitable form which is similiar to our last definition is Taylor series of analytic function on c. If $f(x)$ is an analytic function near c, then we can write f as Taylor series on some neighborhood c as below

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n. \tag{12}$$

Hence, Taylor series of f can be written as c-type ops transformation as

$$f(x) = Ops^c\left\{\frac{f^{(n)}(c)}{n!}\right\}(x). \tag{13}$$

For some reason, we just write

$$f = Ops^c\left\{\frac{f^{(n)}(c)}{n!}\right\}. \tag{14}$$

Please pay more attention here. Upper index c is viewed as variable (not necessary fixed). We can write

$$Ops^a\left\{\frac{f^{(n)}(c)}{n!}\right\} = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

i.e when we change the value of upper index c by a, the value of $\frac{f^{(n)}(c)}{n!}$ doesn't change but the center of power series on the right side changes to a.

Its clear that ops transformation is a special case of c-type of ops transformation by taking value c = 0 [14], i.e

$$Ops^0\{a_n\} = Ops\{a_n\}. \tag{15}$$

For more brief information. We shall discuss some more examples as following.

Example 1. $Ops^c\{1\} = \frac{1}{1-(x-c)}$.

Proof: Observe that $Ops^c\{1\} = \sum(x-c)^n$. Suppose that $y = x - c$ then the right side of equation become $\sum y^n = \frac{1}{1-y}$ for $|y| < 1$. Hence we have for $|x-c| < 1$ or $c - 1 < x < c + 1$ we will get the series $\sum(x-c)^n$ will converge and we have $Ops^c\{1\} = \frac{1}{1-(x-c)}$.

Here is another example

Example 2. $Ops^c\left\{\frac{1}{n!}\right\} = e^{x-c}$.

Now we shall analyze more properties of c-type ops transformation. First we success to preserve the "shifting index" properties as well as previous form [15]

Theorem 7. (shifting-entry) For each $\{a_n\}$ sequence, we have

$$Ops^c\{0, a_0, a_1, \dots\} = (x-c)Ops^c\{a_n\}. \tag{16}$$

Proof: Observe that

$$\begin{aligned} Ops^c\{0, a_0, a_1, \dots\} &= 0 + \sum_{n=1}^{\infty} a_{n-1}\,(x-c)^n \\ &= \sum_{n=0}^{\infty} a_n (x-c)^{n+1} \\ &= (x-c)\sum_{n=0}^{\infty} a_n\,(x-c)^n \\ &= (x-c)Ops^c\{a_n\} \end{aligned}$$

■

Hence by induction we can conclude as corollary below

Corollary 1. For each $\{a_n\}$ sequence, we have

$$Ops^c\left\{\underbrace{0,0,0,\ldots,0}_{k-entries}\, a_0, a_1, \dots\right\} = (x-c)^k Ops^c\{a_n\}. \tag{17}$$

Proof: For $n = 1$ the statement is true due to theorem 7. Lets assume for $n = k$, the statement is also true, i.e (17) holds. For $n = k+1$, we have

$$\begin{aligned} Ops^c\left\{\underbrace{0,0,0,\ldots,0}_{k+1-entries}\, a_0, a_1, \dots\right\} &= (x-c)\, Ops^c\left\{\underbrace{0,0,0,\ldots,0}_{k-entries}\, a_0, a_1, \dots\right\} \\ &= (x-c)((x-c)^k Ops^c\{a_n\}) \\ &= (x-c)^{k+1} Ops^c\{a_n\} \end{aligned}$$

■

Theorem 8. For each $\{a_n\}$ sequence, we have

(18)

$$Ops^{c}(\{a_n\}) - a_0 = (x-c)Ops^{c}\{a_{n+1}\}.$$

Proof: Observe that

$$\begin{aligned} Ops^{c}(\{a_n\}) - a_0 &= \sum_{n=1}^{\infty} a_n (x-c)^n \\ &= \sum_{n=0}^{\infty} a_{n+1} (x-c)^{n+1} \\ &= (x-c)\sum_{n=0}^{\infty} a_{n+1} (x-c)^n \\ &= (x-c)Ops^{c}\{a_{n+1}\} \end{aligned}$$ ■

Fortunately we also sucess to keep linearity properties of ops transformation[16].

Theorem 9. For each $\{a_n\},\{b_n\}$ sequences and any real numbers α, β, we have

$$Ops^{c}(\alpha\{a_n\} + \beta\{b_n\}) = \alpha Ops^{c}\{a_n\} + \beta Ops^{c}\{b_n\}. \tag{19}$$

Proof: Lets observe

$$\begin{aligned} Ops^{c}(\alpha\{a_n\} + \beta\{b_n\}) &= \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-c)^n \\ &= \alpha \sum_{n=0}^{\infty} a_n (x-c)^n + \beta \sum_{n=0}^{\infty} a_n (x-c)^n \\ &= \alpha Ops^{c}\{a_n\} + \beta Ops^{c}\{b_n\} \end{aligned}$$ ■

Although we success to prove linearity of ops transformation, but unfortunately that linearity of upper index, i.e

$$Ops^{\alpha c+\beta d}\{a_n\} \neq Ops^{\alpha c}\{a_n\} + Ops^{\beta d}\{b_n\}.$$

Next we shall observe properties of c-type ops transformation for product of two power series. Its still works similarily as previous result.

Theorem 10. For each $\{a_n\},\{b_n\}$ sequences, we have

$$Ops^{c}\{a_n\}Ops^{c}\{b_n\} = Ops^{c}\left\{\sum_{k=0}^{n} a_k b_{n-k}\right\}. \tag{20}$$

Proof: Let $\{a_n\},\{b_n\}$ any two sequences, we have

$$\begin{aligned}
Ops^c\{a_n\}Ops^c\{b_n\} &= (a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots)(b_0 + b_1(x-c) + b_2(x-c)^2 + \ldots) \\
&= (a_0(b_0 + b_1(x-c) + \ldots) + a_1(x-c)(b_0 + b_1(x-c) + \ldots) + \ldots \\
&= a_0b_0 + a_0b_1(x-c) + a_1b_0(x-c) + a_0b_2(x-c)^2 + .a_1b_1(x-c)^2 \ldots \\
&= a_0b_0 + (a_0b_1 + a_1b_0)(x-c) + (a_0b_2 + a_1b_1 + a_2b_0)(x-c)^2 + \ldots \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(x-c)^n \\
&= Ops^c\left\{\sum_{k=0}^{n} a_k b_{n-k}\right\} \qquad \blacksquare
\end{aligned}$$

From above theorem, we can conclude easily the following fact

Example 3. $(Ops^c\{1\})^2 = Ops^c\{n+1\}$.

Translating the notation into sigma notation we get

$$\sum_{n=0}^{\infty}(n+1)(x-c)^n = \left(\frac{1}{1-(x-c)}\right)^2$$

From example 1 we can observe that how c-type ops transformations helps us to calculate, or manipulate some power series.
Here is another example

Example 4. $\left(Ops^c\left\{\frac{1}{n!}\right\}\right)^{\mathbf{2}} = Ops^c\left\{\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\right\}$.
Therefore we concluce that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{1}{k!\,(n-k)!}\right)(x-c)^n = e^{2x-2c}$$

Hence we can take another form of e^x as following equation

$$e^x = \left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{e^c}{k!\,(n-k)!}\right)(x-c)^n\right)^{\frac{1}{2}}$$

As for last discussion we shall observe how c-type ops transformation properties when we take its derivatives. We still use $D_x Ops^c\{a_n\}$ to notate the derivative of ops transformation on its radius convergence.

Consider the fact that the form $x-c$ has the same derivative with *x* itself [17]. Therefore we still able to adapt the formula for derivative of previous ops transformation as below

Theorem 11. For each $\{a_n\}$ sequence, we have

$$D_x Ops^c(\{a_n\}) = Ops^c(\{(n+1)a_{n+1}\}). \tag{21}$$

Proof: Observe that

$$\begin{aligned} D_x Ops^c(\{a_n\}) &= \frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-c)^n \\ &= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-c)^n \\ &= Ops^c\{(n+1)a_{n+1}\} \end{aligned}$$

■

Trivialy we also can conclude the next modification theorem

Theorem 12. For each $\{a_n\}$ sequence, we have

$$(x-c) D_x Ops^c(\{a_n\}) = Ops^c\{n a_n\}. \tag{22}$$

Proof: Observe that

$$\begin{aligned} (x-c) D_x Ops^c(\{a_n\}) &= (x-c) Ops^c\{(n+1)a_{n+1}\} \\ &= (x-c)\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-c)^n \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-c)^{n+1} \\ &= \sum_{n=0}^{\infty} n a_n (x-c)^n \\ &= Ops^c\{n a_n\} \end{aligned}$$

From the last two theorems, we also can inductively conclude the following two corollaries.

Corollary 2. For each $\{a_n\}$ sequence, we have

$$D_x^k Ops^c\{a_n\} = Ops^c\left\{\frac{(n+k)!}{n!} a_{n+k}\right\}. \tag{23}$$

where D_x^k is kth-derivative of ops transformation [15].

Corollary 3. For each $\{a_n\}$ sequence, we have

$$Ops^c\{n^k a_n\} = \underbrace{(x-c)D_x\big((x-c)D_x(\dots)\big)(x-c)D_x Ops^c\{a_n\}.}_{k-times} \tag{24}$$

At the end of this research, we will try our transformation to solve some ordinary differential equation with variable coefficient. This solution must be exist due to [18] For example the equation

$$y''-2(x-1)y'+2y=0 \tag{25}$$

We will find the solution of (25) near c = 1 as following:

Step 1: Assume the solution has the form $y=\sum_{n=0}^{\infty} C_n (x-1)^n$.

Step 2: Transform (25) to ops transformation equation.

$$D_x^2 Ops^1\{C_n\}-2(x-1)D_x Ops^1\{C_n\}+2Ops^1\{C_n\}=Ops^1\{0\}$$

Step 3: Solve the equation

$$\begin{aligned} D_x^2 Ops^c\{C_n\}-2(x-1)D_x Ops\{C_n\}+2Ops\{C_n\} &= Ops^1\{(n+2)(n+1)C_{n+2}\}-Ops^1\{2nC_n\} \\ &\quad +Ops^1\{2C_n\} \\ &= Ops^1\{(n+2)(n+1)C_{n+2}-2nC_n+2C_n\} \\ &= Ops^1\{0\} \end{aligned}$$

Step 4: Remove ops transformation from the equation, we have

$$(n+2)(n+1)C_{n+2}=(2n-2)C_n \quad \text{for} \quad n=0,1,2,\ldots$$

Step 5: By evaluating n one by one, we have the solution

$$y=C_0\left(1-(x-1)^2-\frac{1}{6}(x-1)^4+\cdots\right)+C_1(x-1)$$

CONCLUSIONS

Based on the discussion above, it can be concluded that c-type ops transformation is a generalization of the ordinary ops transformation with additional c as the upper index. All properties of ordinary ops transformations can still apply in c-type ops transformations. The c-type ops transformation can also be applied to solve ordinary differential equations for variable coefficients.

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