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A Fractional-Order Leslie-Gower Model with Fear and Allee Effect

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ABSTRACT

To describe the interaction of prey and predator, we consider a predator-prey model based on the Leslie-Gower model. The model is formed by assuming fear effect in the prey and Allee effect in predators. In order to account for the memory effect, we apply the Caputo fractional-order derivative. The model has four possible equilibrium points, namely the origin, the predator extinction point, the prey extinction point, and all population exist point. Here, we show that two local stable points and two unstable points. Furthermore, we also investigate the stability changing caused by Hopf bifurcation when the order of fractional derivative changes. Finally, we perform several simulations to support our analysis results. We observe numerically by using the predictor-corrector method for the local stability, the existence of Hopf bifurcation, and the influence of fear factor and Allee effect to prey and predator.

Keywords: Hopf Bifurcation; Leslie-Gower Model; Local Stability

INTRODUCTION

The relationship between predators and prey is still a special thing in the development of ecological modeling. The main concern of this situation is how to maintain the availability of the ecosystem resources. In ecology, the presence of prey depends on how they can protect themselves, and the presence of predators depends on the availability of prey [1]. These relationships make researchers interested in forming predator-prey interactions into mathematical models. Several modifications have been made by researchers to build models that are more suitable for biological behavior. For example, the interaction between prey and predator considering the fear factor in prey [2]–[5], the influence of Allee effect on the availability of prey and predator [6]–[9], the impact of refuge in prey to the existence of predator [10]–[12], and the exploitation of prey and predator by harvesting [6], [12], [13].

In the last decade, the Allee effect has received great attention for the population dynamics. The Allee effect is divided into two types, namely demographic Allee effect and component Allee effect. According to [14], [15], the component Allee effect is a scenario at the lower population density affected by the positive interaction between growth rate and population density so that can increase their extinction. This effect can make a demographic Allee effect to a small population density. To be specific, if the population has a high density, then the competition for food will increase and the growth rate of population will decrease. Therefore, the demographic Allee effect does not hold for large population density [16]. The scenario occurs as a result of several conditions, such as

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obligate cooperators, mate finding problem, and anti-predator strategies in prey. We give an example, namely if a population reproduces sexually, then it can increase its density and an individual can find a mate easily. Thus, it can reduce the risk of inbreeding. In contrast to the large population density, the small population density has the risk of inbreeding. In the ecological model, the Allee effect can be constructed on prey population [6], [7], [17], predator population [9], [16], [18], or both population [8]. According to [16], for the literature on predator-prey model with the Allee effect, predators are more prone than their prey because predators are usually smaller than prey population. For example, the spotted owls *(Strix Occidentalis Caurina)* lost their habitat causing them to be unable to find their mates [19].

In addition, several experimental studies have shown that the presence of predators can change prey behavior even more strongly than the direct predation effect [2]–[4]. Fear of prey can affect the physiological state of juvenile prey (e.g., reduce prey reproduction) and be harmful to their survival as adults [3], [5], [20], [21]. For example, sparrows *(Melospiza Melodia)* during their breeding season without direct predation using electric fences, and it is found that there is a 40% reduction in the density of offspring due to predation risk [21].

Recently, we study the dynamical behavior of predator-prey interaction by assuming that (i) the Allee effect occurs in predators and (ii) prey is afraid of predators because prey is always alert to possible predator attacks. Biologically, the growth rate of individual should involve previous and current conditions [22]. That is, all current conditions of population density depend on all previous conditions [23]. Therefore, we also consider the memory effect which means the effect of all previous biological conditions to the present condition by replacing the first-order derivative with the fractional-order derivative [1], [9], [17], [24]. The memory effect shows that the population dynamics of present condition depend on all previous conditions stored in their memory system such as the experience in foraging, the best place to take shelter, the perfect time to migrate, and so on [1]. To play the superlative form of model, ordinary calculus is less effective in describing complex phenomenon involving memory effect and hereditary biological properties [25]. Thus, the fractional calculus is applied to solve the problem. Because, it has the ability to describe biological conditions related to the memory effect [26]. There are several well-known fractional-order derivatives that are used as operators in the predator-prey model. By considering the availability of analytical tools, we choose the Caputo fractional-order for our model as done by [1], [9], [17], [24]. According to [22], the Caputo fractional-order can be used on the classic initial condition as in the integer order equations. It has rich analytical tools in observing the dynamic of predator-prey systems.

In this manuscript, we organize several contents as follows. The section 1 presents several methods to solve the model. In the section 2, the mathematical model is formulated to obtain the first-order model and replacing it with the fractional-order derivative operator. In the section 3 and 4, the model is solved to explore the dynamical behaviors by investigating the equilibrium points, local stability, and Hopf bifurcation. In the section 4, the analytical results are demonstrated through several numerical simulations. We end this discussion by giving the conclusion in the section 5.

METHOD

In observing the interaction of the predator-prey population, we perform several steps to modify a modified Leslie-Gower predator-prey model. The steps are presented as follows.

1) Reviewing and studying the previous literature related to the problems taken.
2) Formulating the modified Leslie-Gower model by adding the fear factor and Allee effect. Next, we transform it into a fractional-order model.
3) Identifying equilibrium points and local stability in the model.
4) Investigating Hopf bifurcation in the model.
5) Performing several numerical simulations to observe dynamical behavior in the model. The numerical method used in this paper is the predictor-corrector method for fractional-order equations.

RESULTS AND DISCUSSION

Mathematical Model

The interaction between prey and predator population is presented in a modified Leslie-Gower model proposed by Aziz-Alaoui and Okiye [27] and Yu [28]. For the next research, Yu [29] considers a modified Leslie-Gower model incorporating the Beddington-DeAngelis functional response. The modified Leslie-Gower model with Beddington-DeAngelis function response can be written as follows.

$$\begin{aligned} \frac{dN}{dT} &= rN\left(1-\frac{N}{K}\right)-\frac{aNP}{1+bN+cP}, \\ \frac{dP}{dT} &= sP\left(1-\frac{eP}{k+N}\right), \end{aligned} \tag{1}$$

where $N = N(T)$ and $P = P(T)$ are the density of prey and predator population at time t. The parameters r, K, b, c, w, s, e, k are positive values. In the particular, the biological meaning of parameters can be shown in **Table 1**.

Table 1. The biological meaning of parameters in system (1)

Parameter	Biological Meaning
r	The intrinsic growth rate of prey
s	The intrinsic growth rate of predator
a	The capture rate by predator against prey
b	The measure of handling time by predator against prey
c	The amount of disturbance among predator
e	The reproduction rate of predator
K	The carrying capacity of prey
k	The environmental protection of predator

According to [30], the predation of the predator can influence the behavior of prey indirectly resulting in fear. Consequently, the protection of frightened prey diminishes and leaves their newborn [20]. Therefore, we consider the fear factor multiplying the intrinsic growth rate of prey with $f(u,P) = \frac{1}{1+uP}$, where the parameter u is the fear rate of prey. Biologically, the fear factor satisfies several conditions as follows.

$$f(0,P) = 1, f(u,0) = 1, \lim_{u\to\infty} f(u,P) = 0,$$
$$\lim_{u\to\infty} f(u,P) = 0, \frac{\partial f(u,P)}{\partial u} < 0, \frac{\partial f(u,P)}{\partial P} < 0.$$

The biological meaning of conditions can be shown in [31]. In this article, we are interested to observe the Allee effect as done by Feng and Kang [8] and assume that the Allee effect occurs only in predators. Therefore, our system becomes the following system.

$$\begin{aligned} \frac{dN}{dT} &= \frac{rN}{1+uP} - \frac{rN^2}{K} - \frac{aNP}{1+bN+cP}, \\ \frac{dP}{dT} &= sP\left(\frac{P}{P+n} - \frac{eP}{k+N}\right), \end{aligned} \tag{2}$$

where the parameter n is the measure of the Allee effect in predator.

For simplicity, system (2) is formed into a non-dimensional system by using parameters $(x,y,t) \to \left(\frac{N}{K}, \frac{eP}{K}, rT\right)$. Thus, we obtain the following system.

$$\begin{aligned} \frac{dx}{dt} &= \frac{x}{1+\rho y} - x^2 - \frac{xy}{\delta+\beta x+\gamma y}, \\ \frac{dy}{dt} &= \theta y\left(\frac{y}{y+\mu} - \frac{y}{\nu+x}\right), \end{aligned} \tag{3}$$

where $\rho = \frac{uK}{e}, \delta = \frac{re}{aK}, \beta = \frac{ber}{a}, \gamma = \frac{cr}{a}, \theta = \frac{s}{r}, \mu = \frac{ne}{K}, \nu = \frac{k}{K}$. From system (3), we can see that the system only has 7 dimensionless parameters, that is $\rho, \delta, \beta, \gamma, \theta, \mu$, and ν. Non-dimensionalization can reduce the number of parameters by grouping them in a meaningful way. In general, the groupings can provide a relative measure of the influence of dimension parameters [32]. For example, θ is the ratio of growth rates of predator to prey such that $\theta > 1$ and $\theta < 1$ have definite ecological significance. That is, prey can reproduce more rapidly than predator.

Model with Caputo Operator

To form system (3) into a fractional-order model, we use the similar manner as in [1], [22], [24], [33]. By replacing the left-hand sides of the system (3) with the Caputo fractional-order derivative, we obtain the following model with α is the memory effect.

$$\begin{aligned} D_*^\alpha x &= \frac{x}{1+\rho y} - x^2 - \frac{xy}{\delta+\beta x+\gamma y}, \\ D_*^\alpha y &= \theta y\left(\frac{y}{y+\mu} - \frac{y}{\nu+x}\right), \end{aligned} \tag{4}$$

where D_*^α shows the Caputo fractional-order derivative for a real-valued function f defined as follows.

$$D_*^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau,$$

with $t \geq 0, \Gamma(\cdot)$ is Euler's Gamma function, $f \in C^n([0,+\infty),\mathbb{R})$, and $\alpha \in [0,1)$ [34].

When we replace the operator with the Caputo fractional-order, our model has an inconsistency time's dimension between the left-hand sides of model with its right-hand side. To overcome this condition, we can adjust by rescaling all favorable parameters. Thus, let $\hat{\theta} = \theta^\alpha, \hat{\mu} = \mu^\alpha, \hat{\nu} = \nu, \hat{\rho} = \rho^\alpha, \hat{\delta} = \delta^\alpha, \hat{\beta} = \beta^\alpha$, and $\hat{\gamma} = \gamma^\alpha$, we have

$$\begin{aligned} D_*^\alpha x &= \frac{x}{1+\hat{\rho}y} - x^2 - \frac{xy}{\hat{\delta}+\hat{\beta}x+\hat{\gamma}y}, \\ D_*^\alpha y &= \hat{\theta}y\left(\frac{y}{y+\hat{\mu}} - \frac{y}{\hat{\nu}+x}\right). \end{aligned} \tag{5}$$

Form model (5), we can simplify by re-denoting all parameters. We remove the hat $\hat{}$ on each parameters. Thus, we have the final model as follows.

$$\begin{aligned} D_*^\alpha x &= \frac{x}{1+\rho y} - x^2 - \frac{xy}{\delta+\beta x+\gamma y}, \\ D_*^\alpha y &= \theta y\left(\frac{y}{y+\mu} - \frac{y}{\nu+x}\right). \end{aligned} \tag{6}$$

Equilibrium Points and Local Stability

To investigate the existence and stability of the equilibrium points, we use the Magtinon condition presented in the following condition.

Lemma 1. *(See* [35]*) Consider the Caputo fractional-order system with initial conditions as follows.*

$$D_*^\alpha \vec{x}(t) = \vec{f}(t,\vec{x}), \vec{x}(0) = \vec{x}_0,$$

where $x \in \mathbb{R}^n, n \in \mathbb{N}$, *and* $\alpha = (0,1]$. *The point* $\vec{x}^*$ *can be called an equilibrium point when* $\vec{x}^*$ *satisfies* $\vec{f}(t,\vec{x}^*) = 0$. *Moreover, the point* $\vec{x}^*$ *is locally asymptotically stable when the eigenvalues* $\lambda_i, i = 1,2,\dots,n$ *of the Jacobian matrix* $J(\vec{x}^*)$ *satisfies* $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

From **Lemma 1**, we identify the equilibrium points of the system (6) by setting $D_*^\alpha x = D_*^\alpha y = 0$. Therefore, we obtain the equilibrium points and their existence condition as follows.

1) The point $E_0(0,0)$ is always exists. It means that both populations are extinct.
2) The point $E_1(1,0)$ is always exists. It explains that the predator is extinct.
3) The point $E_2(0,\nu-\mu)$ exists when $\nu > \mu$. It shows that prey is extinct.
4) The interior point $E^*(x^*,y^*)$, where $x^* = y^* + \mu - \nu$ and exists if $y^* > \nu - \mu$ with y^* is positive roots of the cubic equations as follows.

$$(y^*)^3 + \omega_1(y^*)^2 + \omega_2 y^* + \omega_3 = 0, \tag{7}$$

where

$$\omega_1 = \frac{\rho(\mu-\nu)(2\beta+\gamma)+\delta\rho+\beta+\gamma+\rho}{3\rho(\gamma+\beta)},$$

$$\omega_2 = \frac{\beta\rho(\mu^2 - 2\mu\nu + \nu^2) + (\mu - \nu)(\delta\rho + 2\beta + \gamma) - \beta + \delta - \gamma + 1}{3\rho(\gamma + \beta)},$$
$$\omega_3 = \frac{(\mu - \nu - 1)(\beta\mu - \beta\nu + \delta)}{\rho(\gamma + \beta)}.$$

Let $s = y^* + \omega_1$, we obtain

$$g(s) = s^3 + 3ps + q = 0, \tag{8}$$

where $p = \omega_2 - \omega_1^2$ and $q = \omega_3 - 3\omega_1\omega_2 + 2\omega_1^3$. By using Cardan's method as in [7], we have the existence condition of an equilibrium point as follows.

Lemma 2. *Let $y^* > \nu - \mu$. The point $E^*(x^*, y^*)$ is a positive equilibrium point with y^* is positive roots of (7) when it satisfies one of the following conditions.*

1) *If $q < 0$, then (8) has a single positive root. Thus, system (6) has a unique equilibrium point $E^*(s_1 - \omega_1 + \mu - \nu, s_1 - \omega_1)$ with $s_1 > \omega_1$.*
2) *If $q > 0$ and $p < 0$, then*
   a) *If $q^2 + 4p^3 = 0$, then (8) has a positive root of multiplicity two. Thus, system (4) has a unique equilibrium point $E^*\left(\sqrt{-p} + \mu - \nu, \sqrt{-p}\right)$.*
   b) *If $q^2 + 4p^3 < 0$, then (8) has two positive points. Thus, system (6) has two possible equilibrium points, that is, $E_1^*(s_1 - \omega_1 + \mu - \nu, s_1 - \omega_1)$ and/or $E_2^*(s_2 - \omega_1 + \mu - \nu, s_2 - \omega_1)$ with $s_{1,2} > \omega_1$.*
3) *If $q = 0$ and $p < 0$, then (8) has an unique positive root. Thus, system (6) has an unique equilibrium point $E^*\left(\sqrt{-3p} + \mu - \nu, \sqrt{-3p}\right)$.*

It is known that if (8) has two positive roots, then their positive roots are $s_1 = \frac{\left(-4q+4\sqrt{4p^3+q^2}\right)^{\frac{2}{3}}-4p}{2\left(-4q+4\sqrt{4p^3+q^2}\right)^{\frac{1}{3}}}$ and $s_2 = -\frac{s_1}{2} + \frac{\sqrt{s_1^3+4q}}{2\sqrt{s_1}}$. Meanwhile, if (8) has a positive root, then the positive root is $s_1 = \frac{\left(-4q+4\sqrt{4p^3+q^2}\right)^{\frac{2}{3}}-4p}{2\left(-4q+4\sqrt{4p^3+q^2}\right)^{\frac{1}{3}}}$.

The local stability analysis is done by identifying the eigenvalue of the Jacobian matrix. The Jacobian matrix $J(E)$ for all equilibrium points (x, y) can be presented as follows.

$$J(E) = \begin{bmatrix} \frac{1}{1+\rho y} - 2x - \frac{y(\delta + \gamma y)}{(\delta + \beta x + \gamma y)^2} & -\frac{x\rho}{(1+\rho y)^2} - \frac{x(\delta + \beta x)}{(\delta + \beta x + \gamma y)^2} \\ \theta\left(\frac{y}{x+\nu}\right)^2 & \frac{\theta y(y + 2\mu)}{(y+\mu)^2} - \frac{2\theta y}{x+\nu} \end{bmatrix}. \tag{9}$$

The local stability of any point is ensured by the following theorems.

Theorem 1. *$E_0(0,0)$ is unstable and $E_1(1,0)$ is a non-hyperbolic point.*
Proof: The Jacobian matrix for $E_0(0,0)$ and $E_1(1,0)$ is presented as follows.

$$J(E_0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$J(E_1) = \begin{bmatrix} -1 & -\rho - \dfrac{1}{\beta + \delta} \\ 0 & 0 \end{bmatrix}.$$

Based on the Jacobian matrix $J(E_0)$, we know that the eigenvalues of $J(E_0)$ are $\lambda_1 = 1$ and $\lambda_2 = 0$. Thus, the point E_0 is unstable because $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$. Moreover, for $E_1(1,0)$, the eigenvalues of $J(E_1)$ are $\lambda_1 = -1$ and $\lambda_2 = 0$. It is shown that $|\arg(\lambda_2)| = \frac{\alpha\pi}{2}$. Thus, the point E_1 is a non-hyperbolic point.

Theorem 2. *$E_2(0, \nu - \mu)$ is locally asymptotically stable.*
Proof: By substituting E_2 into (9), we obtain

$$J(E_2) = \begin{bmatrix} \dfrac{1}{1 + \rho(\nu - \mu)} - \dfrac{(\nu - \mu)}{\delta + (\nu - \mu)\gamma} & 0 \\ \theta\left(\dfrac{\nu - \mu}{\nu}\right)^2 & -\theta\left(\dfrac{\nu - \mu}{\nu}\right)^2 \end{bmatrix}.$$

From Jacobian matrix $J(E_2)$, we obtain the eigenvalues, that is $\lambda_1 = \frac{1}{1+\rho(\nu-\mu)} - \frac{(\nu-\mu)}{\delta+(\nu-\mu)\gamma}$ and $\lambda_2 = -\theta\left(\frac{\nu-\mu}{\nu}\right)^2$. By using the existence condition of E_2, it is clear that $\lambda_1 < 0$ when $\frac{1}{1+\rho(\nu-\mu)} < \frac{(\nu-\mu)}{\delta+(\nu-\mu)\gamma}$ and $\lambda_2 < 0$. Thus, $\left|\arg(\lambda_{1,2})\right| = \pi > \frac{\alpha\pi}{2}$. In other words, the point E_2 is locally asymptotically stable.

Theorem 3. *Suppose that*

$$\begin{aligned}
\varphi_1 &= \frac{y^*(\delta + 2\beta x^* + \gamma y^*)}{(\delta + \beta x^* + \gamma y^*)^2} - \theta\left(\frac{y^*}{y^* + \mu}\right)^2 - \frac{1}{1 + \rho y^*}, \\
\varphi_2 &= \theta\left(\frac{y^*}{y^* + \mu}\right)^2 \left(\frac{x^*(\delta + \beta x^*) - y^*(\delta + 2\beta x^* + \gamma y^*)}{(\delta + \beta x^* + \gamma y^*)^2} + \frac{1 + \rho(x^* + y^*)}{(1 + \rho y^*)^2}\right), \\
\alpha^* &= \frac{2}{\pi}\left|\tan^{-1}\left(\frac{\sqrt{4\varphi_2 - \varphi_1^2}}{\varphi_1}\right)\right|.
\end{aligned}$$

$E^*(x^*, y^*)$ is locally asymptotically stable if it satisfies one of the following conditions.
a) $\varphi_1^2 \geq 4\varphi_2, \varphi_1 < 0,$ *and* $\varphi_2 > 0$.
b) $\varphi_1^2 < 4\varphi_2,$ *and if* $\varphi_1 < 0,$ *or* $\varphi_1 > 0$ *and* $\alpha < \alpha^*$.
Proof: The Jacobian matrix at $E^*(x^*, y^*)$ is presented as follows.

$$J(E^*) = \begin{bmatrix} -\dfrac{1}{1 + \rho y^*} + \dfrac{y^*(\delta + 2\beta x^* + \gamma y^*)}{(\delta + \beta x^* + \gamma y^*)^2} & -x^*\left(\dfrac{\rho}{(1 + \rho y^*)^2} + \dfrac{(\delta + \beta x^*)}{(\delta + \beta x^* + \gamma y^*)^2}\right) \\ \theta\left(\dfrac{y^*}{y^* + \mu}\right)^2 & -\theta\left(\dfrac{y^*}{y^* + \mu}\right)^2 \end{bmatrix},$$

From Jacobian matrix $J(E^*)$, we have the characteristic equation $\lambda^2 - \varphi_1\lambda + \varphi_2 = 0$. Therefore, we obtain the eigenvalues $\lambda_{1,2} = \frac{\varphi_1 \pm \sqrt{\Lambda}}{2}$ with $\Lambda = \varphi_1^2 - 4\varphi_2$. Suppose $\varphi_1^2 \geq 4\varphi_2$, we have $\Lambda \geq 0$. In this condition, we observe that $\lambda_{1,2} < 0$ when $\varphi_1 < 0$ and $\varphi_2 > 0$. Consequently, $\left|\arg(\lambda_{1,2})\right| > \frac{\alpha\pi}{2}$. Thus, the point E^* is locally asymptoticaly stable. However, if $\varphi_1^2 < 4\varphi_2$, then $\Lambda < 0$. The eigenvalues $\lambda_{1,2}$ are a pair of complex conjugate. By using **Lemma 1**, $\left|\arg(\lambda_{1,2})\right| > \frac{\alpha\pi}{2}$ when $\alpha < \alpha^*$ and one of two conditions, that is $\varphi_1 < 0$ or $\varphi_1 > 0$. Thus, E^* is locally asymptoticaly stable. Finally, the stability condition of $E^*(x^*, y^*)$ is proven.

Hopf Bifurcation

Hopf bifurcation for a fractional-order system is a change in the stability of systems that enters a limit cycle when there is a pair of complex eigenvalues. Based on **Theorem 3**, the order of derivatives can affect the stability of the interior equilibrium point with $\varphi_1^2 < 4\varphi_2$ and $\varphi_1 > 0$. In this article, we take the order of derivative as the bifurcation parameter. Therefore, to ensure the existence of Hopf bifurcation, we present the following theorem.

Theorem 4. *(Existence of Hopf Bifurcation) Suppose that $\varphi_1^2 < 4\varphi_2$ and $\varphi_1 > 0$. The point E^* enters a Hopf bifurcation when α passes through α^*.*

Proof: Based on **Theorem 3**, the roots of the characteristic equation for E^* are a pair of complex conjugate eigenvalues with positive real parts. It is easy to confirm that $m(\alpha^*) = 0$ and $\left.\frac{dm(\alpha^*)}{d\alpha}\right|_{\alpha=\alpha^*} \neq 0$ with $m(\alpha) = \frac{\alpha\pi}{2} - \min_{1 \leq i \leq 2} |\arg(\lambda_i)|$. According to [36], the point E^* undergoes a Hopf bifurcation when α crosses α^*.

Numerical Simulations

To demonstrate the dynamical analysis results, we perform several numerical simulations. In this article, we use the predictor-corrector approximation for fractional-order equations developed by Diethelm, et. al. [37]. Because the field data is not available, we choose the parameter values that satisfy the results of the stability conditions obtained from the previous discussion. For the first simulation, we choose several parameter values as follows: $\rho = 1.2, \delta = 0.1, \beta = 0.8, \gamma = 0.3, \theta = 0.6, \mu = 0.3, \nu = 0.5, \alpha = 0.9$. In this work, we obtain that $E_2(0,0.2)$ is a unique equilibrium point because there are two unstable points, that is $E_0(0,0)$ and $E_1(1,0)$, and a local stable point $E_2(0,0.2)$ which is shown by all solutions converged to E_2. We can see its phase portrait in **Figure 1**. It means that the predator can live for a long time even though the prey has become extinct. It is caused because the great fear of prey causes prey to become extinct. Meanwhile, the predator can survive against the Allee effect because the environmental protection of predator is very good.

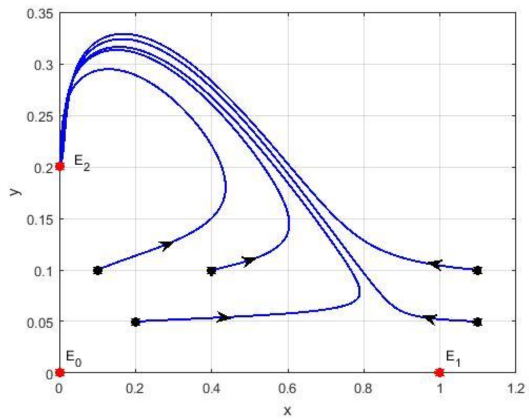
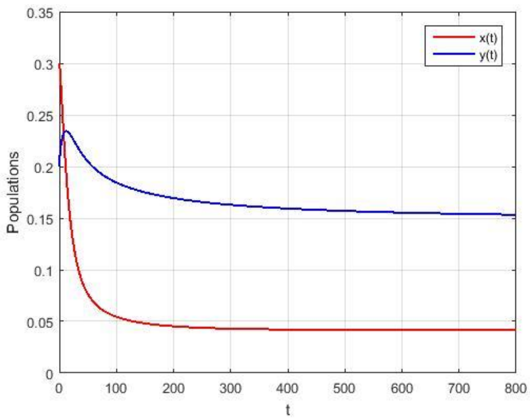
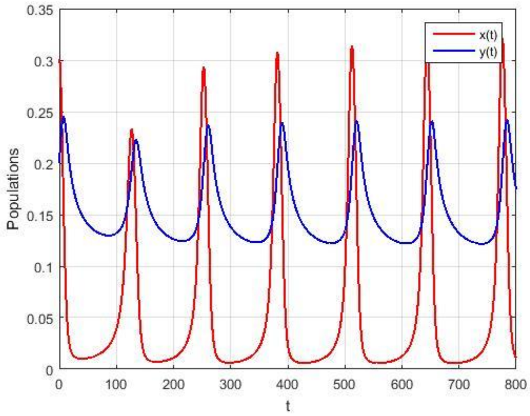


Figure 1. Phase portrait of system (6) using the parameter values as follows: $\rho = 1.2, \delta = 0.1, \beta = 0.8, \gamma = 0.3, \theta = 0.6, \mu = 0.3, \nu = 0.5, \alpha = 0.9$.

For the second simulation, we set $\mu = 0.4$ and the other parameter values are made the same as in the previous simulation. Based on **Lemma 2**, the point E^* lies in the interior plane. Therefore, we can interpret that all populations can live together for a long time because some predator populations die due to increased the Allee effect. Thus, the population density of predator will descrease. Consequently, prey population can survive. Now, if we take $\alpha = 0.6$, the solution converges to the equilibrium point E^* as in **Figure 2(a)**. In other words, the point E^* is a local stable point. Meanwhile, when we take $\alpha = 0.9$, the solution loses stability and oscillates as in **Figure 2(b).** It shows that the solution enters limit cycles or undergoes a Hopf bifurcation which corresponds to **Theorem 4**. Therefore, the population density fluctuates when the memory effect is large, that is $\alpha > 0.9$.



(a) $\alpha = 0.6$



(b) $\alpha = 0.9$

Figure 2. Time series of system (6) using the parameter values as follows: $\rho = 1.2, \delta = 0.1, \beta = 0.8, \gamma = 0.3, \theta = 0.6, \mu = 0.4, \nu = 0.5$

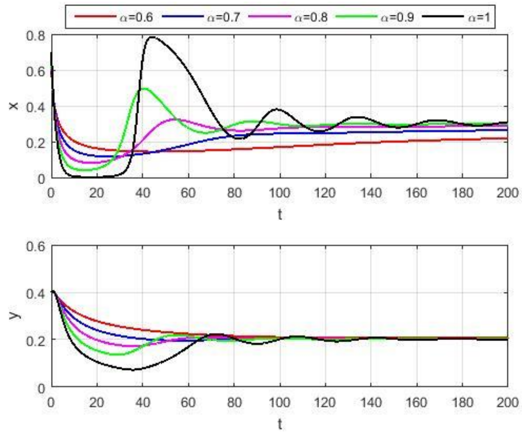


Figure 3. Time series of system (6) using the parameter values as follows: $\rho = 1.2, \delta = 0.1, \beta = 0.8, \gamma = 0.3, \theta = 0.6, \mu = 0.6, \nu = 0.5$

In this work, we want to observe the effect of the order derivative on the fractional-order system. By using $\rho = 1.2, \delta = 0.1, \beta = 0.8, \gamma = 0.3, \theta = 0.6, \mu = 0.6, \nu = 0.5$, we obtain the simulation as in **Figure 3** with $\alpha = \{0.6, 0.7, 0.8, 0.9, 1\}$. It is obseved that all solutions oscillate and converge to the interior point E^*. Moreover, if we assign the order of derivative with larger values, then we can observe that the solution of system converges more quickly to the equilibrium point. Therefore, the order of derivative affects the rate of convergence. It means that the when memory effect is large, then in this case, the prey population can increase for a long time. Moreover, the predator populations tend to remain constant.

From the previous experiment, we observe that the Allee effect greatly affects the dynamical behaviour. Therefore, we want to observe the impact of Allee effect on the fractional-order system. By using $\alpha = 0.6$ and setting the parameter μ with varying values, we get the simulation as in **Figure 4.** In this work, it is shown that the number of prey increases when the parameter μ is large. But, when we take a large parameter μ, the number of predator decreases. Thus, the Allee effect constant is inversely proportional to the number of predator and directly proportional to the number of prey. This result is the same as the conslusion in [9]. Now, we observe the influence of fear effect on the system. By assigning $\mu = 0.4$ and using the varying parameters ρ, we have the simulation as in **Figure 5**. It is obeserved that all populations decrease when the parameter ρ is huge. This means that the fear effect constant is inversely proportional to the density of two populations.

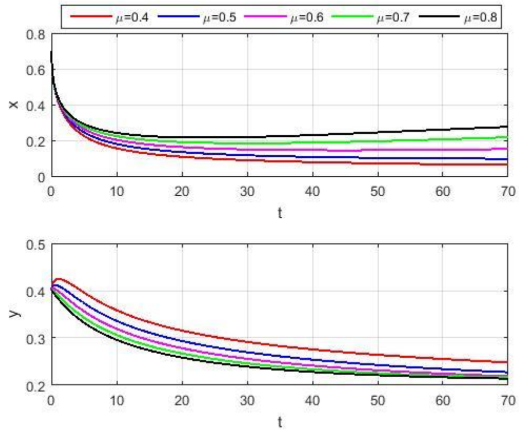


Figure 4. The time series of system (6) using the parameter values as follows: $\rho = 1.2, \delta = 0.1, \beta = 0.8, \gamma = 0.3, \theta = 0.6, \nu = 0.5, \alpha = 0.6$

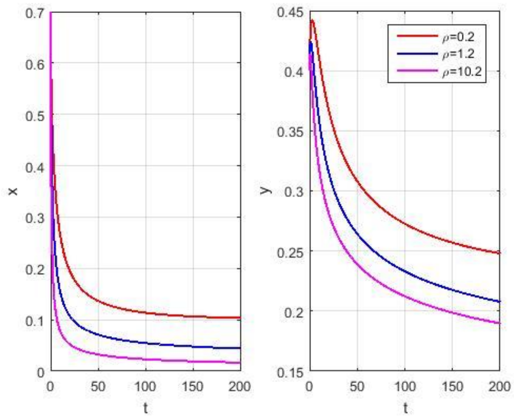


Figure 5. The time series of system (6) using the parameter values as follows: $\delta = 0.1, \beta = 0.8, \gamma = 0.3, \theta = 0.6, \nu = 0.5, \mu = 0.4, \alpha = 0.6$

CONCLUSIONS

This manuscript has studied the predator-prey interaction formed into a fractional-order Leslie-Gower model with Allee and fear effect. We find two local stable points and two unstable points with certain conditions of existence and stability. We also present the existence of Hopf bifurcation by assigning the order of derivative as the bifurcation parameter both analytically and numerically. From the results obtained, when the order of fractional derivative is large, then the solution converges faster. Here, we observe that when $\alpha < 0.9$, then the solutions of system are locally asymptotically stable. However, when $\alpha > 0.9$, then the solutions of system are isolated by limit cycle via Hopf bifurcation. We also conclude that the Allee effect constant is directly proportional to the number of prey and inversely proportional to the number of predator. Furthermore, the fear effect constant is inversely proportional to the number of the two populations.

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