CAUCHY –Jurnal Matematika Murni dan Aplikasi
Volume 7(4) (2023), Pages 559-567
p-ISSN: 2086-0382; e-ISSN: 2477-3344

Characteristic of Quaternion Algebra Over Fields

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ABSTRACT

Quaternion is an extension of the complex number system. Quaternion are discovered by formulating 4 points in 4-dimensional vector space using the cross product between two standard vectors. Quaternion algebra over a field is a 4-dimensional vector space with bases $\{1, i, j, k\}$. Quaternion algebra is an algebra that is not commutative but has an identity element and an inverse element in each element, so it is often referred to as a skew field. Skew field is an interesting field structure to be studied further. The purpose of this study is to obtain the characteristics of split quaternion algebra and determine how it interacts with central simple algebra. The research method used in this paper is literature study on quaternion algebra, field and central simple algebra. The results of this study establish the equivalence of split quaternion algebra as well as the theorem relating central simple algebra and quaternion algebra. The conclusion obtained from this study is that split quaternion algebra has five different characteristics and quaternion algebra is a central simple algebra with dimensions less than equal to four. This research can be further applied to modeling movements in $\mathbb{R}^3$ such as robots, 3D carboard film animation, and drone control.

Keywords: quaternion algebra; characteristics; split; central simple algebra

INTRODUCTION

Quaternion is an extension of the complex number system. In order to define 3 points in the vector space $\mathbb{R}^3$ in the same way as 2 points defined in complex field $\mathbb{C}$, William Rowan Hamilton sought a solution in that space [1]. The results of multiplication at 3 points in $\mathbb{R}^3$ did not work because Hamilton could not find the multiplication value of ij and ji. By defining 4 points in the vector space $\mathbb{R}^4$ in the same way, Hamilton expanded his investigation. Then he found the solution by using the cross product between the standard vectors i, j, and k. It is essential to compute the product of ij, jk, ik, ji, kj, and ki with $i^2 = j^2 = k^2 = -1$. To form the quaternion number, these products must be non-commutative, in other word, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$ [2]. Astronautics, robotics, computer graphics and visualization, animation, special effects in movies, navigation, and many other fields can all benefit from the use of quaternion numbers.

Let K represent a field with characteristic not equal to 2. If $\alpha, \beta \in K^* = K\backslash\{0\}$, then exists a four-dimensional K-algebra with basis $\{1, i, j, k\}$. The symbol for this K-algebra is written as $H_K(\alpha, \beta)$. K-algebra is an algebra over a field K and isomorphic with it for some $\alpha, \beta \in K^*$. An example for quaternion algebra is $H_K(-1, -1) = \mathbb{H}$ which is Hamilton's

Submitted: September 21, 2022 Reviewed: October 21, 2022 Accepted: November 18, 2022
DOI: http://dx.doi.org/10.18860/ca.v7i4.17625

Quaternion. With the group center at K, quaternion algebra is a central simple algebra over field K that is associative, non-commutative, and devoid of two-sided ideals [3]. For parameterizing smooth rotation and spatial transformations in vector space, quaternion representations are greatly favored. They eliminate the issue of gimbal locks and are typically regarded as being resilient to sheer/scaling noise and perturbations (i.e., numerically stable rotations). The model has additional degrees of freedom because quaternion rotations operate on two planes of rotation, whereas complex rotations only operate on one [4].

Tarnauceanu's research [5] discusses the characteristics of quaternion numbers. This research studies about the properties of quaternion numbers in general, but has not discussed further about the characteristics of quaternion numbers over the field. Savin's research [6] discusses quaternion algebra which is split over a certain field. This research learns more about the property of split in quaternion algebra over field in general. Savin's research [7] discusses about the class of division quaternion algebras over the field. This research studies the special properties of division algebra in quaternion algebra. Acciaro's research [3] discusses about the characteristics of quaternion algebras over quadratic field. This research is a development of research results from previous studies. This study discusses the characteristics of quaternion algebra over a field, the characteristics of split quaternion algebra, and the use of division algebra to prove the theorems discussed in this study. The author's research discusses the development of Acciaro's research by adding and completing the proof of the theorem used in Acciaro's research.

According to study in [8], several characteristics of quaternion are described which are not a division algebra and have isotropic elements. According to study [3], additional characteristics of quaternion algebra, including its relationship to the Hilbert symbol and isomorphic with square matrix over a field, are discussed in this paper. In both studies, the theorem which discusses the characteristics of quaternion algebra is not shown to be proven directly. In this study, we will discuss about the characteristics of quaternion algebra, its full proof, and how it relates to central simple algebra. The main goals of this study were obtained the characteristics of split quaternion algebra and obtained the connection between quaternion algebra and central simple algebra. This research is expected to provide benefits for the development of mathematics about split quaternion algebra over a field and central simple algebra.

METHODS

In this section, we will discuss the materials and methods used and discuss about the definitions used in this study related to quaternion algebra. The research used in this study is a literature study on quaternion algebra over a field and its properties. The following are definitions and reference theories used in this study to answer the existing problems.

Definition 1. (See [9]) **(Quaternion)**
Quaternion can be expressed mathematically as follows.

$$\mathbb{H} = \{a + bi + cj + dk; a, b, c, d \in \mathbb{R}\}$$

where the following multiplication rule conditions apply:

1. $i^2 = j^2 = k^2 = -1$.
2. $ij = k,\ ji = -k,\ jk = i,\ kj = -i,\ ki = j,\ ik = -j$.
3. For any element $a \in \mathbb{R}$ is commutative with i, j, and k.

Definition 2 (See [10]) **(Algebra over field)**
Suppose that K is a field and A is a vector space over field K equipped with binary addition operations from $A \times A$ to A. If A meets the axioms below, it is an algebra over a field K, or so-called K-algebra:
$$(\lambda a)b = a(\lambda b) = \lambda(ab), \ \ \forall \lambda \in K \ and \ \forall a, b \in A$$

Definition 3 (See [3]) **(Zero Divisor)**

The associative algebra A over a field is a division algebra if and only if it contains a multiplication identity element $1 \neq 0$ and every non-zero element in A has a left and right multiplication inverse. If A is an algebra which has finite dimensions, it can only be a division algebra if it does not have a zero divisor.

Definition 4 (See [11]) **(Division Ring)**

Division ring is a ring K where each non-zero element has an inverse (two sides) i.e. $K\backslash\{0\}$.

Definition 5 (See [11]) **(Division Algebra)**

Division algebra is an algebra over division ring K (every non-zero element has an inverse).

Definition 6 (See [3]) **(Isotropic)**

Vector is said to be isotropic if norm of a non-zero vector $q \in H_K(\alpha, \beta)$ is zero ($N(q) = 0$).

Definition 7 (See [12]) **(Central Simple Algebra)**

Let K be a field, L be its field extension, and A be a K-algebra with finite dimensions. A is said to be central simple algebra over K if it satisfies one of the following equivalent conditions:

1. A is a simple ring with center K.
2. There exists an isomorphism $A \cong M_n(D)$ where D is a division algebra with center K.
3. A_L isomorphic for some d with ring matrix $M_d(L)$.

Definition 8 (See [13]) **(Center Group)**

Let A represent K-algebra. The center of a group A is defined by the set
$$Z(A) = \{z \in A | \ az = za, \forall a \in A\}$$
A is said to be center or central if $Z(A) = K$.

Definition 9 (See [11]) **(Simple Algebra)**

Let A represent K-algebra. A is said to be simple if it does not have a non-trivial two-sided ideal, i.e. A only has $\{0\}$ and A which is a two-sided ideal.

Definition 10 (See [13]) **(Central Simple Algebra)**

Central simple algebra is K-algebra which is central and simple.

Theorem 11 (See [8])

Let A be a finite dimensional K-algebra and K be a field. If and only if there are integers $n > 0$ and a finite field extension $K|k$ such that $A \otimes_k K$ is isomorphic with a ring matrix $M_n(K)$, then A is a central simple algebra.

Definition 12 (See [14]) **(Field Extension)**

Consider that E and F are fields. E is a field extension of F if and only if F is a subfield of E. The vector space E over a field F is identified by the symbols E/F and its dimensions are denoted by $[E:F]$.

Definition 13 (See [3]) **(Split Algebra)**

Let A represent a central simple algebra over a field K. If A is isomorphic to the square matrix $M_2(K)$ over K, then A is said to be split over field K.

Definition 14 (See [15]) **(Hilbert Equation)**

For any field F and any element $a, b \in \mathbb{F}^* = \mathbb{F}\backslash\{0\}$. We say that

$$aX^2 + bY^2 = Z^2$$

is the Hilbert equation, and the solution $(x, y, z) \in \mathbb{F}^3$ is trivial if and only if $x = y = z = 0$, and non-trivial otherwise.

Definition 15 (See [15]) **(Hilbert Symbol)**

Let $R \subseteq \mathbb{F}$ be a subring and $\mathbb{F}$ be a field. Define the mapping that resolves the quadratic diagonal equation in three variables with coefficients in $\mathbb{F}$ with solvency R non-trivial:

$$h_{R,\mathbb{F}}(\cdot,\cdot) \ : \mathbb{F}^* \times \mathbb{F}^* \to \{-1,1\}$$
$$(a,b) \mapsto \begin{cases} 1, & if\ \exists(x,y,z) \in R^3\backslash\{0\} :\ ax^2 + by^2 = z^2 \\ -1, & otherwise \end{cases}$$

The definitions and theorems written above will be used in the results and discussion section to answer the research problems asked in the introduction section.

The steps in this research are as follows.

1. The literature study is searched using publish or perish software by entering keywords, year of publication, and maximum article limit. The keyword entered is quaternion algebra and quadratic field. The year of publication included is the last five years of research on quaternion algebra and quadratic fields. The maximum article limit is 500 articles.
2. The literature obtained from publish or perish software was selected by eliminating literature in the form of books, doctoral thesis, proceedings, and other topics that were not relevant to this research.
3. Screening articles through abstracts that are irrelevant to the topic who want to discuss.
4. Read the entire article and choose one article that you want to cover further.
5. One article that has been selected, understood in depth and developed the result of the research.

6. Write the progress of the research results of previous paper in this research and complete the proof of the theorems in the previous paper.

RESULTS AND DISCUSSION

A quaternion known as a ring which has an inverse for each non-zero element and is not commutative with regard to multiplication. It is well known that quaternion algebra possesses central and simple features since it is a central simple algebra over a field. Quaternion algebra has special characteristics in its properties and in this study, the properties of quaternion algebra over fields with characteristics other than two will be studied in more detail. The characteristics of split quaternion algebra over a field and its equivalents are demonstrated in the following theorem.

Theorem 16 [3]
Let K represent a field with characteristic is not equal to 2 ($char(K) \neq 2$) and Let $\alpha, \beta \in K\backslash\{0\}$ and A is an quaternion algebra ($A = H_K(\alpha, \beta)$). The following statements are equivalent.

1. $A \cong H_K(1, -1) \cong M_2(K)$;
2. A is not a division algebra;
3. A contains an isotropic element;
4. The equation $\alpha x^2 + \beta y^2 = 1$ has a solution $(x, y) \in K \times K$;
5. α is a norm of $K(\sqrt{\beta})$;

If any of the statements above apply to A, then A said to be split over a field K.

Proof:
The equivalence of this theorem will be demonstrated by searching for the implications of the statements $(1) \Longrightarrow (2), (2) \Longrightarrow (3), (3) \Longrightarrow (4), (4) \Longrightarrow (5)$, and $(5) \Longrightarrow (1)$. We reviewed 5 cases of implications for solving the equivalence of the theorem.

1. $\mathbf{(1) \Longrightarrow (2)}$
   First, it is known that the quaternion algebra $H_K(1, -1)$ can be defined as follows.
   $$H_K(1, -1) = \{a + bi + cj + dk \mid i^2 = 1, j^2 = -1, k^2 = 1\}$$
   with $ij = k, ji = -k$.
   Then, it is known that $H_K(1, -1)$ is isomorphic with $M_2(K)$ where the base is $\{1, i, j, k\}$ and the square matrix of the base can be defined as follows.
   $$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
   For instance, choose two non-zero elements in $H_K(1, -1)$, namely $1 + i = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $1 - i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Multiply the two arbitrary elements so that
   $$\begin{aligned}(1 + i)(1 - i) &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 1 & -1 + 1 \\ 1 - 1 & -1 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$
   The fact that zero results from the product of two non-zero elements means that $H_K(1, -1)$ has a zero divisor. Since $\exists (1 + i), (1 - i) \neq 0 \in H_K(1, -1) \ni (1 + i)(1 - i) = 0$, it may be deduced from Definition 3 that a quaternion algebra is not a division algebra.

Based on the evidence above, it can be concluded that $H_K(1,-1) \cong M_2(K)$ is not a division algebra.

2. $(\mathbf{2}) \Rightarrow (\mathbf{3})$
First, it is known that $H_K(\alpha,\beta)$ is not a division algebra which means that $H_K(\alpha,\beta)$ has a zero divisor. An element $a \in H_K(\alpha,\beta)$ is said to have an inverse if its norm is not equal to zero ($N(a) \neq 0$). $H_K(\alpha,\beta)$ is not a division algebra if it is connected to the concept of norm, which means that some of its member lack an inverse or in other word, it has a zero divisors element. Consider choosing the element $b \neq 0 \in H_K(\alpha,\beta)$ in such a way that $H_K(\alpha,\beta)$ is not a division algebra. The norm of element b is zero ($N(b) = 0$) because the element b lacks an inverse.
Based on the evidence above, it can be concluded that there are isotropic elements in $H_K(\alpha,\beta)$.

3. $(\mathbf{3}) \Rightarrow (\mathbf{4})$
First, it is well known that quaternion algebra has isotropic elements. Suppose we choose $g = 1 + i \in H_K(\alpha,\beta)$ which is an isotropic element. So according to Definition 6, the norm of isotropic element is as follows.

$$N(g) = g\bar{g} = 0$$
$$(1+i)(1-i) = 0$$
$$1^2 - i^2 = 0$$
$$1 - \alpha = 0 \rightarrow \alpha = 1$$

The equation $\alpha x^2 + \beta y^2 = 1$ can be changed to $x^2 + \beta y^2 = 1$. There are two cases of this quadratic equation, namely

- If $\beta = 0$, then the value of $x = \sqrt{1} = \pm 1$
- If $\beta \neq 0$, then the value of $y = \pm \frac{\sqrt{1-x^2}}{\sqrt{\beta}}$

We have reviewed these two cases and concluded that for any value of β, the equation $x^2 + \beta y^2 = 1$ must have a solution $(x,y) \in K \times K$ so that based on the proof above, the equation $\alpha x^2 + \beta y^2 = 1$ must have a solution on the field $K \times K$ for algebra having isotropic elements.

4. $(\mathbf{4}) \Rightarrow (\mathbf{5})$
First, it is known that the equation $\alpha x^2 + \beta y^2 = 1$ has a solution $(x,y) \in K \times K$.
Assume that β is not a square in $K\backslash\{0\}$. Since β is not a square in $K\backslash\{0\}$ then $x \neq 0$.
Then the equation can be formed as

$$\alpha x^2 + \beta y^2 = 1$$
$$\alpha + \beta\left(\frac{y}{x}\right)^2 = \frac{1}{x^2}$$
$$\alpha = \left(\frac{1}{x}\right)^2 - \beta\left(\frac{y}{x}\right)^2$$
$$\left(\frac{1}{x}\right)^2 - \beta\left(\frac{y}{x}\right)^2 = N\left(\frac{1}{x} + \left(\frac{y}{x}\right)\sqrt{\beta}\right)$$

The aforementioned evidence leads to the conclusion that α is the norm of field $K(\sqrt{\beta})$.

5. $(\mathbf{5}) \Rightarrow (\mathbf{1})$

First, it is known that $\alpha = N\left(K(\sqrt{\beta})\right)$ such that

$$\alpha = \frac{1}{x^2} - \beta\left(\frac{y^2}{x^2}\right)$$
$$\alpha + \beta\left(\frac{y^2}{x^2}\right) = \frac{1}{x^2}$$
$$\alpha x^2 + \beta y^2 = 1$$

If α, β are not both negative values, then α, β have a solution in this case. Let's say we select $\alpha = 1, \beta = -1$ such that we obtain $H_K(1,-1)$. Based on the evidence above, it can be concluded that the norm of the quadratic field contains the quaternion algebra $H_K(1,-1)$.

Based on the evidence from the five cases and based on the nature of the implications, it can be concluded that statements $(1) - (5)$ are equivalent.

Another characteristic of quaternion algebra is that it discusses its relationship to central simple algebra. It is known that quaternion algebra is a central simple algebra. This theorem explains the structure of central simple algebra and establishes a connection between quaternion algebra and central simple algebra.

Theorem 17

If A is a central simple algebra over a field F with dimension less than 4, then A is isomorphic to the quaternion algebra over a field F.

Proof:

Let A represent a central simple algebra with dimension less than 4. Since A is the central simple algebra, $A \cong M_n(D)$, where D is a division algebra over field F, according to Definition 7, Since $A \cong M_n(D)$, the dimension of A is n^2. A has dimension less than 4 so that the possible values of n are 1 or 2. Consider the following two scenarios: $n = 1$ and $n = 2$.

- If $n = 1$, then $A \cong D$ so that A is a division algebra over field F.
- If $n = 2$, then $A \cong M_2(D)$ so that $= F$. $A \cong M_2(F)$.

Assume that A is a division algebra. Take a field $K = F(i) \subseteq A$ and the non-central element $i \in A$. Since $F \subseteq K \subseteq A$ so that $\dim_F K = 2$ dan $\dim_F A = 4$. Thus, K is a quadratic field extension of F and assume that $i \in K$ has been chosen such that $i^2 = a \in F$.

Let f be the inner automorphism in A defined by i. Following that, $f^2 =$ identity and eigenspace decomposition $A = A^+ \oplus A^-$, where

$$A^+ = \{x \in A: f(x) = x\} = \{x \in A: ix = xi\}, \qquad and$$
$$A^- = \{x \in A:\ f(x) = -x\} = \{x \in A:\ ix = -xi\}$$

Consider a non-zero element $j \in A^-$. Since $K \subseteq A^+$ and $K \cdot j \in A^-$, we get $K = A^+$ dan $K \cdot j = A^-$. We have $j^2 i = ij^2$ because of $ij = -ji$ which means that $j^2 \in A^+ = K$. On other hand, $F(j)$ is also an extension of the quadratic field of F, so that j contains quadratic equation $j^2 + cj - b = 0$, for $b, c \in F$. The equation $cj = b - j^2 \in K$ suggests that $c = 0$ and $j^2 = b \in F$ from these equations, thus

$$A = K \oplus K \cdot j = F \oplus Fi \oplus Fj \oplus Fij$$

A isomorphic with quaternion algebra $\left(\frac{a,b}{F}\right)$. Based on the above evidence, it can be concluded that A is isomorphic with quaternion algebra over a field F.

CONCLUSIONS

The conclusion of this study is that there are five characteristics of quaternion algebra that are split over a field, which are equivalent to a square matrix with element of field, is not a division algebra, has isotropic elements, is equivalent to the Hilbert Symbol which has a solution to the field, and is equivalent to the norm of a quadratic field. Besides that, quaternion algebra over a field isomorphic with a central simple algebra with dimensions less than four.

ACKNOWLEDGEMENTS

The author would like to thank to the Indonesian Ministry of Education, Culture, Research, and Technology for funding this research under Master Thesis Research Project in 2022 entitled "Aljabar Quaternion Atas Komposit Lapangan Kuadratik", grant number 1318/UN6.3.1/PT.00/2022.

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