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On Rainbow Antimagic Coloring of Join Product of Graphs

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ABSTRACT

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A bijection f from $V(G)$ to the set $\{1,2,\dots,|V(G)|\}$ is a labeling of graph G. The bijection f is called rainbow antimagic labeling if for any two edge xy and $x'y'$ in path $u-v, w(xy) \neq w(x'y')$, where $w(xy) = f(x) + f(y)$ and $u, v \in V(G)$. Rainbow antimagic coloring is a coloring of graph G which has a rainbow antimagic labeling. Thus, every rainbow antimagic labeling induces a rainbow coloring G where the edge weight $w(xy)$ is the color of the edge xy. The rainbow antimagic connection number of graph G is the smallest number of colors of all rainbow antimagic colorings of graph G, denoted by $rac(G)$. In this study, we studied rainbow antimagic coloring and have an exact value of rainbow antimagic connection number of join product of graph $G + K_1$ where G is graph mP_n, graph mS_n, graph $m\mathcal{F}_n$, graph mF_n and graph $mS_{n,n}$.

Keywords: antimagic labeling; join product of graphs; rainbow coloring; rainbow antimagic connection number

INTRODUCTION

In this paper, we use a simple and connected graphs. The graph definition used in this study refferes to Chartrand *et al.* [7]. The join product of graphs G_1 and G_2 which is denoted by $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1), y \in V(G_2)\}$ [13]. While rainbow antimagic coloring of graph is new concept with combining the concept of rainbow coloring and antimagic labeling of graph. Rainbow coloring was first introduced by Chartrand *et al*. [8]. Let G be a connected graph, the edge coloring of G with the function $c: E(G) \to \{1,2,\dots,k\}$ $k \in \mathbb{N}$, is k-coloring of the graph G where the two adjacent edges can be colored with the same color. Rainbow $u-v$ path is the path in G if no two edges are the same color. The graph G is a rainbow connection if every $u, v \in V(G)$ has a rainbow path. The edge coloring on G has a rainbow connection called rainbow coloring. The minimum colors to make G rainbow-connected is called the rainbow connection number of G and denoted by $rc(G)$. The research on rainbow connection number has gained many results, including [18] and [19].

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The study of rainbow coloring has variants including rainbow vertex coloring and total rainbow coloring. Rainbow vertex coloring was first introduced in [17]. The results of rainbow vertex coloring are found in [20], [22]. Another variant of rainbow coloring is total rainbow coloring and the results can be seen in [15] and [23]. Graph labeling was introduced in [24] by Wallis *et al.* (2001). Hartsfield and Ringel introduced antimagic labeling for the first time in [14]. Antimagic labeling has had several results including by Baca *et al.* in [2,3,4,5]. Then Dafik *et al.* has contributed to the antimagic labeling in [11]. In addition, antimagic labeling results can also be found in [9] and [10].

Arumugam *et al.* [1] initiated the study of combining graph coloring and graph labeling. The bijective function from edge set $E(G)$ to $\{1,2,\ldots,|E(G)|\}$ and $w(v) = \sum_{xy\in E(v)} f(xy)$ and $E(v)$ is the set of edges that are incident to vertices v, for every $v \in V(G)$. The bijective function f if for two adjacent vertices $u,v \in V(G)$, $w(u) \neq w(v)$ is called antimagic labeling. The coloring of the vertices on G with the vertices of v colored with $w(v)$ is the local antimagic labeling. If we consider the chromatic number of the local antimagic labeling, then this notion is called local antimagic coloring. Motivated the combination done by Arumugam, Dafik $et\,al.$ in [12] defined the combination of the concepts of rainbow coloring and antimagic labeling into a new concept called rainbow antimagic coloring. Dafik *et al.* determine the theorem about the existence of rainbow $x-y$ path of any graph of $diam(G) \leq 2$.

Theorem 1. Let G be a connected graph of diameter $diam(G) \leq 2$. Let $f: V(G) \to \{1,2,\ldots,|V(G)|\}$ be any bijective function. For any $u,v \in V(G)$ there exists a rainbow $u-v$ path.

Rainbow antimagic coloring results have been studied in [6], [12], [16] and [21]. In this study, we studied rainbow antimagic coloring and have an exact value of rainbow antimagic connection number of join product of graph $G + K_1$ where G is graph mP_n, graph mS_n, graph $m\mathcal{F}_n$, graph mF_n and graph $mS_{n,n}$.

METHOD

To determine the number of rainbow antimagic coloring of graph, we use the following steps:

1. For any graph G, identify the set of vertices $V(G)$ and set of edges $E(G)$.
2. Analyze the lower bound of rainbow antimagic connection number (rac) based on Lemma: $rac(G) \geq \max\{rc(G), \Delta(G)\}$.
3. Label the vertices of the graph G with the function: $V(G) \to \{1,2,3,\ldots,|V(G)|\}$.
4. Determine the edge weight based on the sum of vertex label which incident with the edge. To calculate edge weight we give the function, $w(uv) = f(u) + f(v)$ for $u, v \in V(G)$.
5. Verify that every two vertex in the graph G have rainbow paths. If not, repeat the step 3.
6. Determine the upper bound of $rac(G)$ from the number of different edge weight.
7. The exact value of rainbow antimagic connection number can be determined if lower bound is the same with upper bound of rainbow antimagic connection number.

RESULTS AND DISCUSSION

In this section, we will show our new results on join product of graphs stated in a lemma and theorem. First we create a lower bound for the graph $mG + K_1$ then use it to determine the result of the rainbow antimagic connection number of the graph $mP_n + K_1$, graph $mS_n + K_1$, graph, graph $mF_n + K_1$ and graph $mS_{n,n} + K_1$.

Lemma 1. *For $m \geq 3$ and G is connected graph, $rac\ (mG + K_1) \geq \Delta(mG + K_1)$.*

Proof. The graph $mG + K_1$ is the comb product of graphs mG and K_1. It is obtained by taking m copy of G and one copy of K_1 and joining the vertex of K_1 to every vertex in the m copy of G. $V(mG + K_1) = V(mG) \cup V(K_1)$ and $E(mG + K_1) = E(mG) \cup E(K_1) \cup \{xy\ |x \in V(mG), y \in V(K_1)\}$. By this definition, we have $d(y) = \Delta(mG + K_1)$ and $f: V\ (mG + K_1) \rightarrow \{1,2,3, \dots, |V(mG + K_1)|\}$ is a bijection function. Since f is a bijection function we have $f(y) \neq f(x)$, for every $y, x \in V\ (mG + K_1)$ so that for every $yx, yz \in E(mG + K_1), w(yx) \neq w(yz)$. Therefore, $rac(mG + K_1) \geq \Delta(mG + K_1)$. Based on the description above, $rac(mG + K_1) \geq \Delta(mG + K_1)$. ■

Theorem 2. *For $n, m \geq 3$, $rac(mP_n + K_1) = mn$.*

Proof. Let $(mP_n + K_1)$ be a graph with vertex set $V(mP_n + K_1) = \{x_{ij}, 1 \leq i \leq m, 1 \leq i \leq n\} \cup \{v\}$ and edge set $E(mP_n + K_1) = \{vx_{ij}, 1 \leq i \leq m, 1 \leq n\} \cup \{x_{ij}x_{ij+1}, 1 \leq i \leq m, 1 \leq n - 1\}$. The cardinality of $V(mP_n + K_1)$ is $|V(mP_n + K_1)| = mn + 1$ and the cardinality of $E(mP_n + K_1)$ is $|E(mP_n + K_1)| = m(2n - 1)$. To have the rainbow antimagic connection number of $mP_n + K_1$, first we need to show the lower bound of $rac(mP_n + K_1)$. Clearly, according to **Lemma 1** we have $rac\ (mP_n + K_1) \geq mn$.

Secondly, to have the exact value we have to show the upper bound of $rac(mP_n + K_1)$. Let $f : V(mP_n + K_1) \rightarrow \{1,2, \dots, mn + 1\}$ be a vertex labeling of graph $mP_n + K_1$ defined as follows.

$$f(v) = 2m + 1$$

$$f(x_{ij}) = \begin{cases} i & , \text{ for } 1 \leq i \leq m, j = 1 \\ i + j + 1 & , \text{ for } 1 \leq i \leq m, j = 3 \\ 2j + i - 1 & , \text{ for } j \text{ is odd}, 1 \leq i \leq m, 5 \leq j \leq n \\ mn - i + 2 & , \text{ for } 1 \leq i \leq m, j = 2 \\ n - i - j + 2 & , \text{ for } j \text{ is even}, 1 \leq i \leq m, 4 \leq j \leq n \end{cases}$$

For the edge weights, we have :

$$w(vx_{ij}) = \begin{cases} 2m + i + 1 & , \text{ for } 1 \leq i \leq m, j = 1 \\ 2m + i + j + 2 & , \text{ for } 1 \leq i \leq m, j = 3 \\ 2m + i + 2j & , \text{ for } j \text{ is odd}, 1 \leq i \leq m, 5 \leq j \leq n \\ 2m + nm - i - j & , \text{ for } 1 \leq i \leq m, j = 2 \\ 2m + nm - i - j + 3 & , \text{ for } j \text{ is even}, 1 \leq i \leq m, 4 \leq j \leq n \end{cases}$$

$$w(x_{ij}x_{ij+1}) = \begin{cases} mn + 2 & , \text{ for } j \text{ is odd}, 1 \leq i \leq m \\ mn + n + 1 & , \text{ for } j \text{ is even}, 1 \leq i \leq m, j \neq n - 1 \\ mn + n + 2 & , \text{ for } 1 \leq i \leq m, j = n - 1 \end{cases}$$

It is easy to see the edge weights of $f : V(mP_n + K_1) \rightarrow \{1,2, \dots, mn + 1\}$ induces a rainbow antimagic coloring of mn colors. Thus $rac(mP_n + K_1) \leq mn$. Comparing the two bounds, we have the exact value of $rac\ (mP_n + K_1) = mn$. The last is to evaluate the existence of rainbow $x - y$ path of $mP_n + K_1$. Since $diam(mP_n + K_1) = 2$, based on **Theorem 1**, for every two vertices $x, y \in V(mP_n + K_1)$, there exists a rainbow $x - y$ path. It completes the proof. ■

For an illustration, a rainbow antimagic coloring of graph $mP_n + K_1$ can be seen in Figure 1.

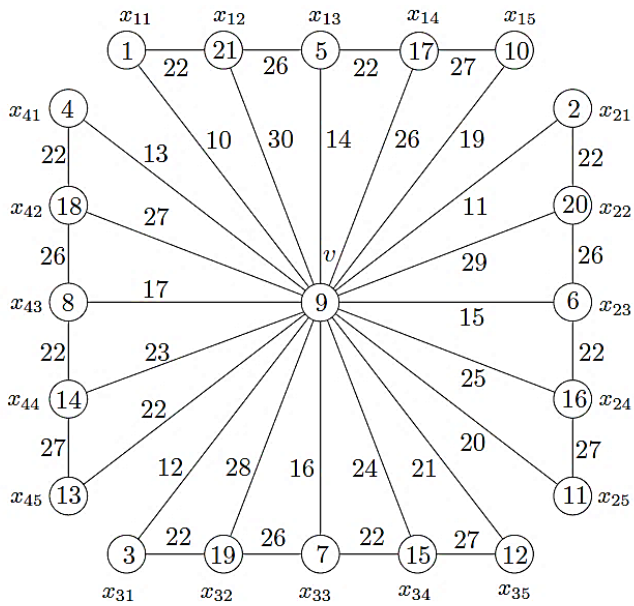

Figure 1. A rainbow antimagic coloring of of join product of graph $4P_5 + K_1$.

Theorem 3. *For* $n, m \geq 3$, $rac(mS_n + K_1) = mn$.

Proof. Let $mS_n + K_1$ be a graph with vertex set $V(mS_n + K_1) = \{v\} \cup \{x_i, 1 \leq i \leq m\} \cup \{x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and edge set $E(mS_n + K_1) = \{vx_{ij}, 1 \leq i \leq m, 1 \leq j \leq n-1\} \cup \{x_i x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n-1\} \cup \{vx_i, 1 \leq i \leq m\}$. The cardinality of $V(mS_n + K_1)$ is $|V(mS_n + K_1)| = mn + 1$ and the cardinality of $E(mS_n + K_1)$ is $|E(mS_n + K_1)| = 2mn - m$. To have the rainbow antimagic connection number of $mS_n + K_1$, first we need to show the lower bound of $rac(mS_n + K_1)$. Clearly, according to **Lemma 1** we have $rac\ (mS_n + K_1) \geq mn$.

Secondly, to have the exact value we have to show the upper bound of $rac(mS_n + K_1)$. Let $f : V(mS_n + K_1) \to \{1,2,\ldots,mn+1\}$ be a vertex labeling of graph $mS_n + K_1$ defined as follows.

$$f(v) = m + 2$$

$$f(x_i) = i \quad , \quad \text{for } 1 \leq i \leq m$$

$$f(x_{ij}) = \begin{cases} m+1 & , \quad \text{for } i = m, j = 1 \\ m - i + mj + 2 & , \quad \text{for } 1 \leq i \leq m-1, 2 \leq j \leq n \end{cases}$$

For the edge weights, we have :

$$w(x_i x_{ij}) = \begin{cases} m+i+1 & , \quad \text{for } i = m, j = 1 \\ m + mj + 2 & , \quad \text{for } 1 \leq i \leq m-1, 2 \leq j \leq n \end{cases}$$

$$w(vx_i) = m + i + 2 \quad , \quad \text{for } 1 \leq i \leq m$$

$$w(vx_{ij}) = \begin{cases} 2m+3 & , \quad \text{for } i = m, j = 1 \\ 2m - i + mj + 4 & , \quad \text{for } 1 \leq i \leq m-1, 2 \leq j \leq n \end{cases}$$

It is easy to see the edge weights of $f : V(mS_n + K_1) \to \{1,2,\ldots,mn+1\}$ induces a rainbow antimagic coloring of mn colors. Thus $rac(mS_n + K_1) \leq mn$. Comparing the two bounds, we have the exact value of $rac\ (mS_n + K_1) = mn$. The last is to evaluate the existence of rainbow $x - y$ path of $mS_n + K_1$. Since $diam(mS_n + K_1) =$

2, based on **Theorem 1**, for every two vertices $x, y \in V(mS_n + K_1)$, there exists a rainbow $x - y$ path. It completes the proof. ■

For an illustration, a rainbow antimagic coloring of graph $mS_n + K_1$ can be seen in Figure 2.

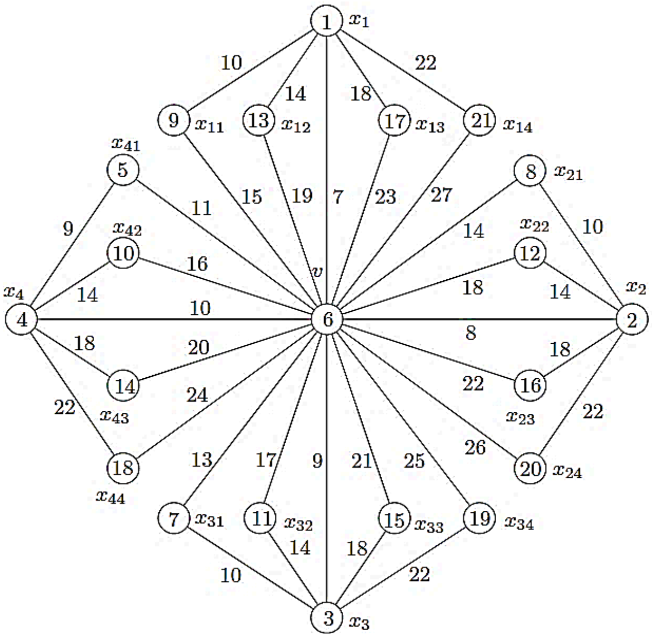

Figure 2. A rainbow antimagic coloring of of join product of graph $4S_5 + K_1$.

Theorem 4. *For* $n, m \geq 3$, $rac(m\mathcal{F}_n + K_1) = 2mn + m$.

Proof. Let $m\mathcal{F}_n + K_1$ be a graph with vertex set $V(m\mathcal{F}_n + K_1) = \{v\} \cup \{x_i, 1 \leq i \leq m\} \cup \{x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{y_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(m\mathcal{F}_n + K_1) = \{x_i x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_{ij} y_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{v x_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{v y_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$. The cardinality of $V(m\mathcal{F}_n + K_1)$ is $|V(m\mathcal{F}_n + K_1)| = 2mn + m + 1$ and the cardinality of $E(m\mathcal{F}_n + K_1)$ is $|E(m\mathcal{F}_n + K_1)| = 5mn$. To have the rainbow antimagic connection number of $m\mathcal{F}_n + K_1$, first we need to show the lower bound of $rac(m\mathcal{F}_n + K_1)$. Clearly, according to **Lemma 1** we have $rac(m\mathcal{F}_n + K_1) \geq 2mn + m$.

Secondly, to have the exact value we have to show the upper bound of $rac(m\mathcal{F}_n + K_1)$. Let $f : V(m\mathcal{F}_n + K_1) \rightarrow \{1, 2, \ldots, 2mn + m + 1\}$ be a vertex labeling of graph $m\mathcal{F}_n + K_1$ defined as follows.

$$f(v) = 3m$$

$$f(x_i) = \begin{cases} 2m + i & , \quad \text{for } 1 \leq i \leq m - 1 \\ 2m + i + 1 & , \quad \text{for } i = m \end{cases}$$

$$f(x_{ij}) = \begin{cases} mj - i + m - 3 & , \quad \text{for } 1 \leq i \leq m, j = 1,2 \\ mj - i + 2m - 2 & , \quad \text{for } 1 \leq i \leq m, 3 \leq j \leq n \end{cases}$$

$$f(y_{ij}) = 2mn + m - i - mj + 6 \quad , \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n$$

For the edge weights, we have :

$$w(vx_i) = \begin{cases} 5m + i & , \quad \text{for } 1 \leq i \leq m - 1 \\ 5m + i + 1 & , \quad \text{for } i = m \end{cases}$$

$$w(vx_{ij}) = \begin{cases} mn + mj - i + 1 & , \text{ for } 1 \le i \le m, 1 \le j \le n-1 \\ 2mn + 6 - m & , \text{ for } 1 \le i \le m, j = n \end{cases}$$
$$w(vy_{ij}) = 4mn - i - mj - 2 \quad , \text{ for } 1 \le i \le m, 1 \le j \le n$$
$$w(x_i x_{ij}) = \begin{cases} 3m + mj - 3 & , \text{ for } 1 \le i \le m-1, j = 1,2 \\ 4m + mj - 2 & , \text{ for } 1 \le i \le m-1, 3 \le j \le n \\ 3m + mj - 2 & , \text{ for } i = m, j = 1,2 \\ 3m + mj - 1 & , \text{ for } i = m, 3 \le j \le n \end{cases}$$
$$w(x_i y_{ij}) = \begin{cases} 2mn + 3m - mj + 6 & , \text{ for } 1 \le i \le m-1, 1 \le j \le n \\ 2mn + 3m - mj + 7 & , \text{ for } i = m, 1 \le j \le n \end{cases}$$
$$w(x_{ij} y_{ij}) = \begin{cases} 3mn - 2i - 1 & , \text{ for } 1 \le i \le m, 1 \le j \le n \\ 3mn - 2i + 4 & , \text{ for } 1 \le i \le m, j = n \end{cases}$$

It is easy to see the edge weights of $f : V(m\mathcal{F}_n + K_1) \to \{1,2,\dots,2mn + m + 1\}$ induces a rainbow antimagic coloring of mn colors. Thus $rac(m\mathcal{F}_n + K_1) \le 2mn + m$. Comparing the two bounds, we have the exact value of $rac(m\mathcal{F}_n + K_1) = 2mn + m$. The last is to evaluate the existence of rainbow $x - y$ path of $m\mathcal{F}_n + K_1$. Since $diam(m\mathcal{F}_n + K_1) = 2$, based on **Theorem 1**, for every two vertices $x, y \in V(m\mathcal{F}_n + K_1)$, there exists a rainbow $x - y$ path. It completes the proof. ■

For an illustration, a rainbow antimagic coloring of graph $m\mathcal{F}_n + K_1$ can be seen in Figure 3.

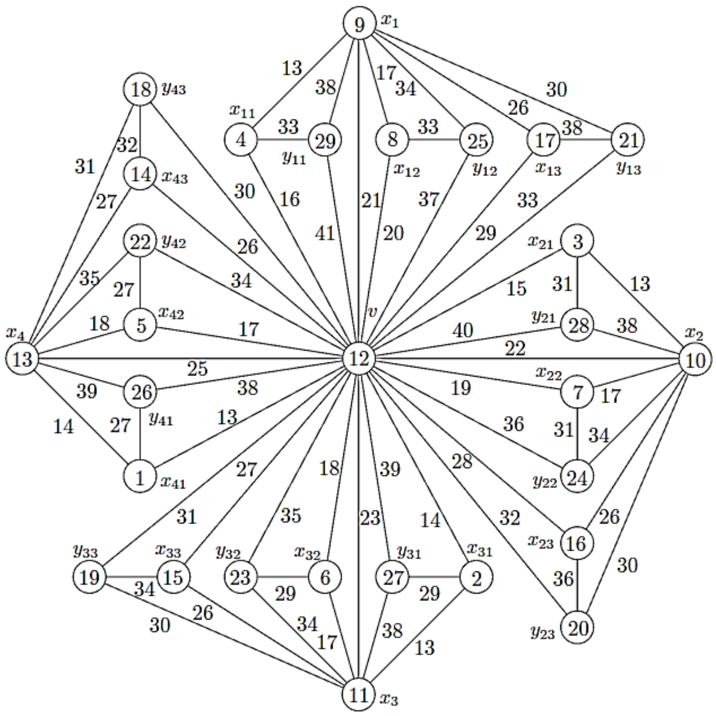

Figure 3. A rainbow antimagic coloring of join product of graph $4\mathcal{F}_3 + K_1$.

Theorem 5. *For* $n, m \ge 3$, $rac(mF_n + K_1) = mn + m$.

Proof. Let $mF_n + K_1$ be a graph with vertex set $V(mF_n + K_1) = \{v\} \cup \{x_i, 1 \le i \le m\} \cup \{x_{ij}, 1 \le j \le m, 1 \le j \le n\}$ and edge set $E(mF_n + K_1) = \{vx_i, 1 \le i \le m\} \cup \{vx_{ij}, 1 \le i \le m, 1 \le j \le n\} \cup \{x_i x_{ij}, 1 \le i \le m, 1 \le j \le n\} \cup \{x_{ij} x_{ij+1}, 1 \le i \le m, 1 \le j \le n-1\}$. The cardinality of $V(mF_n + K_1)$ is $|V(mF_n + K_1)| = mn + m + 1$ and the cardinality of $E(mF_n + K_1)$ is $|E(mF_n + K_1)| = 3mn$. To have the rainbow antimagic connection number of $mF_n + K_1$, first we need to show the lower bound of $rac(mF_n + K_1)$. Clearly, according to **Lemma 1** we have $rac\ (mF_n + K_1) \geq mn + m$.

Secondly, to have the exact value we have to show the upper bound of $rac(mF_n + K_1)$. Let $f : V(mF_n + K_1) \rightarrow \{1,2,\dots,mn + m + 1\}$ be a vertex labeling of graph $mF_n + K_1$ defined as follows.

$$f(v) = 2m + \left\lceil \frac{n}{2} \right\rceil$$

$$f(x_i) = \begin{cases} 2m + i & , \text{ for } 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil - 1 \\ 2m + i + 1 & , \text{ for } \left\lceil \frac{m}{2} \right\rceil \leq i \leq m \end{cases}$$

$$f(x_{ij}) = \begin{cases} mn + 2m - i - j\left(\left\lceil \frac{m}{2} \right\rceil\right) & , \text{ for } i \text{ is odd}, 1 \leq i \leq m, 1 \leq j \leq n \\ j\left(\left\lceil \frac{m}{2} \right\rceil\right) - i + 1 & , \text{ for } i \text{ is even}, 1 \leq i \leq m, 1 \leq j \leq n \end{cases}$$

For the edge weights, we have:

$$w(vx_i) = \begin{cases} 4m + \left\lceil \frac{m}{2} \right\rceil + i & , \text{ for } 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil - 1 \\ 4m + \left\lceil \frac{m}{2} \right\rceil + i + 1 & , \text{ for } \left\lceil \frac{m}{2} \right\rceil \leq i \leq m \end{cases}$$

$$w(vx_{ij}) = \begin{cases} mn + 4m - i - j\left(\left\lceil \frac{m}{2} \right\rceil\right) + \left\lceil \frac{m}{2} \right\rceil & , \text{ for } i \text{ is odd}, 1 \leq i \leq m, 1 \leq j \leq n \\ 2m - i + j\left(\left\lceil \frac{m}{2} \right\rceil\right) - \left\lceil \frac{m}{2} \right\rceil & , \text{ for } i \text{ is even}, 1 \leq i \leq m, 1 \leq j \leq n \end{cases}$$

$$w(x_i x_{ij}) = \begin{cases} mn + 4m - j\left(\left\lceil \frac{m}{2} \right\rceil\right) & , \text{ for } i \text{ is odd}, 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil - 1, 1 \leq j \leq n \\ mn + 4m - j\left(\left\lceil \frac{m}{2} \right\rceil\right) + 1 & , \text{ for } i \text{ is odd}, \left\lceil \frac{m}{2} \right\rceil \leq i \leq m, 1 \leq j \leq n \\ 2m + j\left(\left\lceil \frac{m}{2} \right\rceil\right) + 1 & , \text{ for } i \text{ is even}, 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil - 1, 1 \leq j \leq n \\ 2m + j\left(\left\lceil \frac{m}{2} \right\rceil\right) + 2 & , \text{ for } i \text{ is odd}, 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil - 1, 1 \leq j \leq n \end{cases}$$

$$w(x_{ij} x_{ij+1}) = \begin{cases} mn + m - 2i + 7 & , \text{ for } i \text{ is odd}, 1 \leq i \leq m, 1 \leq j \leq n \\ mn - 2i + 7 & , \text{ for } i \text{ is even}, 1 \leq i \leq m, 1 \leq j \leq n \end{cases}$$

It is easy to see the edge weights of $f : V(mF_n + K_1) \rightarrow \{1,2,\dots,mn + m + 1\}$ induces a rainbow antimagic coloring of $mn + m$ colors. Thus $rac(mF_n + K_1) \leq mn + m$. Comparing the two bounds, we have the exact value of $rac(mF_n + K_1) = mn + m$. The last is to evaluate the existence of rainbow $x - y$ path of $mF_n + K_1$. Since $diam(mF_n + K_1) = 2$, based on **Theorem 1**, for every two vertices $x, y \in V(mF_n + K_1)$, there exists a rainbow $x - y$ path. It completes the proof. ∎

For an illustration, a rainbow antimagic coloring of graph $mF_n + K_1$ can be seen in Figure 4.

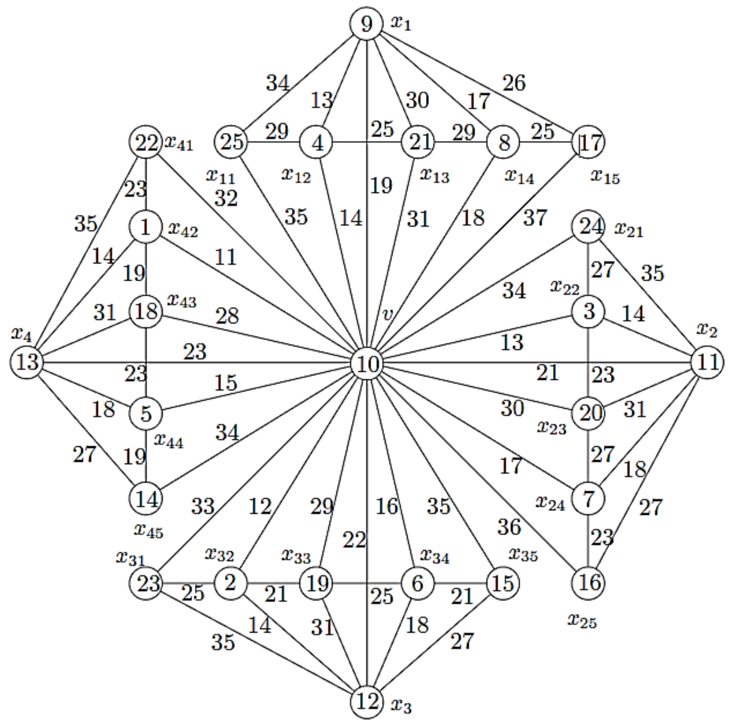

Figure 4. A rainbow antimagic coloring of join product of graph $4F_5 + K_1$.

Theorem 6. *For* $n, m \geq 3$, $rac(mS_{n,n} + K_1) = 2mn + m$.

Proof. Let $mS_{n,n} + K_1$ be a graph with vertex set $V(mS_{n,n} + K_1) = \{v\} \cup \{x_i, y_i, 1 \leq i \leq m\} \cup \{x_{ij}, y_{ij}, 1 \leq j \leq m, 1 \leq j \leq n\}$ and edge set $E(mF_n + K_1) = \{vx_i, vy_i, x_iy_i, 1 \leq i \leq m\} \cup \{vx_{ij}, vy_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_ix_{ij}, y_iy_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$. The cardinality of $V(mS_{n,n} + K_1)$ is $|V(mS_{n,n} + K_1)| = 2mn + m + 1$ and the cardinality of $E(mS_{n,n} + K_1)$ is $|E(mS_{n,n} + K_1)| = 3m + 4mn$. To have the rainbow antimagic connection number of $mS_{n,n} + K_1$, first we need to show the lower bound of $rac(mS_{n,n} + K_1)$. Clearly, according to **Lemma 1** we have $rac\left(mS_{n,n} + K_1\right) \geq 2mn + m$.

Secondly, to have the exact value we have to show the upper bound of $rac(mS_{n,n} + K_1)$. Let $f : V\left(mS_{n,n} + K_1\right) \rightarrow \{1,2,\ldots,2mn + m + 1\}$ be a vertex labeling of graph $mS_{n,n} + K_1$ defined as follows.

$$f(v) = mn + m + 1$$
$$f(x_i) = mn + m + i + 1 \quad , \quad \text{for } 1 \leq i \leq m$$
$$f(y_i) = mn + i \quad , \quad \text{for } 1 \leq i \leq m$$
$$f\left(x_{ij}\right) = (i-1)n + j \quad , \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n$$
$$f\left(y_{ij}\right) = mn + 2m + (i-1)n + j + 1 \quad , \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n$$

For the edge weights, we have:

$$w(vx_i) = 2mn + m + i + 2 \quad , \quad \text{for } 1 \leq i \leq m$$
$$w(vy_i) = 2mn + m + i + 1 \quad , \quad \text{for } 1 \leq i \leq m$$
$$w\left(vx_{ij}\right) = mn + m + (i-1)n + j + 1 \quad , \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n$$
$$w\left(vy_{ij}\right) = 2mn + 3m + (i-1)n + j + 2 \quad , \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n$$
$$w(x_iy_i) = 2mn + m + 2i + 1 \quad , \quad \text{for } 1 \leq i \leq m$$

$$w(x_i x_{ij}) = mn + m + i + (i-1)n + j + 1 \quad , \quad \text{for } 1 \le i \le m, 1 \le j \le n$$
$$w(y_i y_{ij}) = 2mn + 2m + i + (i-1)n + j + 1 \quad , \quad \text{for } 1 \le i \le m, 1 \le j \le n$$

It is easy to see the edge weights of $f : V(mS_{n,n} + K_1) \rightarrow \{1,2,\dots,2mn + m + 1\}$ induces a rainbow antimagic coloring of $2mn + m$ colors. Thus $rac(mS_{n,n} + K_1) \le 2mn + m$. Comparing the two bounds, we have the exact value of $rac\,(mS_{n,n} + K_1) = 2mn + m$. The last is to evaluate the existence of rainbow $x - y$ path of $mS_{n,n} + K_1$. Since $diam(mS_{n,n} + K_1) = 2$, based on **Theorem 1**, for any two vertices $x, y \in V(mS_{nn} + K_1)$, there exists a rainbow $x - y$ path. It completes the proof. ■

For an illustration, a rainbow antimagic coloring of graph $mS_{n,n} + K_1$ can be seen in Figure 5.

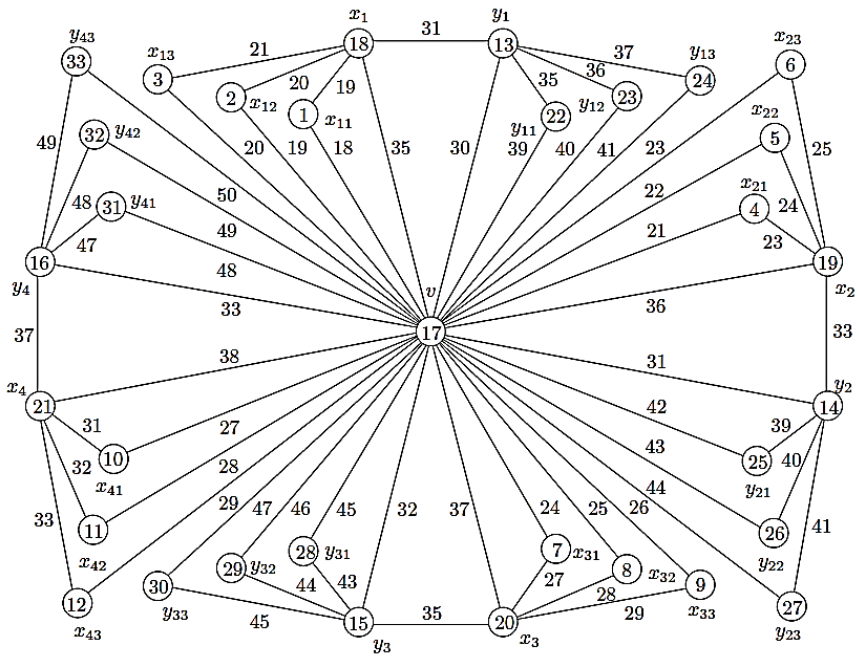

Figure 5. A rainbow antimagic coloring of join product of graph $4S_{3,3} + K_1$

CONCLUSIONS

We studied rainbow antimagic coloring of join product graph $G + K_1$ where G is $mP_n, mS_n, m\mathcal{F}_n, mF_n, mS_{n,n}$. This research generates a lower bound of rainbow antimagic connection number for the joint product of graph $G + K_1$ and based on the lower bound, we have the exact value of rainbow antimagic connection number of graph $mP_n + K_1$, $mS_n + K_1$, $m\mathcal{F}_n + K_1$, $mF_n + K_1$ and graph $mS_{n,n} + K_1$. Based on these results, rainbow antimagic connection number of joint product of graphs $G + K_1$ depends on maximum degree of $G + K_1$.

However, if is not a graph K_1, it is still difficult to determine the exact value of the rainbow antimagic connection number. Therefore, this study raises an open problem. Determine the rainbow antimagic connection number of join product of graph $G + H$ where H is not a graph K_1.

REFERENCES

[1] Arumugam S, Premalatha K, Baˇca M, Semaniˇcov´a-Feˇnovˇc´ıkov´a A, Local antimagic vertex coloring of a graph, *Graphs Combin.* **33** (2017), 275-285.

[2] Baˇca M, Antimagic labelings of antiprisms, Journal of combinatorial mathematics and combinatorial computing. **35** (2000), 217-224.

[3] Baˇca M, Baskoro E T, Jendrol S, Miler M, Antimagic labelings of hexagonal plane maps, *Utilitas mathematica*. **66** (2004), 231-238.

[4] Baˇca M, Lin Y, Miler M, Antimagic labelings of grids, *Utilitas mathematica*. **72** (2007), 65-75.

[5] Baˇca M, Dafik, Ryan J, Antimagic labelings of disjoint union of s-crowns, *Utilitas mathematica*. **79** (2009), 193-205.

[6] Budi H S, Dafik, Tirta I M, Agustin I H, Kristiana A I, On rainbow antimagic coloring of graphs, *Journal of Physics: Conf. Series* **1832** (2021), 012016.

[7] Chartrand G, Lesniak L, Zhang P, Graphs & Digraphs, sixth ed., *Taylor & Francis Group*, New York, 2016.

[8] Chartrand G, Johns G L, Mckeon K A, Zhang P, Rainbow connection in graphs, *Math. Bohemica* **133** (2008), 85-98.

[9] Chang F, Liang Y C, Pan Z, Zhu X, Antimagic labeling of reguler graphs, *Journal of Graph Theory*} **82** (4) (2016) 339-349.

[10] Cranston D W, Reguler bipartite graphs are antimagic, *Journal of Graph Theory* **60** (3) (2009) 173-182.

[11] Dafik, Miler M, Ryan J, Baˇca M, Antimagic labeling of the union of two stars, *Australasian Journal of combinatorics* **42** (2018), 35-44.

[12] Dafik, Susanto F, Alfarisi R, Septory B J, Agustin I H, Venkatachalam M, On rainbow antimagic coloring of graphs. *Advanced Mathematical Models and Aplication*. **6** (3) (2021) 278-291.

[13] F. Harary, Graphs Theory, *Wesley*. New London, 1969.

[14] Hartsfield N, Ringel G, Pearls in Graph Theory, *Academic Press*, San Diego, 1990.

[15] Hasan M S, Slamin, Dafik, Agustin I H, Alfarisi R, On the total rainbow connection of the wheel related graphs, *Journal of Physics: Conf. Series* **1008** (2018), 012054.

[16] Joedo J C, Dafik, Kristiana A I, Agustin I H, Nisviasari R, On the rainbow antimagic coloring of vertex almagamation of graphs, *Journal of Physics: Conf. Series* **2157** (2022), 012014.

[17] Krivelevich M, Yuster R, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, *J. Graph Theory* **63** (3) (2010), 185-191.

[18] Li H, Li X, Liu S, Rainbow connection of graphs with diameter 2, *Discrete Mathematics* **312** (8) (2012), 1453-1457.

[19] Li H, Li X, Sun Y, Rainbow connection of graphs with diameter 3, *Discussiones Mathematicae Graph Theory* **37** (2) (2017), 141-154.

[20] Li X, Shi Y, On the rainbow vertex connection, *Graph Theory* **33** (2013), 307-313.

[21] Septory B J, Utoyo M I, Dafik, Sulistiyono B, Agustin I H, On rainbow antimagic coloring of special graphs. *Journal of Physics: Conference Series*. **1836** (2021) 012016.

[22] Simamora D N S, Salman A N M, The rainbow (vertex) connection number of pencil graphs, *Procedia Computer Science* **74** (2010), 138-142.

[23] Sun Y, On rainbow total coloring of a graph, *Discrete Applied Mathematics* **194** (2015), 171-177.

[24] Waliis W D, Magic graphs, *Boston*. Birkhauser. 2001.