



The Distance Irregular Reflexive k -Labeling of Graphs

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ABSTRACT

A total k -labeling is a function f_e from the edge set to the set $\{1, 2, \dots, k_e\}$ and a function f_v from the vertex set to the set $\{0, 2, 4, \dots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. A distance irregular reflexive k -labeling of graph G is total k -labeling. If for every two different vertices u and u' of G , $w(u) \neq w(u')$, where $w(u) = \sum_{u_i \in N(u)} f_v(u_i) + \sum_{uv \in E(G)} f_e(uv)$. The minimum k for graph G which has a distance irregular reflexive k -labeling is called distance reflexive strength of the graph G , denoted by $D_{ref}(G)$. In this paper, we determine the lower bound of distance reflexive strength of any graph and the exact value of distance reflexive strength of path, star, and friendship graph.

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Keywords: distance irregular reflexive k -labeling; distance reflexive strength; path; star; friendship.

INTRODUCTION

A simple graph G is a pair $(V(G), E(G))$, where $V(G)$ denotes a non-empty vertex set, whereas $E(G)$ denotes an unordered pair of two distinct vertices $u, v \in V(G)$. If there is a path between every pair of distinct vertices in G , the graph is said to be connected. Among the various types of problems that arise while studying graph theory, one that has grown in popularity in recent years is graph labeling. Throughout this paper all graphs are simple, connected and undirected $G(V, E)$. The vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively [1]. A graph labeling is a map that connects graph elements to numbers (usually to the positive or non-negative integers). Gallian [2] describes a recent concept of graph labeling. The most common choices of domain are the set of labels to the vertex set (called vertex labeling), the edge set (called edge labeling), or to both vertices and edges (called total labeling). In many cases it is quite interesting to consider the sum of all labels associated with a graph element called the weight of the element which is denoted by w .

In this research, we concentrate on studying a novel kind of labeling called irregular labeling. If each vertex has a different weight (label sum), when we assign positive integer labels to the edges or vertices of a connected graph with at least three vertices,

they become irregular. The irregularity strength of a graph G is the minimum of all such irregular assignments' maximum weights. On [3-13], you can view the results of a graph's irregularity strength. The overall vertex irregularity strength on graph G was evaluated by Baca et al.; for more information, see [15].

We define a k -labeling for a graph G and assign the numbers $\{1, 2, \dots, k\}$ to the graph's elements by using a vertex irregular reflexive labeling. Let k be a natural number; total k -irregular labeling is a function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, k\}$. A vertex irregular reflexive total k -labeling is one of the many types of irregular k -labeling. Vertex irregular reflexive labeling is described in detail by Tanna et al. in their article [14]. Total k -labeling is a function f_e from $E(G)$ to the first natural number up to k_e and a function f_v from $V(G)$ to the non-negative even number up to $2k_v$, where $k = \max\{k_e, 2k_v\}$. A vertex irregular reflexive k -labeling of the graph G is the total k -labeling, if for every two different vertices x and x' of G , $wt(x) \neq wt(x')$, where $wt(x) = f_v(x) + \sum_{xy \in E(G)} f_e(xy)$. The reflexive vertex strength of the graph G , denoted by $rvs(G)$, is the minimum k for a graph G with a vertex irregular reflexive k -labeling. The minimum number of edges between two vertices u and v is called the **distance** between these vertices and is denoted as $d(u, v)$.

Motivated by the definition of **distance irregular labeling** [16] and **irregular reflexive labeling** of graphs, we develop a new concept namely **distance irregular reflexive labeling**. We have given a formal definition as follows: A total k -labeling is a function f_e from $E(G)$ to the set $\{1, 2, \dots, k_e\}$ and a function f_v from $V(G)$ to the set $\{0, 2, 4, \dots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. A distance irregular reflexive k -labeling of the graph G is the total k -labeling, if for every $u, u' \in V(G), u \neq u'$, $w(u) \neq w(u')$, where $w(u) = \sum_{u_i \in N(u)} f_v(u_i) + \sum_{uv \in E(G)} f_e(uv)$ and $N(u)$ is the neighbourhood of distance 1. The minimum k for graph G which has a distance irregular reflexive k -labeling is called distance reflexive strength of the graph G , denoted by $D_{ref}(G)$. In this paper, we determine the lower bound of distance reflexive strength of any graph and the exact value of distance reflexive strength for three types of connected graphs: path, star, and friendship graph.

METHOD

The method used in determining the distance reflexive strength of a graph is as follows:

1. Defines a graph as data or a research object.
2. Determine the vertex set and edge set of the graph.
3. Determine the lower bound distance reflexive strength of a graph with the following Lemma and Corollary.

Lemma: $D_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil$ and Corollary: $D_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil + 1$, if $p + \delta - 1 > \Delta(2t - 1)$.

4. Construct vertex and edge labels based on the definition of distance irregular reflexive labeling.
5. Determine the upper bound of the distance reflexive strength of a graph with the obtained function in point 4.
6. If the upper bound of distance reflexive strength of a graph is the same as the lower bound of distance reflexive strength, then the value of distance reflexive strength can be determined from the graph.

7. If the upper bound of distance reflexive strength of the graph is not the same as the lower bound of distance reflexive strength, then point 4 is repeated until the upper bound of distance reflexive strength is equal to the lower bound of distance reflexive strength.

RESULTS AND DISCUSSION

Before determining the distance reflexive strength of a graph, we need to determine the lower bound which is used as a basic reference and proof in the theorem. In addition, the lower bound is used to guarantee the obtained distance reflexive strength is the smallest or minimum value. The lower bound of distance reflexive strength is presented in the following Lemma.

Lemma 1. *Let G be a graph which has $p, \delta,$ and Δ . For any graph G ,*

$$D_{ref}(G) \geq \left\lceil \frac{p + \delta - 1}{2\Delta} \right\rceil$$

where $p, \delta,$ and Δ are order, minimum degree, and maximum degree, respectively.

Proof. Assume that G is a graph of order p , the minimum degree δ , and the maximum degree Δ . The greatest label on the graph for every distance irregular reflexive k -labeling is $k' = D_{ref}(G)$. We choose u as a vertex of degree δ , in such a case that the vertex weight of u is at least:

$$w(u) = \sum_{u_i \in N(u)} f_v(u_i) + \sum_{uv \in E(G)} f_e(uv) = 0 + 1(\delta) = \delta.$$

Moreover, to get the minimum k' the vertex weight must form an arithmetic sequence, namely $\delta, \delta + 1, \delta + 2, \dots, \delta + p - 1$.

In other hand, we choose u' as a vertex of degree Δ , such that

$$w(u') = \sum_{u'_i \in N(u')} f_v(u'_i) + \sum_{u'v' \in E(G)} f_e(u'v') = 2k_v(\Delta) + k_e(\Delta).$$

Since $k' = \max\{k_e, 2k_v\}$, then

$$w(u') = 2k_v(\Delta) + k_e(\Delta) \leq k'(2\Delta).$$

It is obtained the largest vertex weight is $p + \delta - 1$. Since $p + \delta - 1 \leq wt(u')$, then $k'(2\Delta) \geq p + \delta - 1$, such that $k' \geq \frac{p+\delta-1}{2\Delta}$. Since $D_{ref}(G)$ should be integer and we need a sharpest lower bound, it implies $k' \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil$ or $D_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil$. ■

From the **Lemma 1**, we establish the corollary that can be used to calculate the exact value of distance reflexive strength as follows.

Corollary 1. *Let G be a graph which has $p, \delta,$ and Δ . If $p + \delta - 1 > \Delta(2t - 1)$, then*

$$D_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil + 1,$$

where $p, \delta, \Delta,$ and t are order, minimum degree, maximum degree, and an odd integer, respectively.

Proof. Let G be a graph which has $p, \delta,$ and Δ . Let $p + \delta - 1 = t(2\Delta)$. According to

Lemma 1, we have $D_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil > \left\lceil \frac{\Delta(2t-1)}{2\Delta} \right\rceil = \left\lceil \frac{2t-1}{2} \right\rceil = t$.

Moreover, suppose t is $D_{ref}(G)$, t will be the label with the largest size under any distance irregular reflexive k -labeling. On the vertex with the maximum degree, the greatest weight can be attained. Let u' be the vertex of greatest degree, with u' is vertex weight being

$$w(u') = \sum_{u'_i \in N(u')} f_v(u'_i) + \sum_{\{u'v' \in E(G)\}} \leq (t - 1)(\Delta) + t(\Delta) = \Delta(2t - 1).$$

Based on **Lemma 1**, we obtain $p + \delta - 1 \leq \Delta(2t - 1)$. It contradicts with $p + \delta - 1 > \Delta(2t - 1)$. Hence $t \neq D_{ref}(G)$, such that $D_{ref}(G) > \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil > \left\lceil \frac{\Delta(2t-1)}{2\Delta} \right\rceil = \left\lceil \frac{2t-1}{2} \right\rceil = t$. Thus, it concludes that $D_{ref}(G) \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil + 1$. ■

We use those lemma and corollary above to determine the distance reflexive strength of path, star, and friendship graph which presented in the following theorems.

Theorem 1. Let G_n be a graph isomorphic with a path graph. For $\forall n \geq 3$,

$$D_{ref}(G_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil + 1, & \text{if } n \equiv 3,4(mod 8) \\ \left\lceil \frac{n}{4} \right\rceil, & \text{if } n \equiv 0,5,6,7(mod 8) \end{cases}$$

where n is natural number.

Proof. The graph G_n , $n \geq 3$ has a vertex set is $V(G_n) = \{a_i; 1 \leq i \leq n\}$ and an edge set $E(G_n) = \{a_i a_{i+1}; 1 \leq i \leq n - 1\}$. As such $\delta(P_n) = 1$ and $\Delta(P_n) = 2$, for $n \equiv 0,5,6,7(mod 8)$ base on **Lemma 1** we have

$$D_{ref}(G_n) \geq \left\lceil \frac{p + \delta - 1}{2\Delta} \right\rceil = \left\lceil \frac{n + 1 - 1}{2(2)} \right\rceil = \left\lceil \frac{n}{4} \right\rceil.$$

For $n \equiv 3(mod 8)$, it gives $D_{ref}(G_n) \geq \left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{8b+3}{4} \right\rceil = 2b + 1$ and for $n \equiv 4(mod 8)$, it gives $D_{ref}(G_n) \geq \left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{8b+4}{4} \right\rceil = 2b + 1$, where b is natural number. Since $\left\lceil \frac{n}{4} \right\rceil = 2b + 1$ is odd number, $2(2 \left\lceil \frac{n}{4} \right\rceil - 1)$ will be the biggest vertex label on path. In other hand, $n > 2(2 \left\lceil \frac{n}{4} \right\rceil - 1)$, this condition satisfies with **Corollary 1**, such that we have

$$D_{ref}(G_n) \geq \left\lceil \frac{n}{4} \right\rceil + 1.$$

Furthermore, we show the upper bound of $D_{ref}(G_n)$ by constructing and defining the function as follows. We give an illustration in Figure 1 to show the upper bound of P_3, P_4, P_5, P_6 , and P_7 .

Now we give the construction of vertex and edge labels in the following steps:

- (i) Give the label on vertices with the function as follows. For $1 \leq i \leq 2k + 1$,

$$f_v(a_i) = k$$

where

$$k = \begin{cases} \left\lceil \frac{n}{4} \right\rceil + 1, & \text{if } n \equiv 3,4(mod 8) \\ \left\lceil \frac{n}{4} \right\rceil, & \text{if } n \equiv 0,5,6,7(mod 8) \end{cases}$$

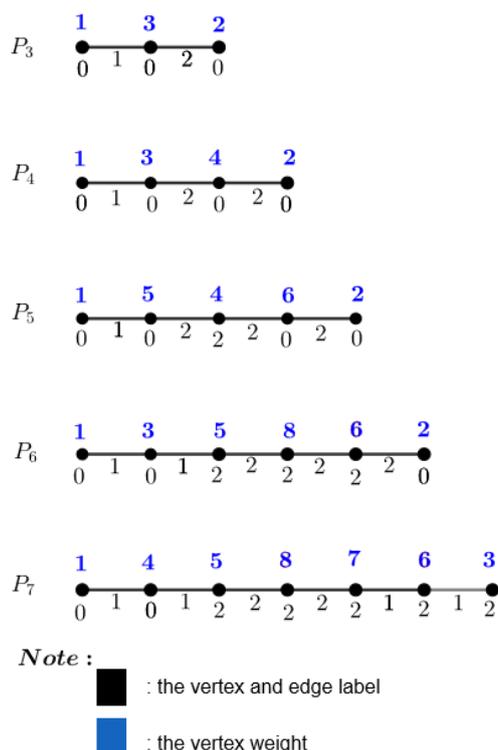


Figure 1. The construction of vertex and edge labels on $P_3, P_4, P_5, P_6,$ and P_7 .

(ii) Give the label on edges with the function as follows. For $1 \leq i \leq 2k$,

$$f_e(a_i a_{i+1}) = k + 1 - \left\lfloor \frac{i}{2} \right\rfloor.$$

(iii) The vertex weight, $w(a_1) = 2k$ and give the vertex weight $w(a_i) = 4k + 2 - i$, for $2 \leq i \leq 2k$.

(iv) Determine $w(a_{2k+1}) = 2k + 1$, and for $2k + 2 \leq i \leq n - 1, w(a_i) = 4k + 1 - i$.

(v) For $2k + 2 \leq i \leq n$, label each vertex a_i with $f_v(a_{i-1})$ and label edge $a_{i-1} a_i$ with $w(a_{i-1}) - f_v(a_{i-2}) - f_e(a_{i-2} a_{i-1}) - f_v(a_i)$.

(vi) If the edge label of point [4] is 0, then label each vertex a_i with $f_v(a_{i-1}) - 2$ and label edge $a_{i-1} a_i$ with $w(a_{i-1}) - f_v(a_{i-2}) - f_e(a_{i-2} a_{i-1}) - f_v(a_i)$.

(vii) The vertex weight of a_n as follows.

$$w(a_n) = f_v(a_{n-1}) + f_e(a_{n-1} a_n),$$

since $w(a_n)$ is the sum of two labels (vertex and edge), then $w(a_n)$ must be different from the others.

Since the all vertices are distinct, $D_{ref}(G_n) \geq k$ and $D_{ref}(G_n) \leq k$, it concludes that $D_{ref}(G_n) = k$ for $n \geq 3$. ■

Theorem 2. Let H_n be a graph isomorphic with a star graph. For $\forall n \geq 3$, $D_{ref}(H_n) = n$.

Proof. The graph $H_n, n \geq 3$ has a vertex set $V(H_n) = \{\alpha, x_i; 1 \leq i \leq n\}$ and an edge set $E(H_n) = \{\alpha x_i; 1 \leq i \leq n\}$. The star graph has n vertices of degree 1 and one dominant vertex. Since $\delta(H_n) = 1$ and the dominant vertex always contributes the constant label to other vertices, the smallest vertex weight is at least 1 and the largest vertex weight on vertices of degree 1 is at least n . Since α is the dominant vertex and we can label it with 0, the vertex weight on vertices of degree 1 depends on 1 label (only edge), such that $D_{ref}(H_n) \geq n$. Since $\Delta(H_n) = n$, then $D_{ref}(H_n) \geq n \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil$ and satisfies the **Lemma 1**. Furthermore, we show the upper bound of $D_{ref}(H_n)$ by defining the following function:

$$f_v(\alpha) = f_v(x_i) = 0$$

$$f_e(\alpha x_i) = i: 1 \leq i \leq n$$

By those vertex and edge labels, we have the vertex weight as follows.

$$w(\alpha) = \sum_{i=1}^n [f_v(x_i) + f_e(\alpha x_i)] = \frac{n(n+1)}{2}$$

$$w(x_i) = i: 1 \leq i \leq n$$

We know that all vertex weights of graph H_n are distinct and it gives a distance reflexive strength on the star graph. Since $D_{ref}(H_n) \leq n$ and $D_{ref}(H_n) \geq n$, it concludes that $D_{ref}(H_n) = n$ for $n \geq 3$. ■

Theorem 3. Let M_n be a graph isomorphic with friendship graph (F_n). For $\forall n \geq 3$,

$$D_{ref}(M_n) = \begin{cases} \left\lceil \frac{2n+1}{3} \right\rceil + 1, & \text{if } n \equiv 1(\text{mod } 3) \\ \left\lceil \frac{2n+1}{3} \right\rceil, & \text{otherwise} \end{cases}$$

where n is natural number.

Proof. The graph $M_n, n \geq 3$ has a vertex set $V(M_n) = \{a, u_i, v_i; 1 \leq i \leq n\}$ and an edge set $E(M_n) = \{au_i; 1 \leq i \leq n\} \cup \{av_i; 1 \leq i \leq n\} \cup \{u_i v_i; 1 \leq i \leq n\}$. The graph M_n has $2n$ vertices of degree 2 and one dominant vertex. Since $\delta(M_n) = 2$ and the dominant vertex always contributes the constant label to other vertices, the smallest vertex weight is at least 2 and the biggest vertex weight on vertices of degree 2 is at least $2n + 1$. Since a is the dominant vertex and we can label it with 0, the vertex weight on vertices of degree 2 depends on 3 labels (2 edges and 1 adjacent vertex). Since label of vertex must be an even number, in such a case that $D_{ref}(M_n) \geq \left\lceil \frac{2n+1}{3} \right\rceil$. Since $\Delta(M_n) = 2n$, then $D_{ref}(M_n) \geq \left\lceil \frac{2n+1}{3} \right\rceil \geq \left\lceil \frac{p+\delta-1}{2\Delta} \right\rceil$ and satisfies the **Lemma 1**. For $n \equiv 1(\text{mod } 3)$, there is an integer s such that $2n = 2(3s + 1) = 6s + 2$, then $\left\lceil \frac{2n+1}{3} \right\rceil = \left\lceil \frac{6s+3}{3} \right\rceil = 2s + 1$. The vertex weight $2n + 1 = 6s + 3$ is the sum of 3 labels where each label is not more than $2s + 1$, such that $2n + 1 = 6s + 3 = (2s + 1) + (2s + 1) + (2s + 1)$. As such the vertex label should be an even number, for $n \equiv 1(\text{mod } 3)$ we get $D_{ref}(M_n) \geq \left\lceil \frac{2n+1}{3} \right\rceil + 1$.

Furthermore, we show the upper bound of $D_{ref}(M_n)$ by defining the following function:

$$\begin{aligned}
 f_v(a) &= 0 \\
 f(u_i) &= \begin{cases} 0, & \text{if } i = 1 \\ 2 \left\lfloor \frac{i-1}{3} \right\rfloor, & \text{if } 2 \leq i \leq n \end{cases} \\
 f_e(u_i v_i) &= \begin{cases} f_v(u_i) = f_v(v_i) \\ \frac{2}{3}i, & \text{if } i \equiv 0 \pmod{3} \\ \frac{1}{3}(2i+1), & \text{if } i \equiv 1 \pmod{3} \\ \frac{1}{3}(2i-1), & \text{if } i \equiv 2 \pmod{3} \end{cases} \\
 f_e(au_i) &= f_e(u_i v_i) \\
 f_e(av_i) &= \begin{cases} \frac{2}{3}i + 1, & \text{if } i \equiv 0 \pmod{3} \\ \frac{1}{3}(2i+1) + 1, & \text{if } i \equiv 1 \pmod{3} \\ \frac{1}{3}(2i-1) + 1, & \text{if } i \equiv 2 \pmod{3} \end{cases}
 \end{aligned}$$

By those vertex and edge labels, we have the vertex weight as follows.

$$w(a) = \sum_{i=1}^n [f_v(u_i) + f_v(v_i) + f_e(au_i) + f_e(av_i)]$$

$$w(u_i) = 2i : 1 \leq i \leq n$$

$$w(v_i) = 2i + 1 : 1 \leq i \leq n$$

We know that the all vertex weights of graph M_n are distinct. Thus, it gives a distance reflexive strength on M_n for $n \geq 3$. ■

CONCLUSIONS

We determined the lower bound and $D_{ref}(G)$, where G be several graphs which isomorphic with path, star, and friendship graphs. Since every graph has different characteristics, the distance reflexive strength varies. Based on the characteristics of these graphs, we can characterize the distance reflexive strength. Characterizing the distance reflexive strength is a difficult problem that requires extensive research. As a result, we present the following open problems.

Open problem 1. Find out the exact value of $D_{ref}(P_n)$ for $n \equiv 1, 2 \pmod{8}$ and any graphs apart from those families.

Open problem 2. Characterize the distance reflexive strength of graph.

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