An Inclusive Local Irregularity Vertex Coloring of Dutch Windmill Graph

Arika Indah Kristiana*, **Lusi Rizzami Prahastiwi, Rafiantika Megahnia Prihandini**

1Mathematics Education, Faculty of Teacher Training and Education, University of Jember, Indonesia

Email: arika.fkip@unej.ac.id

ABSTRACT

Let $G(V,E)$ is a simple and connected graph with $V(G)$ as vertex set and $E(G)$ as edge set. An inclusive local irregularity vertex coloring is a development of the topic of local irregularity vertex coloring. An inclusive local irregularity vertex coloring is defined by coloring the graph so that its weight value is obtained by adding up the labels of the neighboring vertex and its label. The inclusive local irregularity chromatic number is defined as the minimum number of colors obtained from coloring the vertex of the inclusive local irregularity in graph G. In this paper, we find the inclusive local irregularity vertex coloring and determine the chromatic number on the Dutch windmill graph using axiomatic deductive and pattern recognition methods. The results of this study are expected to be used as a basis for studies in the development of knowledge related to the inclusive local irregularity vertex coloring.

Keywords: Inclusive Local Irregularity Vertex Coloring; Dutch Windmill Graph.

INTRODUCTION

A graph is a pair of sets consisting of a finite non-empty set and elements called vertices, and a (possibly empty) set of ordered pairs of vertices called edges [1]. One of the main topics in graph theory is graph coloring. Graph coloring can be interpreted as giving color to each element of the graph. The coloring of vertex gives a different color to two adjacent vertices [2]. The chromatic number is the minimum number of colors obtained from a graph coloring, which is denoted by $\chi(G)$ [3]. One of the special cases in graph coloring is an inclusive local irregularity vertex coloring [4],[5]. Several studies related to the local irregular coloring can be seen in the following studies [6]-[10].

An Inclusive local irregularity vertex coloring is an extension of the topic of local irregularity vertex coloring. In local irregularity vertex coloring, the vertex weights are obtained by adding up the neighboring vertices. Then on inclusive local irregularity vertex coloring, the vertex weight is obtained by adding up the labels of the neighboring vertices and their label. The coloring of vertices of inclusive local irregularity as follows [11].

Definition 1. *Given* $l:V(G)\rightarrow\{1,2,...,k\}$ *is a function of the label and* $w^i:V(G)\rightarrow N$ *is a weight function of inclusive local irregularity vertex coloring, with* $w^i(v)=l(v)+\sum_{u\in N(v)}l(u)$.

This labeling is an inclusive local irregularity vertex coloring if:
   (i) $Opt(l)=min\{max\{l_a\}\};l_a$; *is an inclusive local irregularity vertex coloring*
   (ii) For every vertex $uv\in E(G),w(u)\neq w(v)$

Definition 2. [11] *The minimum number of colors obtained by inclusive local irregularity vertex coloring of a graph G is called inclusive local irregularity vertex chromatic number, denoted by* $\chi_{lis}^i(G)$.

To make it easier to find an inclusive local irregularity vertex coloring in this study, the following lemma and observations will be used:

Lemma 1. [11] *Given that the graph G is a simple and connected graph, then* $\chi_{lis}^i(G)\geq\chi_{lis}(G)$

Observation 1. *Local irregularity chromatic number of the Dutch windmill graph* $(D_{m,n})$ *with* $m\geq2$ *and* $n=even$ *is* $\chi_{lis}(D_{m,n})=3$

Observation 2. *Local irregularity chromatic number of the Dutch windmill graph* $(D_{m,n})$ *with* $m\geq2$ *and* $n=odd$ *is* $\chi_{lis}(D_{m,n})=4$

Previous research related to the inclusive local irregularity vertex coloring on several simple graphs, namely on path graphs, circle graphs, and star graphs [11]. then followed by other researchers with thatch graph, H-star graph, and double star graph [12], with fan graph, firecracker graph, and sun graph [13], and with cricket graph, tadpole graph, peach graph, a comb star graph $m+1$ and $n+1$, bipartite complete comb graph and star graph, and friendship comb graph and star graph[10], [14].

Dutch windmill graph denoted by $(D_{m,n})$ is a graph formed from m cycle graph with $m\geq2$, where m is the number of cycle graphs, and n is the number of vertices of each cycle graph and $n\geq3$, which has one common center vertex [15]. The research related to the inclusive local irregularity vertex coloring is still relatively new, the researchers are interested in conducting research on the inclusive local irregularity vertex coloring on Dutch windmill graphs because it has never been studied before.

METHODS

This research uses the axiomatic deductive method and pattern detection method. The axiomatic deductive method is defined as a research method that uses a theorem or axiom with the principles of deductive proof in mathematical logic to then be used in local irregularity vertex coloring on the Dutch windmill graph. While the pattern detection method in this study is used to formulate patterns and look for inclusive local irregularity chromatic numbers in a Dutch windmill graph.

RESULTS AND DISCUSSION

In this study, the researcher discusses the inclusive local irregularity vertex coloring on the Dutch windmill graph and the theorem is generated as follows.

Theorem 1. *Given a Dutch windmill graf $\left(D_{m,n} \right)$, with $m \geq 2$, and n is even, then the inclusive local irregularity chromatic number of the graph is $\chi_{lis}^{i} \left(D_{m,n} \right) = 3$*

Proof. *Let $\left(D_{m,n} \right)$ be a Dutch windmill graph with $m \geq 2$, and n is even, has the vertex set* $V \left(D_{m,n} \right) = \left\{ y \right\} \cup \left\{ x_{i,j}; 1 \leq i \leq m, 1 \leq j \leq n-1 \right\}$ and the edge set $E \left(D_{m,n} \right) = \left\{ x_{i,1} y; 1 \leq i \leq m \right\} \cup \left\{ x_{i,j} x_{i,j+1}; 1 \leq i \leq m, 1 \leq j \leq n-2 \right\} \cup \left\{ x_{i,n-1} y; 1 \leq i \leq m \right\}$ so that the cardinality of the vertex and edge set are respectively $\left| V \left(D_{m,n} \right) \right| = (n-1)m+1$ and $\left| E \left(D_{m,n} \right) \right| = mn$. For each $v \in V \left(D_{m,n} \right)$ if each vertex is labeled with 1, then the neighboring vertex will have the same weight, like $w^{i} \left(x_{i,j} \right) = l(y) + l \left(x_{i,j} \right) + l \left(x_{i,j+1} \right) = 3$ and $w^{i} \left(x_{i,j+1} \right) = l \left(x_{i,j} \right) + l \left(x_{i,j+1} \right) + l \left(x_{i,j+2} \right) = 3$ while the vertex $x_{i,j}$ and $x_{i,j+1}$ are neighboring vertex. This condition is contrary to the definition where $w^{i} \left(x_{i,j} \right) \neq w^{i} \left(x_{i,j+1} \right)$ so in the proof of this case $Opt(l) \geq 2$. Based on lemma 1 and observation 1, it is known that the lower bound of inclusive local irregularity chromatic number of the Dutch windmill graph $\left(D_{m,n} \right)$ with $m \geq 2$, and $n = even$, is $\chi_{lis}^{i} \left(D_{m,n} \right) \geq \chi_{lis} \left(D_{m,n} \right) = 3$. The upper bound of inclusive local irregularity defined by $l : V \left(D_{m,n} \right) \rightarrow \{1,2\}$. The labeling function is presented as follows.

$$l \left(x_{i,j} \right) = \begin{cases} 1, & \text{for } 1 \leq i \leq m, \ j = odd \\ 2, & \text{for } 1 \leq i \leq m, \ j = even \end{cases}$$
$$l(y) = 2$$

So, if the labels of the neighboring vertex and the own label are added together, the vertex weights are obtained as follows:

$$w^{i} \left(x_{i,j} \right) = \begin{cases} 4, & \text{for } 1 \leq i \leq m, \ j = odd \\ 5, & \text{for } 1 \leq i \leq m, \ j = even \end{cases}$$
$$w^{i}(y) = 2m+2$$

From the vertex weights, we get $\left| w \left(D_{m,n} \right) \right| = 3$ which is the chromatic upper bound of the Dutch windmill graph $\left(D_{m,n} \right)$ with $m \geq 2$, and $n = even$. Based on the chromatic lower bound through observation 1 and the chromatic upper bound through the labeling above, obtained $3 \leq \chi_{lis}^{i} \left(D_{m,n} \right) \leq 3$ so that the chromatic number of the Dutch windmill graph $\left(D_{m,n} \right)$ with $m \geq 2$, and $n = even$ is $\chi_{lis}^{i} \left(D_{m,n} \right) = 3$. ■

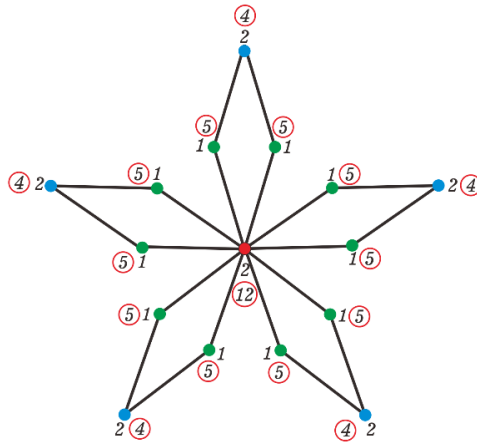**Figure 1.** Dutch Windmill Graph $m=5$ dan $n=4$ $\chi_{lis}^i(D_{5,4})=3$

Figure 1 is the result of inclusive local irregularity vertex coloring on the Dutch windmill graph $(D_{5,4})$, According to theorem 1, the inclusive local irregularity chromatic number is $\chi_{lis}^i(D_{5,4})=3$.

Theorem 2. *Given a Dutch windmill graf $(D_{m,n})$, with $m\geq 2$, and n is odd, then the inclusive local irregularity chromatic number of the graph is $\chi_{lis}^i(D_{m,n})=4$*

Proof. *Let $(D_{m,n})$ is Dutch windmill graph with $m\geq 2$, and n is odd, has the vertex set* $V(D_{m,n})=\{y\}\cup\{x_{i,j};1\leq i\leq m,1\leq j\leq n-1\}$ and the edge set $E(D_{m,n})=\{x_{i,1}y;1\leq i\leq m\}\cup\{x_{i,j}x_{i,j+1};1\leq i\leq m,1\leq j\leq n-2\}\cup\{x_{i,n-1}y;1\leq i\leq m\}$ so that the cardinality of the vertex and edge are respectively $|V(D_{m,n})|=(n-1)m+1$ and $|E(D_{m,n})|=mn$. For each $v\in V(D_{m,n})$ if each vertex is labeled with one, then the neighboring vertex will have the same weight, like $w^i(x_{i,j})=l(y)+l(x_{i,j})+l(x_{i,j+1})=3$ and $w^i(x_{i,j+1})=l(x_{i,j})+l(x_{i,j+1})+l(x_{i,j+2})=3$ while the vertex $x_{i,j}$ and $x_{i,j+1}$ are neighboring vertex. This condition is contrary to the definition where $w^i(x_{i,j})\neq w^i(x_{i,j+1})$ so in the proof of this case $Opt(l)\geq 3$. Based on lemma 1 and observation 2, it is known that the lower bound of inclusive local irregularity chromatic number of the Dutch windmill graph $(D_{m,n})$ with $m\geq 2$, and $n=odd$, is $\chi_{lis}^i(D_{m,n})\geq\chi_{lis}(D_{m,n})=4$. The upper bound of inclusive local irregularity defined by $l:V(D_{m,4})\rightarrow\{1,2,3\}$. The labeling function is presented as follows.

$$l(x_{i,j})=\begin{cases}1, & \text{for } 1\leq i\leq m, \ j=even\\2, & \text{for } 1\leq i\leq m, \ j=odd\end{cases}$$

$$l(y)=3$$

So, if the labels of the neighboring vertex and the own label are added together, the vertex weights are obtained as follows:

$$w^i\left(x_{i,j}\right) = \begin{cases} 4, & \text{for } 1 \le i \le m, \ j = 3 \\ 5, & \text{for } 1 \le i \le m, \ j = 2 \\ 6, & \text{for } 1 \le i \le m, \ j = 1,4 \end{cases}$$

$$w^i\left(y\right) = 3m + 3$$

From the vertex weights, we get $\left|w\left(D_{m,n}\right)\right| = 4$ which is the chromatic upper bound of the Dutch windmill graph $\left(D_{m,\,n}\right)$ *with* $m \ge 2$, and $n = odd$. Based on the chromatic lower bound through observation 2 and the chromatic upper bound through the labeling above, obtained $4 \le \chi^i_{lis}\left(D_{m,n}\right) \le 4$ so that the chromatic number of the Dutch windmill graph $\left(D_{m,\,n}\right)$ with $m \ge 2$, *and* $n = odd$ is $\chi^i_{lis}\left(D_{m,n}\right) = 4$. ∎
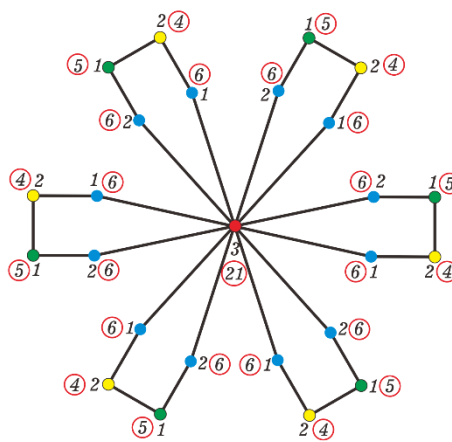

Figure 2. Dutch Windmill Graph $m = 6$ dan $n = 5$ $\chi^i_{lis}\left(D_{6,5}\right) = 4$

Figure 2 is the result of inclusive local irregularity vertex coloring on the Dutch windmill graph $\left(D_{6,5}\right)$, According to theorem 2, the inclusive local irregularity chromatic number is $\chi^i_{lis}\left(D_{6,5}\right) = 4$.

CONCLUSIONS

Based on the results and discussion above, two new theorems were obtained regarding the topic of inclusive local irregularity vertex coloring of the Dutch windmill graph. While the inclusive local irregularity chromatic number of the Dutch windmill graph is as follows.

(i) *Inclusive local irregularity chromatic number of Dutch windmill graph* $\left(D_{m,\,n}\right)$,

   with $m \ge 2$, *and* $n = even$ *is* $\chi^i_{lis}\left(D_{m,n}\right) = 3$

(ii) *Inclusive local irregularity chromatic number of Dutch windmill graph* $\left(D_{m,\,n}\right)$,

   with $m \ge 2$, *and* $n = odd$ *is* $\chi^i_{lis}\left(D_{m,n}\right) = 4$

REFERENCES

[1]   Slamin, *Teori Graf dan Aplikasinya*. Malang: CV. Dream Litera Buana, 2019.

[2]   G. Chartrand and P. Zhang, *A first course in graph theory*. New York: Dover Publications, Inc., 2012.

[3]   Balakrishnan R and Ranganathan K. "A Textbook of Graph Theory,"2012. [online]. Available: http://www.springer.com/series/233

[4]   G. Chartrand and P. Zhang, *Chromatic graph theory*. Chapman and Hall/CRC, 2008.

[5]   P. Zhang, *A Kaleidoscopic View of Graph Colorings*. New York: Springer, 2016.

[6]   A. Indah Kristiana, M. I. Utoyo, Dafik, I. Hesti Agustin, R. Alfarisi, and E. Waluyo, "On the chromatic number local irregularity of related wheel graph," *J. Phys. Conf. Ser.*, vol. 1211, no. 1, 2019, http://dx.doi.org/10.1088/1742-6596/1211/1/012003.

[7]   A. I. Kristiana, N. Nikmah, Dafik, R. Alfarisi, M. A. Hasan, and Slamin, "On the local irregularity vertex coloring of volcano, broom, parachute, double broom and complete multipartite graphs," *Discret. Math. Algorithms Appl.*, vol. 14, no. 6, p. 2250022, 2022.

[8]   A. I. Kristiana, R. Alfarisi, Dafik, and N. Azahra, "Local irregular vertex coloring of some families graph," *J. Discret. Math. Sci. Cryptogr.*, vol. 25, no. 1, pp. 15–30, 2022.

[9]   A. I. Kristiana, M. Hidayat, R. Adawiyah, Dafik, S. Setiawani, and R. Alfarisi, "On Local Irregularity of the Vertex Coloring of the Corona Product of a Tree Graph.," 2022.

[10]   A. I. Kristiana, I. L. Mursyidah, Dafik, R. Adawiyah, and R. Alfarisi, "Local irregular vertex coloring of comb product by path graph and star graph," *Discret. Math. Algorithms Appl.*, vol. 2250148, 2022.

[11]   A. I. Kristiana, Dafik, R. Alfarisi, U. A. Anwar, and S. M. Citra, "An Inclusive Lokal Irregularity Coloring of Graphs. Advances in," *Math. Sci. J.*, vol. 10, pp. 8941–8946, 2020.

[12]   U. A. Anwar, A. I. Kristiana, A. Fatahillah, D. Dafik, and R. Alfarisi, "Pewarnaan Ketakteraturan Lokal Inklusif pada Keluarga Graf Pohon Tree," *Cgant J. Math. Appl.*, vol. 2, no. 1, pp. 24–30, 2021, doi: 10.25037/cgantjma.v2i1.49.

[13]   A. Ma'arif, M. G. Halim, S. Indriani, A. I. Kristiana, and R. Alfarisi, "Pewarnaan Titik Ketakteraturan Lokal Inklusif pada Graf Kipas, Graf Petasan, dan Graf Matahari," *BAREKENG J. Ilmu Mat. Dan Terap.*, vol. 15, no. 1, 2021.

[14]   A. I. Kristiana, M. G. Halim, and R. Adawiyah, "Pewarnaan Titik Ketakteraturan Lokal Inklusif pada Keluarga Graf Unicyclic," *Contemp. Math. Appl.*, vol. 4, no. 1, pp. 15–27, 2022.

[15]   M. R. R. Kanna, R. P. Kumar, and R. Jagadeesh, "Computation of Topological Indices of Dutch Windmill Graph," *Open J. Discret. Math.*, vol. 06, no. 02, pp. 74–81, 2016, doi: 10.4236/ojdm.2016.62007.