



# On Irregular Colorings of Unicyclic Graph Family

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## ABSTRACT

An Irregular coloring is a proper coloring where each vertex on a graph must have a different code. The color code of a vertex  $v$  is  $code(v) = (a_0, a_1, a_2, \dots, a_k)$  where  $a_0 = c(v)$  and  $a_i = i, 1 \leq i \leq k$  is the number of vertices that are adjacent to  $v$  and colored  $i$ . The minimum  $k$ -color used in irregular coloring is called the irregular chromatic number and denoted by  $\chi_{ir}$ . The type of research used in this research is exploratory research. In this paper, we discuss the irregular chromatic number of the bull graph, pan graph, sun graph, peach graph, and caveman graph.

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**Keywords:** irregular coloring; irregular chromatic number; unicyclic graph

## INTRODUCTION

A graph  $G = (V, E)$  is a pair of a vertex set denoted by  $V(G)$  and an edge set (may be empty) denoted by  $E(G)$ . More detail definitions and some properties can be seen in [1]. In that year, a four-color theorem which discusses maps coloring was discovered. This theorem states that there is a separation of a plane into adjacent areas resulting in an image called a map. The maximum colors which is needed to color the area on the map so that two neighboring areas do not have the same color is four [2].

$$code(v) = (a_0, a_1, a_2, \dots, a_k)$$

where  $a_0 = c(v)$  and  $a_i = i, (1 \leq i \leq k)$  is the number of vertices that are adjacent to  $v$ , and colored  $v$ , where  $c(v)$  is the color at vertex  $v$ . The minimum color used in irregular coloring is called the irregular chromatic number and denoted by  $\chi_{ir}$  [2]. An Irregular coloring is included proper coloring of  $G$ , it follows that [2]

$$\chi(G) \leq \chi_{ir}(G).$$

The following are the theorems, corollaries, and observations that will be used in this paper.

**Lemma 1.** [2] For every pair  $a, b$  of integers with  $2 \leq a \leq b$ , there is a connected graph  $G$  with  $\chi(G) = a$  and  $\chi_{ir}(G) = b$ .

**Corollary 1.** [2] For every graph  $G$ ,  $\omega(G) \leq \chi(G)$

The clique number  $\omega(G)$  of a graph  $G$  is the maximum order of a complete subgraph of  $G$ .

**Observation 1.**[2] Let  $c$  be a (proper) vertex coloring of a nontrivial graph  $G$  and let  $u$  and  $v$  be two distinct vertices of  $G$ .

- If  $c(u) \neq c(v)$ , then  $code(u) \neq code(v)$
- If  $deg_G u \neq deg_G v$ , then  $code(u) \neq code(v)$
- If  $c$  is irregular coloring and  $N(u) = N(v)$ ,  $c(u) \neq c(v)$

**Lemma 2.** [3] Let  $c$  be an irregular  $k$ -coloring of the vertices of a graph  $G$ . The number of different possible color codes of the vertices of degree  $r$  in  $G$  is

$$k \binom{r + (k + 1) - 1}{r} = k \binom{r + k - 2}{r}$$

**Corollary 2.** [3] If  $c$  is an irregular  $k$ -coloring of a nontrivial connected graph  $G$ , then  $G$  contains at most  $k \binom{r + k - 2}{r}$  vertices of degree  $r$ .

**Corollary 3.** [3] If  $\chi_{ir}(G) = k$  where  $n \geq 3$ , then  $n \leq \binom{k}{2} = \frac{k^2(k-1)}{2}$

**Corollary 4.**[3] Let  $k \geq 3$  be an integer,  $\chi_{ir}(C_n) \geq k$  for all integers  $n$ . Such that

$$\frac{(k - 1)^2(k - 2) + 2}{2} \leq n \leq \frac{k^2(k - 1)}{2}$$

**Proposition 1.**[3]  $\chi_{ir}(C_n) = 4$  if  $n$  is even and  $\chi_{ir}(C_n) = 3$  if  $n$  is odd for  $3 \leq n \leq 9$ .

**Lemma 3.** [3] Let  $k \geq 3$ , if  $n = \frac{k^2(k-1)}{2}$ , then  $\chi_{ir}(C_{n-1}) \geq k + 1$

A. Rohoni et. al have discussed irregular coloring in the double wheel graph family [4], fan graphs families [5], and wheel related graphs [6]. Avudainayaki et. al [7] has discussed irregular coloring on central and middle graphs of double star graphs, Anderson et. Al [8] discussed irregular coloring on regular graphs, and Kristiana et. al [9] has discussed local irregularity coloring of some family graphs. There are some previous results of irregular coloring of bipartite graph dan tree graph family [10], star families [11], some generalized graph [12], and Mycielskian Graphs [13].

In this paper, we observed the irregular coloring of bull graph, pan graph, sun graph, peach graph, and caveman graph. The bull Graph is a planar undirected graph with 5 vertices and 5 edges, in the form of a triangle with two disjoint pendant edges [14]. The Pan  $Pa_n$  graph is a graph obtained by combining a cycle graph  $C_n$  with a singleton star graph  $K_1$ [15]. The sun graph of order  $2n$  is a cycle  $C_n$  with an edge terminating in a pendent vertex attached to each vertex[16]. A peach graph is a circular graph  $C_m$  which has  $n$  pendants at one vertex, which is at vertex  $x_1$  contained in the cycle graph[17]. A caveman graph arises by modifying a set of fully connected clusters (caves) by removing one edge from each cluster and using it to connect to a neighboring one such that the clusters form a single loop [18].

## METHOD

The type of research used in this research is exploratory research. This research is research conducted to dig up data and find new things that you want to know so that the results obtained can be used as knowledge development. The following is a description of the steps taken to determine irregular chromatic numbers: (1) determining the graph that will be researched; (2) determining the cardinality of the graph that is used as research; (3) do vertex coloring of the researched graph; (4) create a code from each vertex in the graph. Specifically,  $code(v) = (a_0, a_1, \dots, a_k) =$ , where  $a_0 = c(v)$  and  $a_i$  ( $1 \leq i \leq k$ ) are the number of vertices of color  $i$  adjacent to  $v$ ; (5) if the codes are the same for each vertex, proceed to step 3. (6) determining irregular chromatic numbers.

## RESULT AND DISCUSSION

This research is focused on finding irregular chromatic numbers in the unicyclic graph family, including bull graph, pan graph, sun graph, peach graph, and caveman graph.

**Theorem 1.** The irregular chromatic number of bull graph  $B_{3,n}$  is  $n + 1$ .

*Proof:* This graph has the vertex set  $V(B_{3,n}) = \{x_i | 1 \leq i \leq 3\} \cup \{y_{2i} | 1 \leq i \leq n\} \cup \{y_{3i} | 1 \leq i \leq n\}$  and edge set is  $E(B_{3,n}) = \{x_1x_i | 2 \leq i \leq 3\} \cup \{x_2x_3\} \cup \{x_2y_{2i} | 1 \leq i \leq n\} \cup \{x_3y_{3i} | 1 \leq i \leq n\}$ .  $|V(B_{3,n})| = 2n + 3$  and  $|E(B_{3,n})| = 2n + 3$ .

We need to prove the upper bound  $\chi_{ir}(B_{3,n}) \leq n + 1$  and the lower bound  $\chi_{ir}(B_{3,n}) \geq n + 1$ . First, we prove that the lower bound of the irregular chromatic number of bull graph is  $\chi_{ir}(B_{3,n}) \geq n + 1$ . Assume that  $\chi_{ir}(B_{3,n}) < n + 1$ . We have some condition as follows.

- (1)  $c(x_i) = c(y_{2i})$  and  $c(x_i) c(y_{2i}) \in E(B_{3,n})$ . It's contradicting.
- (2)  $c(x_i) = c(y_{3i})$  and  $c(x_i) c(y_{3i}) \in E(B_{3,n})$ . It's contradicting.
- (3)  $c(x_{2k}) = c(y_{2l})$  and  $k \neq l$ . It's contradicting.
- (4)  $c(x_{3k}) = c(y_{3l})$  and  $k \neq l$ . It's contradicting.

Based on (1), (2), (3), and (4), we get the lower bound of the irregular coloring on a bull graph is  $\chi_{ir}(B_{3,n}) \geq n + 1$ . Furthermore, we prove that the upper bound of the irregular chromatic number of the bull graph is  $\chi_{ir}(B_{3,n}) \leq n + 1$ . The color function  $c$  on this graph is defined as follows.

$$\begin{aligned} c(x_i) &= i, 1 \leq i \leq 3 \\ c(y_{2i}) &= \begin{cases} i & i = 1 \\ i + 1 & 2 \leq i \leq n \end{cases} \\ c(y_{3i}) &= \begin{cases} i & 1 \leq i \leq 2 \\ i + 1 & 3 \leq i \leq n \end{cases} \end{aligned}$$

Based on the color function, the code for each vertex in the bull graph is obtained as follows.

$$\begin{aligned} code(x_1) &= \left( c(x_1), \underbrace{0,1,1,0,0, \dots, 0}_{n+1} \right) \\ code(x_2) &= \left( c(x_2), \underbrace{2,0,2,1,1, \dots, 1}_{n+1} \right) \\ code(x_3) &= \left( c(x_3), \underbrace{2,1,0,1,1, \dots, 1}_{n+1} \right) \\ code(y_{2i}) &= \left( c(y_{2i}), \underbrace{0,1,0,0,0, \dots, 0}_{n+1} \right) \\ code(y_{3i}) &= \left( c(y_{3i}), \underbrace{0,0,1,0,0, \dots, 0}_{n+1} \right) \end{aligned}$$

Based on the color function and vertex code obtained, each neighboring vertex has a different color and each vertex in the graph has a different color code. Therefore, the upper bound of the irregular chromatic number on the bull graph is  $\chi_{ir}(B_{3,n}) \leq n + 1$ . The lower and upper bound of the irregular chromatic number of bull graph is  $n + 1 \leq \chi_{ir}(B_{3,n}) \leq n + 1$ . So,  $\chi_{ir}(B_{3,n}) = n + 1$ .

**Theorem 2.** Let  $(k - 1) \binom{k - 1}{2} + 1 \leq n \leq k \binom{k}{2}$  and  $k \geq 4$ . Irregular chromatic number of pan graph  $Pa_n$  is

$$\chi_{ir}(Pa_n) = \begin{cases} 3 & \text{for } n \text{ odd, } 3 \leq n \leq 9 \\ 3 & \text{for } n = 4 \\ 4 & \text{for } n \text{ even, } 6 \leq n \leq 9 \end{cases}$$

*Proof:* This graph has the vertex set  $V(Pa_n) = \{x_i | 1 \leq i \leq n\} \cup \{y\}$  and the edge set is  $E(Pa_n) = \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{x_i y\}$ .  $|V(Pa_n)| = n + 1$  and  $|E(Pa_n)| = n + 1$ . The pan graph contains the  $C_n$  subgraph where  $C_n$  is a cycle graph. In this case, the  $C_n$  subgraph is labeled according to the  $C_n$  graph. There are three cases in this proof.

**Case 1.**  $\chi_{ir}(Pa_n) = 3$ , for  $n$  odd,  $3 \leq n \leq 9$

We need to prove the upper bound  $\chi_{ir}(Pa_n) \leq 3$  and the lower bound  $\chi_{ir}(Pa_n) \geq 3$ . First, we prove that the lower bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \geq 3$ . Assume that  $\chi_{ir}(Pa_n) < 3$ , for example  $\chi_{ir}(Pa_n) = 2$ , we get the color on  $x_i$  is  $1, 2, 1, 2, \dots, 1$  or  $2, 1, 2, \dots, 2$  periodically. Since  $c(x_1) = c(x_n)$  and  $x_1 x_n \in E(Pa_n)$ . This contradicts the definition of proper coloring where each pair of adjacent vertices must have different colors. Therefore,  $\chi_{ir}(Pa_n) \geq 3$ . Furthermore, we prove that the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq 3$  where  $3 \leq n \leq 9$ . Define the color function  $c: v \rightarrow \{1, 2, 3, \dots, \chi_{ir}\}$ ,  $S = \{1, 2, 3, \dots, \chi_{ir}\}$  and  $p, q \in N$ .

$x_{(p \pmod n)+1}, x_{(p+1 \pmod n)+1}, x_{(p+2 \pmod n)+1} \in V(Pa_n)$ , with  $c(x_{(p \pmod n)+1}), c(x_{(p+1 \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \in S$ . If  $c(x_{(p \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \neq c(x_{(q \pmod n)+1}), c(x_{(q+2 \pmod n)+1})$ , then  $c(x_{(p+1 \pmod n)+1}) = c(x_{(q+1 \pmod n)+1})$ . The color of  $y$  is  $c(y)$ , for  $c(y) \in S, c(x_i) \neq c(y), x_i y \in E(Pa_n)$ . Based on the color function, it can be written, if it is known that  $code(x_p) = (a_0, a_1, a_2, \dots, a_k)$ , and  $code(x_q) = (a_0, a_1, a_2, \dots, a_k), code(x_p) \neq code(x_q)$ .

We get the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq 3$ . The lower and upper bound of the irregular chromatic number of pan graph is  $3 \leq \chi_{ir}(Pa_n) \leq 3$ . So,  $\chi_{ir}(Pa_n) = 3$ .

**Case 2.**  $\chi_{ir}(Pa_n) = 3, n = 4$

We need to prove the upper bound  $\chi_{ir}(Pa_n) \leq 3$  and the lower bound  $\chi_{ir}(Pa_n) \geq 3$ . First, we prove that the lower bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \geq 3$ . Assume that  $\chi_{ir}(Pa_n) < 3$ , for example  $\chi_{ir}(Pa_n) = 2$ . Define the color function  $c: v \rightarrow \{1, 2\}$  is  $c(x_1) = 1, c(x_2) = 2, c(x_3) = 1, c(x_4) = 2, c(y) = 2$ . The color code obtained is  $code(x_1) = (1, 0, 3), code(x_2) = (2, 2, 0), code(x_3) = (1, 0, 2), code(x_4) = (2, 2, 0), code(y) = (2, 1, 0)$ . Based on the code obtained, there is the same code, namely  $code(x_2) = code(x_4)$ . This contradicts the definition of irregular colorin that each vertex in a graph must have a different color code. So,  $\chi_{ir}(Pa_n) \geq 3$ . Furthermore, we prove that the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq 3$ . Define the color function  $c: v \rightarrow \{1, 2, 3\}$  is  $c(x_1) = 1, c(x_2) = 3, c(x_3) = 1, c(x_4) = 2, c(y) = 2$ . The color code obtained is  $code(x_1) = (1, 0, 2, 1), code(x_2) = (3, 2, 0, 0), code(x_3) = (1, 0, 1, 1), code(x_4) = (2, 2, 0, 0), code(y) = (2, 1, 0, 0)$ . So, it can be concluded that  $\chi_{ir}(Pa_n) \leq 3$ .

We get the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq 3$ . The lower and upper bound of the irregular chromatic number of pan graph is  $3 \leq \chi_{ir}(Pa_n) \leq 3$ . So,  $\chi_{ir}(Pa_n) = 3$ .

**Case 3.**  $\chi_{ir}(Pa_n) = 4$ , for  $n$  even,  $6 \leq n \leq 9$

We need to prove upper bound  $\chi_{ir}(Pa_n) \leq 4$  and lower bound  $\chi_{ir}(Pa_n) \geq 4$ . First, we prove that the lower bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \geq 4$ . According to Lemma 3, if  $k = 3$ , then  $\chi_{ir}(C_{n-1}) \geq 4$ . We get an odd  $n$ , so

$n - 1$  is even. Since  $Pa_n$  has the subgraph  $C_n$ , where  $C_n$  is a cycle graph labeled as  $C_n$ . So, based on Lemma 3, we can conclude that  $\chi_{ir}(Pa_n) \geq 4$ . Next,  $\chi_{ir}(Pa_n) \geq 4$ . Furthermore, we prove that the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq 4$  where  $3 \leq n \leq 9$ . Define the color function  $c: v \rightarrow \{1, 2, 3, \dots, \chi_{ir}\}$ ,  $S = \{1, 2, 3, \dots, \chi_{ir}\}$  and  $p, q \in N$ .  $x_{(p \pmod n)+1}, x_{(p+1 \pmod n)+1}, x_{(p+2 \pmod n)+1} \in V(Pa_n)$ , with  $c(x_{(p \pmod n)+1}), c(x_{(p+1 \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \in S$ .

If  $c(x_{(p \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \neq c(x_{(q \pmod n)+1}), c(x_{(q+2 \pmod n)+1})$ , then  $c(x_{(p+1 \pmod n)+1}) = c(x_{(q+1 \pmod n)+1})$ . The color on  $y$  is  $c(y)$ , for  $c(y) \in S$ ,  $c(x_i) \neq c(y)$ ,  $x_i y \in E(Pa_n)$ . Based on the color function, it can be written, if it is known that  $code(x_p) = (a_0, a_1, a_2, \dots, a_k)$ , and  $code(x_q) = (a_0, a_1, a_2, \dots, a_k)$ ,  $code(x_p) \neq code(x_q)$ . So, we get the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq 4$ . The lower and upper bound of the irregular chromatic number of pan graph is  $4 \leq \chi_{ir}(Pa_n) \leq 4$ . So,  $\chi_{ir}(Pa_n) = 4$ .

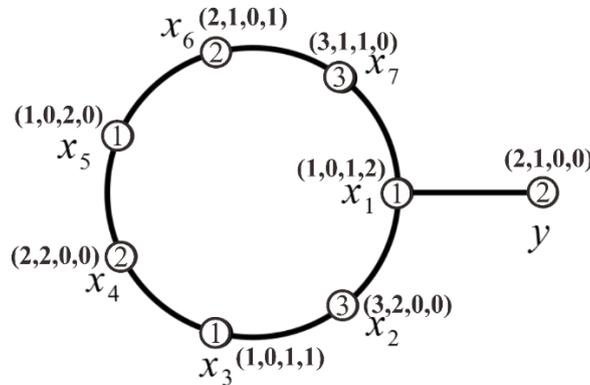


Figure 1.  $\chi_{ir}(Pa_7) = 3$

**Theorem 3.** Let  $(k - 1) \binom{k - 1}{2} + 1 \leq n \leq k \binom{k}{2}$  Irregular chromatic number of pan graph  $Pa_n$  for  $n \geq 3, k \geq 4$  is

$$\chi_{ir}(Pa_n) = \begin{cases} k & \text{for } n \neq k \binom{k}{2} - 1 \\ k + 1 & \text{for } n = k \binom{k}{2} - 1 \end{cases}$$

*Proof:* This graph has the vertex set  $V(Pa_n) = \{x_i | 1 \leq i \leq n\} \cup \{y\}$  and the edge set is  $E(Pa_n) = \{x_i x_{i+1} | 1 \leq i \leq n - 1\} \cup \{x_1 x_n\} \cup \{x_i y\}$ .  $|V(Pa_n)| = n + 1$  and  $|E(Pa_n)| = n + 1$ . The pan graph contains the  $C_n$  subgraph where  $C_n$  is a cycle graph. In this case, the  $C_n$  subgraph is labeled according to the  $C_n$  graph. There are two cases in this proof.

**Case 1.**  $\chi_{ir}(Pa_n) = k, \text{ for } n \neq k \binom{k}{2} - 1$

We need to prove upper bound  $\chi_{ir}(Pa_n) \leq k$  and lower bound  $\chi_{ir}(Pa_n) \geq k$ . First, we prove that the lower bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \geq k$ . According to Corollary 4, if  $k \geq 3$  is an integer, then  $\chi_{ir}(C_n) \geq k$ . Since  $Pa_n$  has the subgraph  $C_n$ , where  $C_n$  is a cycle graph labeled as  $C_n$ . Therefore, we get  $\chi_{ir}(Pa_n) \geq k$ . Next, we prove that the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq k$ . Color function  $c: v \rightarrow \{1, 2, 3, \dots, \chi_{ir}\}$ ,  $S = \{1, 2, 3, \dots, \chi_{ir}\}$  and  $p, q \in N$ .  $x_{(p \pmod n)+1}, x_{(p+1 \pmod n)+1}, x_{(p+2 \pmod n)+1} \in V(Pa_n)$ , with

$$c(x_{(p \pmod n)+1}), c(x_{(p+1 \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \in S.$$

If  $c(x_{(p \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \neq c(x_{(q \pmod n)+1}), c(x_{(q+2 \pmod n)+1})$ , then  $c(x_{(p+1 \pmod n)+1}) = c(x_{(q+1 \pmod n)+1})$ . The color on  $y$  is  $c(y)$ , for  $c(y) \in S, c(x_i) \neq c(y), x_i y \in E(Pa_n)$ . Based on the color function, it can be written, if it is known that  $code(x_p) = (a_0, a_1, a_2, \dots, a_k)$ , and  $code(x_q) = (a_0, a_1, a_2, \dots, a_k), code(x_p) \neq code(x_q)$ . So, we get the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq k$ . The lower and upper bound of the irregular chromatic number of pan graph is  $k \leq \chi_{ir}(Pa_n) \leq k$ . So,  $\chi_{ir}(Pa_n) = k$ .

**Case 2.**  $\chi_{ir}(Pa_n) = k + 1, for n = k \binom{k}{2} - 1$

We need to prove upper bound  $\chi_{ir}(Pa_n) \leq k + 1$  and lower bound  $\chi_{ir}(Pa_n) \geq k + 1$ . First, we prove that the lower bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \geq k + 1$ . According to Lemma 3, if  $n = \frac{k^2(k+1)}{2} = k \binom{k}{2}$ , then  $\chi_{ir}(C_{n-1}) \geq k + 1$ . Since  $Pa_n$  has the subgraph  $C_n$ , where  $C_n$  is a cycle graph labeled as  $C_n$ . Therefore, if  $n - 1$ , then  $k \binom{k}{2} - 1$ . So, if  $n = k \binom{k}{2} - 1$ , then  $\chi_{ir}(Pa_n) \geq k + 1$ .

Next, we prove that the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq k + 1$ . Color function  $c: v \rightarrow \{1, 2, 3, \dots, \chi_{ir}\}, S = \{1, 2, 3, \dots, \chi_{ir}\}$  and  $p, q \in N$ .  $x_{(p \pmod n)+1}, x_{(p+1 \pmod n)+1}, x_{(p+2 \pmod n)+1} \in V(Pa_n)$ , with  $c(x_{(p \pmod n)+1}), c(x_{(p+1 \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \in S$ . If  $c(x_{(p \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \neq c(x_{(q \pmod n)+1}), c(x_{(q+2 \pmod n)+1})$ , then  $c(x_{(p+1 \pmod n)+1}) = c(x_{(q+1 \pmod n)+1})$ . The color on  $y$  is  $c(y)$ , for  $c(y) \in S, c(x_i) \neq c(y), x_i y \in E(Pa_n)$ . Based on the color function, it can be written, if it is known that  $code(x_p) = (a_0, a_1, a_2, \dots, a_k)$ , and  $code(x_q) = (a_0, a_1, a_2, \dots, a_k), code(x_p) \neq code(x_q)$ .

So, we get the upper bound of the irregular chromatic number on the pan graph is  $\chi_{ir}(Pa_n) \leq k + 1$ . The lower and upper bound of the irregular chromatic number of pan graph is  $k + 1 \leq \chi_{ir}(Pa_n) \leq k + 1$ . So,  $\chi_{ir}(Pa_n) = k + 1$ .

**Theorem 4.** The irregular chromatic number of sun graph  $S_n$  for  $k^2 + k + 1 \leq n \leq k^2 + 3k + 2$  and  $k \in N$  is  $k + 2$ .

*Proof:* This graph has the vertex set  $V(S_n) = \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\}$  and the edge set is  $E(S_n) = \{x_i x_{i+1} | 1 \leq i \leq n - 1\} \cup \{x_1 x_n\} \cup \{x_i y_j | 1 \leq i \leq n, 1 \leq j \leq n\}$ .  $|V(S_n)| = 2n$  and  $|E(S_n)| = 2n$ .

We need to prove upper bound  $\chi_{ir}(S_n) \leq k + 2$  and lower bound  $\chi_{ir}(S_n) \geq k + 2$ . First, we prove that the lower bound of the irregular chromatic number on the sun graph is  $\chi_{ir}(S_n) \geq k + 2$ . Assume that  $\chi_{ir}(S_n) < k + 2$ . If  $x_i$  colored with  $k+1$ -color, then at most  $k + 1$  vertex of the same color. Because of  $x_k y_k \in E(S_n)$  and  $y_k$  are colored with  $k+1$ -color, then there are two neighboring vertices having the same color, namely  $c(x_k) = c(y_k)$ . This contradicts the definition of proper coloring, where each neighboring vertex must have a different color. Thus, it can be concluded that the upper bound of the irregular chromatic number of a sun graph is  $\chi_{ir}(S_n) \geq k + 2$ .

Next, we prove that the upper bound of the irregular chromatic number on the sun graph is  $\chi_{ir}(S_n) \leq k + 2$ . The color function  $c$  on this graph is defined as follows.

- For  $x_i$  vertices, where  $n \equiv 1(mod k + 2)$  the color of the vertices  $1, 2, 3, \dots, k + 2, 1, 2, 3, \dots, k + 2$  periodically.
- For  $x_i$  vertices, where  $n$  not equivalent  $1(mod k + 2)$  the color of the vertices  $1, 2, 3, \dots, k + 2, 1, 2, 3, \dots, k + 2, \dots, 1, 2, 3, \dots, k + 2, 2$  periodically.
- For  $y_i$  vertices,  $c(x_i) \neq c(y_j)$  where  $x_i y_j \in E(S_n)$ , and  $c(x_i) \neq c(x_j)$  for  $x_i y_k, x_j y_l \in E(S_n)$ , with  $c(y_k) = c(y_l), k \neq l$ .

We know that the  $code(x_p) = (a_0, a_1, a_2, \dots, a_k)$  and  $code(x_q) = (a_0, a_1, a_2, \dots, a_k)$ , then based on the color function obtained,  $code(x_p) \neq code(x_q)$ . Thus, the upper bound of the irregular chromatic number on a sun graph is  $\chi_{ir}(S_n) \leq k + 2$ . The illustration of irregular coloring of sun graph can be seen in **Figure 2**. The lower and upper bound of the irregular chromatic number of caveman graph is  $k + 2 \leq \chi_{ir}(S_n) \leq k + 2$ . So,  $\chi_{ir}(S_n) = k + 2$ .

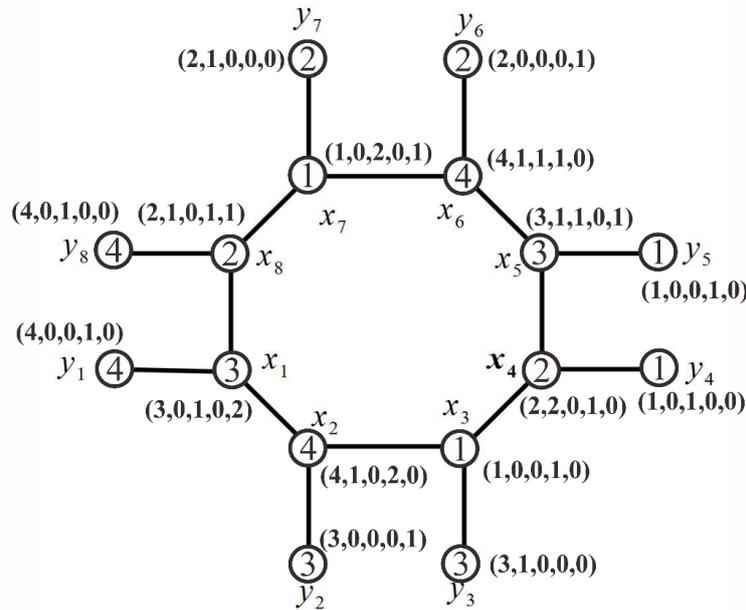


Figure 2.  $\chi_{ir}(S_n) = 4$

**Theorem 5.** Irregular chromatic number of peach graph  $C_m^m$  for  $m \geq 3$  is  $m + 1$ .

*Proof:* This graph has the vertex set  $V(C_m^m) = \{x_i | 1 \leq i \leq m\} \cup \{y_i | 1 \leq i \leq m\}$  and the edge set is  $E(C_m^m) = \{x_i x_{i+1} | 1 \leq i \leq m - 1\} \cup \{x_1 x_m\} \cup \{x_1 y_i | 1 \leq i \leq m\}$ .  $|V(C_m^m)| = 2m$  and  $|E(C_m^m)| = 2m$ .

We need to prove upper bound  $\chi_{ir}(C_m^m) \leq m + 1$  and lower bound  $\chi_{ir}(C_m^m) \geq m + 1$ . First, we prove that the lower bound of the irregular chromatic number on the peach graph is  $\chi_{ir}(C_m^m) \geq m + 1$ . Assume that  $\chi_{ir}(C_m^m) < m + 1$ . We have the following cases.

Case 1.  $c(x_i) = c(y_j), x_i y_j \in E(C_m^m)$ . It's contradicts.

Case 2.  $c(x_k) = c(y_l), k \neq l, y_k x_1, y_j x_1 \in E(C_m^m)$ . It's contradicts.

Based on case 1 and case 2 above, the lower bound is obtained from the irregular chromatic number on the peach graph is  $\chi_{ir}(C_m^m) \geq m + 1$ .

Next, we prove that the upper bound of the irregular chromatic number on the peach graph is  $\chi_{ir}(C_m^m) \leq m + 1$ . The color function  $c$  on this graph is defined as follows.

$$c(x_i) = i, 1 \leq i \leq m$$

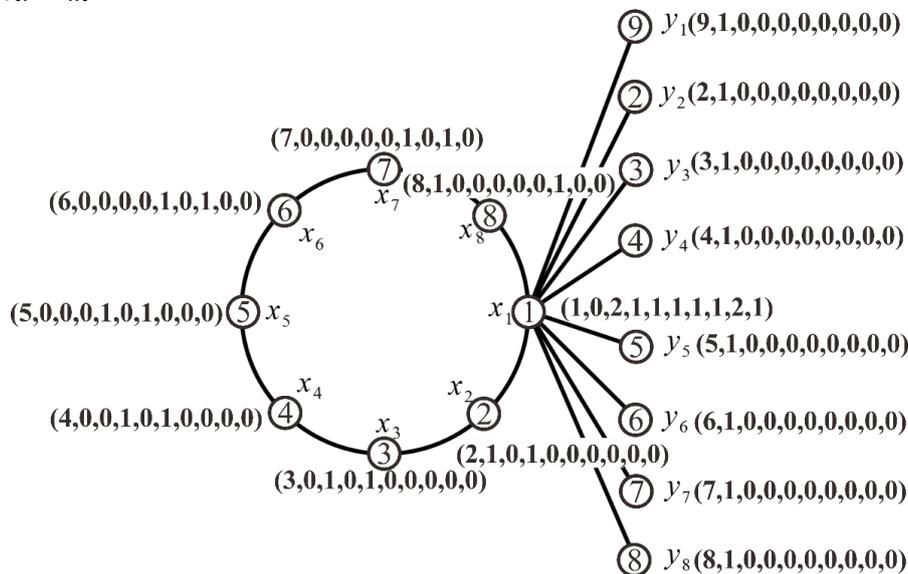
$$c(y_j) = \begin{cases} j & \text{for } 2 \leq j \leq m \\ m + 1 & \text{for } j = 1 \end{cases}$$

Based on the color function above, the code for each vertex in the peach graph is obtained as follows.

$$code(x_i) = \begin{cases} (c(x_i), \underbrace{0,2,1,1,1,1,0,0, \dots, 1,1,2}_{m+1}) & \text{for } i = 1 \\ (c(x_i), \underbrace{0, \dots, 0}_{i-2}, 1, 0, 1, \underbrace{0, \dots, 0}_{m-1}) & \text{for } 2 \leq i \leq m - 1 \\ (c(x_i), \underbrace{1,0,0,0,0,0,0, \dots, 0,0,1,0}_{m+1}) & \text{for } i = m \end{cases}$$

$$code(y_i) = (c(y_i), \underbrace{1,0,0,0, \dots, 0,0,0}_{m+1})$$

Based on the color function and vertex code obtained, each neighboring vertex has a different color and each vertex in the graph has a different color code. Therefore, the upper bound of the irregular chromatic number on the peach graph is  $\chi_{ir}(C_m^m) \leq m + 1$ . The illustration of irregular coloring of sun graph can be seen in **Figure 3**. The lower and upper bound of the irregular chromatic number of peach graph is  $m + 1 \leq \chi_{ir}(C_m^m) \leq m + 1$ . So,  $\chi_{ir}(C_m^m) = m + 1$ .



**Figure 3.**  $\chi_{ir}(C_8^8) = 9$

**Theorem 6.** Irregular chromatic number of caveman graph  $K_{m,3}$  for  $k^2 + k + 1 \leq m \leq k^2 + 3k + 2$  and  $k \in \mathbb{N}$  is  $k + 2$ .

*Proof:* This graph has the vertex set  $V(K_{m,3}) = \{x_i | 1 \leq i \leq 2m\} \cup \{y_j | 1 \leq j \leq m\}$  and the edge set is  $E(K_{m,3}) = \{x_i x_{i+1} | 1 \leq i \leq 2m - 1\} \cup \{x_1 x_{2m}\} \cup \{x_i y_i | 1 \leq i \leq 2m, i = odd, 1 \leq j \leq m\}$ .  $|V(K_{m,3})| = 3m + 6$  and  $|E(K_{m,3})| = 3m + 6$ .

We need to prove upper bound  $\chi_{ir}(K_{m,3}) \leq k + 2$  and lower bound  $\chi_{ir}(K_{m,3}) \geq k + 2$ . First, we prove that the lower bound of the irregular chromatic number on the caveman graph is  $\chi_{ir}(K_{m,3}) \geq k + 2$ . Assume that  $\chi_{ir}(K_{m,3}) \leq k + 2$ , for example is  $\chi_{ir}(K_{m,3}) = k + 1$ . If  $x_i$ , where  $\deg(x_i) = 3$  is colored with  $k+1$ -color, then at most  $k+1$  vertex of the same color. Because of  $x_i y_j \in E(K_{m,3})$  and  $y_j$  are colored with  $k+1$ -color, then there are two neighboring vertices having the same color, namely  $c(x_i) = c(y_j)$ . This contradicts the definition of proper coloring, where each neighboring vertex must have a different color.

Thus, it can be concluded that the lower bound of the irregular chromatic number of a caveman graph is  $\chi_{ir}(K_{m,3}) \geq k + 2$ .

Next, we prove that the upper bound of the irregular chromatic number on the caveman graph is  $\chi_{ir}(K_{m,3}) \leq k + 2$ . The color function  $c$  on this graph is defined as follows.

- For  $x_i$  vertices, where  $\deg(x_i) = 3$ , the color of the vertices  $1, 2, 3, \dots, k + 2, 1, 2, 3, \dots, k + 2$  periodically.
- For  $x_i$  and  $x_j$  vertices, with  $\deg(x_i) = \deg(x_j) = 2$ ,  $x_i = x_{(p+1 \pmod n)+1}$  and  $x_j = x_{(p+1 \pmod n)+1}$ . If  $S = \{1, 2, 3, \dots, k + 2\}$ ,  
 $x_{(p \pmod n)+1}, x_{(p+1 \pmod n)+1}, x_{(p+2 \pmod n)+1} \in V(K_{m,3}) \geq k + 1$ , with  
 $c(x_{(p \pmod n)+1}), c(x_{(p+1 \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \in S$ . If  
 $c(x_{(p \pmod n)+1}), c(x_{(p+2 \pmod n)+1}) \neq c(x_{(q \pmod n)+1}), c(x_{(q+2 \pmod n)+1})$ , then  
 $c(x_{(p+1 \pmod n)+1}) = c(x_{(q+1 \pmod n)+1})$ .
- For  $y_j$  vertices,  $c(x_i) \neq c(y_j)$  for  $x_i y_j \in E(K_{m,3})$
- For  $y_j$  vertices,  $c(y_k) = c(y_l)$  when  $x_i y_k, x_j y_l \in E(K_{m,3})$  and  $c(x_i) \neq c(x_j)$

We know that the  $code(x_p) = (a_0, a_1, a_2, \dots, a_k)$  and  $code(x_q) = (a_0, a_1, a_2, \dots, a_k)$  then based on the color function obtained,  $code(x_p) \neq code(x_q)$ . Thus, the upper bound of the irregular chromatic number on a caveman graph is  $\chi_{ir}(K_{m,3}) \leq k + 2$ . The illustration of irregular coloring of caveman graph can be seen in **Figure 4**. The lower and upper bound of the irregular chromatic number of peach graph is  $k + 2 \leq \chi_{ir}(K_{m,3}) \leq k + 2$ . So,  $\chi_{ir}(K_{m,3}) = k + 2$ .

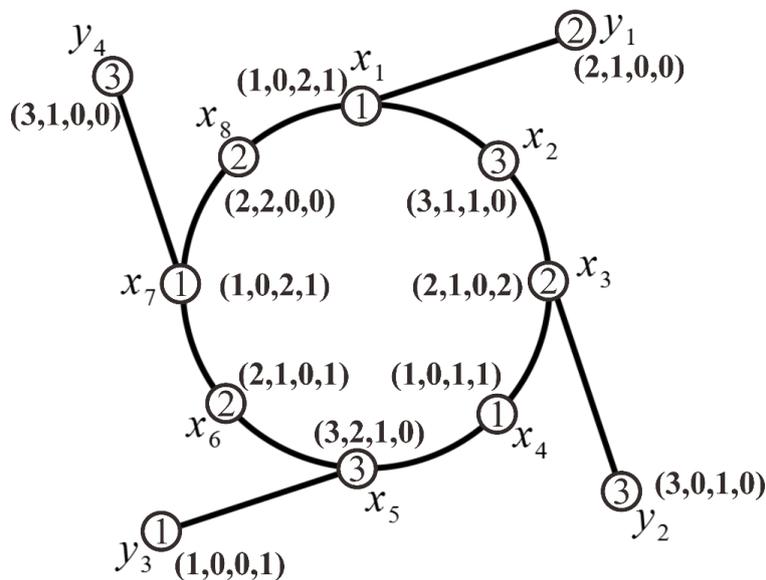


Figure 4.  $\chi_{ir}(K_{4,3}) = 3$

### CONCLUSIONS

In this paper, we get the irregular chromatic number of the unicyclic graph family, namely bull graph, pan graph, sun graph, peach graph, caveman graph.

## REFERENCES

- [1] K. Anitha, B. Selvam, and K. Thirusangu. New Irregular Colourings of Graphs. *Journal of Applied Science and Computations*. vol. 6, no.5, pp.269-279,2019.
- [2] P. Zhang. A Kaleidoscopic View of Graph Colorings (Springer Nature), 2016.
- [3] M. Radcliffe and P. Zhang. Irregular Coloring of Graphs. Kalamazoo: Western Michigan University. 2005.
- [4] A. Rohini and M. Venkatachalam. On Irregular Colorings of Double Wheel Graph Families. *International Conference on Current Scenario in Pure and Applied Mathematics*. vol 68, no. 1, pp. 944-949, 2019.
- [5] A. Rohini and M. Venkatachalam. On irregular colorings of fan graph families. *International Conference on Applied and Computational Mathematics*. 1139, pp 1-6, 2018.
- [6] Rohini, A., and M. Venkatachalam. On irregular coloring of wheel related graphs. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*68.2 (2019): 1462-1472.
- [7] R. Avudainayaki, B. Selvam, and K. Thirusangu. Irregular Coloring of Some Classes of Graphs. *International Journal of Pure and Applied Mathematics*. vol.100, no.10, pp.119-127, 2016.
- [8] M. Anderson, R. P. Vitray, and J. Yellen. Irregular colorings of regular graphs. *Discrete Mathematics*. 312. pp. 2329-2336, 2012.
- [9] A. I. Kristiana, R. Alfarisi, Dafik, & N. Azahra, Local irregular vertex coloring of some families graph. *Journal of discrete mathematical sciences and cryptography*, 25(1), 2022, 15-30.
- [10] Q. A'yun, R. Adawiyah, I. H. Agustin, & E.R. Albirri, On the irregular coloring of bipartite graph and tree graph families. In *Journal of Physics: Conference Series* (Vol. 1836, No. 1, p. 012024). 2021. IOP Publishing.
- [11] R. Saraladevy, P. Nedumaran, and K. Anitha. Irregular set coloring of star families. *AIP Conference Proceedings*. Vol. 2261. No. 1. AIP Publishing LLC, 2020.
- [12] A. Rohini, and M. Venkatachalam. On Irregular Coloring of Some Generalised Graphs. *Advanced Studies in Contemporary Mathematics* 30.1 (2020): 99-120.
- [13] R. Avudainayaki, B. Selvam, K. Thirusangu, & P. P. Ulaganathan, P. P. Irregular Coloring in Mycielskian Graphs. *Recent Trends in Graph Theory & Combinatorics*, 14. 2017.
- [14] P. Ghosh, S. N. Mishra, and A. Pal. Various Labeling on Bull Graph and some Related Graphs. *International Journal of Applications of Fuzzy Sets and Artificial Intelligence*. 5, pp.23-35, 2015.
- [15] Helmi, F. Fran, and D. R. Putra. Prime Labeling of Pan Graph and Its Line, Middle and Duplication Graph. *PRISMA*. vol.3, pp.8-13, 2020.
- [16] M. Mirzakhah and D. Kiani. The Sun Graph is Determined by Its Signless Laplacian Spectrum. *Electronic Journal of Linear Algebra*. 20, pp.610-620, 2010.
- [17] B. H. Xing, G. D. Yu, L. X. Wang and J Cao. The Harary Index of All Unicyclic Graphs with Given Diameter. *Discrete Dynamics in Nature and Society*. pp. 1-6, 2018.
- [18] D. Koutra. *Exploring and Making Sense of Large Graphs*. Pittsburgh. Computer Science Department. Carnegie Mellon University, 2015.