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# m-Polar Fuzzy B-ideal of B-algebra

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**ABSTRACT**

B-algebra is an algebraic structure related to BCI/BCK-algebra. Many researchers have studied fuzzy B-ideal on B-algebra, m-polar fuzzy set on BCI-algebra and B-algebra, m-polar fuzzy subalgebra on BCI-algebra and B-algebra, m-polar fuzzy ideal on BCI-algebra, m-polar $(\in, \in)$-fuzzy p-ideal on BCI-algebra, m-polar $(\in, \in)$-fuzzy q-ideal on BCI-algebra, and m-polar $(\in, \in)$-fuzzy a-ideal on BCI-algebra. We build a new structure, namely m-polar $(\in, \in)$-fuzzy B-ideal on B-algebra. This research aims to extend the knowledge of m-polar fuzzy sets, which can be combined with other algebraic structures, besides BCI-algebra. In this study, we investigate and describe the properties of m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra. We also investigate the connection among m-polar $(\in, \in)$-fuzzy B-ideal, m-polar fuzzy subalgebra, and m-polar fuzzy ideal. We serve a condition that causes an m-polar fuzzy ideal to become an m-polar $(\in, \in)$-fuzzy B-ideal. We also serve expansion properties of an m-polar $(\in, \in)$-fuzzy B-ideal. Futhermore, examples showing the modification of $\pi_i$ formula are added. The properties of m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra are obtained by combining and modifying the properties of m-polar $(\in, \in)$-fuzzy p-ideal, m-polar $(\in, \in)$-fuzzy q-ideal, and m-polar $(\in, \in)$-fuzzy a-ideal of BCI-algebra.

**Keywords**: B-algebra; B-ideal; m-polar fuzzy set; m-polar fuzzy ideal; m-polar fuzzy B-ideal



## INTRODUCTION

Algebra is closely related to sets. The set that has been generally recognized is the crisp set. The crisp set distinguishes its members with a value of zero or one, a member of the set or not. Then, Zadeh [1] proposed a new notion, namely fuzzy sets as a development of crisp sets. In fuzzy sets, membership values for each member lies in [0,1]. A fuzzy set is a set that contain $x$ and the membership degree of $x$. Then, Zhang [2] introduced a new idea, namely bipolar fuzzy sets. In bipolar fuzzy set, membership values for each member lies in [-1,1]. This notion of bipolar information arises because data for problems in real life sometimes comes from 2 factors, positives and negatives.

Chen, et al [3] proposed the m-polar fuzzy set which is an extended version of the bipolar fuzzy set. A mapping $\delta: X \to [0,1]^m$ is named the m-polar fuzzy set of $X$. Chen et al. explained that bipolar fuzzy sets and 2-polar fuzzy sets are identical. This notion of multipolar information arises because data for complex problems in real life sometimes comes from multiple $n$ factors $(n \geq 2)$. For example, Brawijaya University is a good

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university. A university is said to be good if it can fulfill several requirements, such as campus accreditation, adequate facilities, appropriate curriculum, and others. Each of these components has a value in the interval [0,1]. If $n$ denotes the amount of components, then the elements of $[0,1]^n$ denote the values of truth of a fuzzy assertion.

BCK-algebra and BCI-algebra are constructed by Imai and Iseki [4]. B-algebra was first proposed by Neggers and Kim [5] as an algebraic class related to BCK/BCI/algebra. B-algebra is an unempty sets equipped with operations $*$ and 0 as the identity element that satisfies several axioms, then denoted by $(X,*,0)$. Fuzzy B-algebra was first proposed by Jun, et al [6], which is a combination of B-algebra and fuzzy set theory. After that, many researchers developed the notion of fuzzy B-algebra. In 2003, Ahn and Bang [7] investigated the fuzzy subalgebra of B-algebra and classified the subalgebra based on the subalgebra level in B-algebra. Two years later, Yamini and Kailasavalli [8] constructed fuzzy B-ideals and B-ideals of B-algebras.

Al-Masarwah and Ahmad [9] examined m-polar fuzzy sets applied to BCK/BCIalgebras. In the article, they introduced m-polar fuzzy subalgebra and (closed, commutative) m-polar fuzzy ideals, then they investigated the related properties. Continuing the research of Masarwah and Ahmad, Takallo, et al. [10] presented the idea of (normal) m-polar $(\in,\in)$- fuzzy p-ideal and they examined its properties. A year after investigating m-polar $(\in,\in)$-fuzzy p-ideals in BCI-algebra, Miuhuddin, et al [11] built a new structure, the m-polar $(\in,\in)$-fuzzy q-ideals in BCI-algebra and its characteristics are examined. After that, the concept of normal m-polar $(\in,\in)$-fuzzy a-ideal and its properties were established by Borzooei, et al. [12].

In this paper, we combine the ideas of Yamini and Kailasavalli [8], Takallo et al [10], Muhiuddin et al. [11], and Borzooei, et al. [12] to obtain a new idea and structure, namely m-polar $(\in,\in)$-fuzzy B-ideal of B-algebra. This research aims to extend the knowledge of m-polar fuzzy sets, which can be combined with other algebraic structures, besides BCI-algebra. In addition, we investigate the properties of m-polar $(\in,\in)$-fuzzy B-ideal on B-algebra ,i.e general properties obtained by giving a special condition on m-polar $(\in,\in)$-fuzzy B-ideal so that for each m-polar $(\in,\in)$-fuzzy B-ideal a certain inequality holds. We investigate the connection among m-polar $(\in,\in)$-fuzzy B-ideal, m-polar fuzzy ideal and m-polar fuzzy subalgebra. We give a condition that causes an m-polar fuzzy ideal to become an m-polar $(\in,\in)$-fuzzy B-ideal. Futhermore, we give expansion properties of an m-polar $(\in,\in)$-fuzzy B-ideal. This article also introduces a modification of the $\pi_i$ formula in proving whether an m-polar fuzzy subset $A$ is an m-polar $(\in,\in)$-fuzzy B-ideal of B-algebra $(X,*,0)$ or not (in Example 3.6 and Example 3.11), which is missing in the previous research.

## METHODS

We conduct a literature review on [8], [10], [11], and [12] and deepen our understanding of B-algebra, B-ideal on B-algebra, fuzzy B-algebra, fuzzy B-ideal on B-algebra, m-polar fuzzy set, and m-polar fuzzy ideal. We combine the concept of fuzzy B-ideal on B-algebra with m-polar fuzzy set to obtain a new structure, namely m-polar $(\in,\in)$-fuzzy B-ideal of B-algebra. We replace the existing structure in [10], [11], and [12] with B-ideal, so we obtain properties for m-polar $(\in,\in)$-fuzzy B-ideal which are modifications and combinations of the properties in [10], [11], and [12]. Then, we add a new idea which is the addition of a modified $\pi_i$ formula in proving whether an m-polar fuzzy subset $A$ is an m-polar fuzzy B-ideal of B-algebra $(X,*,0)$ or not (in Example 3.6 and Example 3.11).

**Definition 2.1** [5] A set $X$, which is unempty set, with 0 and operation $*$, denoted by $(X,*,0)$, can be considered a B-algebra if it fulfills the below axioms.
B1. $u*u=0$.
B2. $u*0=u$.
B3. $(u*v)*w=u*(w*(0*v))$.
for all $u,v,w\in X$.

**Example 2.2** [5] Suppose $X=\{0,1,2,3,4,5\}$ is a set with operation $*$ which can be shown in the corresponding Cayley table below.

**Table 1**: Definition of operation $*$ for $X=\{0,1,2,3,4,5\}$

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 | 4 | 5 |
| 1 | 1 | 0 | 2 | 4 | 5 | 3 |
| 2 | 2 | 1 | 0 | 5 | 3 | 4 |
| 3 | 3 | 4 | 5 | 0 | 2 | 1 |
| 4 | 4 | 5 | 4 | 1 | 0 | 2 |
| 5 | 5 | 3 | 3 | 2 | 1 | 0 |

Then, $(X,*,0)$ is a B-algebra.

**Propotition 2.3** [14] If $(X,*,0)$ is a B-algebra, then the conditions below are satisfied.
i. $(u*v)*(0*v)=u$.
ii. $v*w=v*(0*(0*w))$.
iii. $u*(v*w)=\big(u*(0*w)\big)*v$ .
iv. $u*w=v*w$, implies $u=v$.
v. If $u*v=0$, then $u=v$.
vi. If $0*u=0*v$, maka $u=v$.
vii. $0*(0*u)=u$.
viii. $0*(u*v)=v*u$.
ix. $(u*w)*(v*w)=u*v$.
for all $u,v,w\in X$.

Assume $(X,*,0)$ is a B-algebra and S is a subset of $X$. $S$ can be said as a subalgebra of $X$ if $u*v\in S$, for all $u,v\in S$ ([14]).

Assume $(X,*,0)$ is a B-algebra and $I$ is a subset of $X$. $I$ can be said as an ideal of $X$ if it fulfills the axioms below.
i. $0\in I$.
ii. If $(u*v)\in I,(v)\in I$, then $(u)\in I$.
for all $u,v,w\in X$ ([14]).

**Definition 2.4** [13] Suppose $(X,*,0)$ is a B-algebra and $I$ is a subset of $X$. $I$ can be said as a B-ideal of $X$ if it fulfills the axioms below.
i. $0\in I$.
ii. If $(u*v)\in I,(w*u)\in I$, then $(v*w)\in I$.
for all $u,v,w\in X$.

**Example 2.5** Suppose $X=\{0,1,2,3,4,5\}$ is a B-algebra by the operation $*$ as in Table 1 and $I=\{0,3\}$ is a subset of B-algebra $(X,*,0)$. So, $I$ is B-ideal of $X$.

**Definition 2.6** [6] Suppose $A$ is a fuzzy subset of B-algebra $(X,*,0)$. $A$ can be said as a fuzzy B-algebra if it fulfills the condition below.

$$\delta_A(u * v) \geq \min\{\delta_A(u), \delta_A(v)\}$$

for all $u, v \in X$.

**Example 2.7** Suppose $(X,*,0)$ with $X = \{0,1,2,3\}$ is a B-algebra with operation $*$ as shown in the corresponding Cayley table below.

**Table 2**: Definition of operation $*$ for $X = \{0,1,2,3\}$

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

Then, $A$ is determined as a fuzzy subset of $X$ which can be written as follows.

$$A = \{(0,0.8), (1,0.7), (2,0.6), (3,0.6)\}$$

Then, $A$ is fuzzy B-algebra.

Suppose $A$ is a fuzzy subset of B-algebra $(X,*,0)$ . $A$ can be said to be a fuzzy ideal of $X$ if it fulfills the axioms below.

i. $\delta_A(0) \geq \delta_A(u)$.

ii. $\delta_A(u) \geq \min\{\delta_A(u * v), \delta_A(v)\}$.

for all $u, v \in X$ ([11]).

**Definition 2.8** [13] Suppose $A$ is a fuzzy subset of B-algebra $(X,*,0)$. $A$ can be considered as a fuzzy B-ideal of $X$ if it fulfills the axioms below.

i. $\delta_A(0) \geq \delta_A(u)$.

ii. $\delta_A(v * w) \geq \min\{\delta_A(u * v), \delta_A(w * u)\}$.

for all $u, v, w \in X$.

**Example 2.9** Suppose $X = \{0,1,2,3\}$ is a B-algebra with the operation $*$ as in Table 2. So, $A$ is determined as a fuzzy subset of $X$ which can be written as follows.

$$A = \{(0,0.6), (1,0.6), (2,0.5), (3,0.5)\}$$

Then, $A$ is fuzzy B-ideal of $X$.

**Definition 2.10** [16] The m-polar fuzzy subset $A$ of $X$ is written down as follows.

$$A = \{(u, \delta_A(u)) | u \in X\}$$

where $\delta_A$ is a function with domain $X$ and codomain m-tuples of real numbers in [0,1]. $\delta_A$ is written down as follows.

$$\delta_A: X \rightarrow [0,1]^m$$

Note that $m$ is a real number. $\tilde{A}(u)$ is the assigned membership value for each $u \in X$ which is written as the statement below.

$$\tilde{A}(u) = ((\pi_1 \circ \delta_A)(u), \dots, (\pi_m \circ \delta_A)(u))$$

where $\pi_i: [0,1]^m \rightarrow [0,1]$ for all $i = 1,2,3,4, \dots, m$ and $\circ$ is composition function.

**Definition 2.11** [17] A m-polar fuzzy set $A$ on set $X$ is given with $\bar{\alpha}_A$ as the membership function of $A$. The m-polar level set of $A$ may be written as condition (1) below.

$$U(\tilde{A}, \hat{t}) = \{u \in X | \tilde{A}(u) \geq \hat{t}\} \quad (1)$$

Based Definition 2.10, condition (1) can be expressed as below.

$$U(\tilde{A}, \hat{t}) = \{u \in X | (\pi_1 \circ \delta_A)(u) \geq t_1, (\pi_2 \circ \delta_A)(u) \geq t_2, \dots, (\pi_m \circ \delta_A)(u) \geq t_m\}$$

**Definition 2.12** [17] m-polar fuzzy set $A$ on set $X$ with $\bar{\alpha}_A$, as the membership function of $A$, in the following form.

$$\tilde{A}(v) = \begin{cases} \hat{r} = (r_1, r_2, \dots, r_m) \in (0,1]^m, v = u \\ 0 = (0,0, \dots, 0) \quad , v \neq u \end{cases}$$

is an m-polar fuzzy point with support $u$ and value $\hat{r}$, denoted by $u_{\hat{r}}$.

If $\tilde{A}(u) \geq \hat{r}$, which based on Definition 2.10 can also be written $(\pi_i \circ \delta_A)(u) \geq r_i$, then m-polar fuzzy point $u_{\hat{r}}$ belongs to m-polar fuzzy set $A$, notated by $u_{\hat{r}} \in A$ for each $i = 1,2,3,4 \dots, m$ (the notion belonginess ($\in$) of an m-polar fuzzy point to an m-polar fuzzy set).

**Definition 2.13** [19] Suppose $A$ is an m-polar fuzzy subset of B-algebra $(X, *, 0)$. $A$ can be said as an m-polar fuzzy subalgebra of $X$ if it fulfills inequality (2) below.

$$\tilde{A}(u * v) \geq \inf\{\tilde{A}(u), \tilde{A}(v)\} \quad (2)$$

for all $u, v \in X$. Based on Definition 2.10, inequality (2) is the same as the inequality below.

$$(\pi_1 \circ \delta_A)(u * v) \geq \inf\{(\pi_1 \circ \delta_A)(u), (\pi_1 \circ \delta_A)(v)\}$$

for all $f, g \in X$ and $i = 1,2,3,4, \dots, m$.

## RESULTS AND DISCUSSION

Here, we will examine the structure of m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra. We give examples and properties of m-polar $(\in, \in)$-fuzzy B-ideal with proofs.

The concept of fuzzy ideal and m-polar fuzzy set of B-algebra $(X,*,0)$ has been explained in the method section. Then, we introduce a new idea, namely m-polar fuzzy ideal of B-algebra $(X,*,0)$ as a combination of the concept of fuzzy ideal and m-polar fuzzy set.

**Definition 3.1** Suppose $A$ is an m-polar fuzzy subset of B-algebra $(X,*,0)$ and $\tilde{A}(u)$ is the membership value for each $u \in X$. $A$ can be said as an m-polar fuzzy ideal of $X$ if it fulfills inequality (3) below.

$$\tilde{A}(0) \geq \tilde{A}(u) \geq \inf\{\tilde{A}(u * v), \tilde{A}(v)\} \tag{3}$$

for all $u, v \in X$. Based on Definition 2.10, inequality (3) can also be written as follows.

$$(\pi_1 \circ \delta_A)(u) \geq \inf\{(\pi_1 \circ \delta_A)(u * v), (\pi_1 \circ \delta_A)(v)\}$$

for all $u, v \in X$ and $i = 1,2,3,4 \dots, m$.

The next lemma explains the notion of belonginess ($\in$) of an m-polar fuzzy point with an m-polar fuzzy subset.

**Lemma 3.2** An m-polar fuzzy subset $A$ of B-algebra $(X,*,0)$ can be said to be an m-polar fuzzy ideal of $X$ if and only if it fulfills the statements below.

i. $(\forall u \in X)(\forall \hat{r} \in [0,1]^m)(u_{\hat{r}} \in A \Rightarrow 0_{\hat{r}} \in A)$.

ii. $(\forall u, v \in X)(\forall \hat{r}, \hat{t} \in [0,1]^m)((u * v)_{\hat{r}} \in A, v_{\hat{t}} \in A \Rightarrow u_{\inf\{\hat{r},\hat{t}\}} \in A)$.

If an m-polar fuzzy set $A$ of B-algebra $(X,*,0)$ satisfies conditions (i) and (ii), then A is called an m-polar $(\in, \in)$-fuzzy ideal of $X$.

After discussing m-polar fuzzy ideal, we define m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra. The following definition of m-polar $(\in, \in)$-fuzzy B-ideal is related to m-polar fuzzy points (note Definition 2.12 and Lemma 3.2).

**Definition 3.3** Suppose $A$ is an m-polar fuzzy subset of B-algebra $(X,*,0)$. $A$ is considered as an m-polar $(\in, \in)$-fuzzy B-ideal of $X$ if it fulfills the statement below.

$$u_{\hat{r}} \in A \Rightarrow 0_{\hat{r}} \in A \tag{4}$$

for all $u \in X$ and $\hat{r} \in [0,1]^m$.

$$(u * v)_{\hat{r}} \in A, (w * u)_{\hat{t}} \in A \Rightarrow (v * w)_{\inf\{\hat{r},\hat{t}\}} \in A \tag{5}$$

for all $u, v, w \in X$ and $\hat{r}, \hat{t} \in [0,1]^m$.

Referring to Definition 2.12 (explaining the notion of belonginess of an m-polar fuzzy point to an m-polar fuzzy set), we construct Proposition 3.4 and Proposition 3.5 which prove that condition (4) and (5) written as m-polar fuzzy points can be written as m-polar fuzzy subset.

**Propotition 3.4** For an m-polar fuzzy set $A$, condition (4) is equivalent to the following inequality.

$$\tilde{A}(0) \geq \tilde{A}(u) \quad (6)$$

for all $u \in X$. Based on Definition 2.10, inequality (6) can also be written as follows.

$$(\pi_i \circ \delta_A)(0) \geq (\pi_i \circ \delta_A)(u) \quad (7)$$

for all $u \in X$and $i = 1,2, \dots, m$.

***Proof.*** Suppose $A$ satisfies condition (4). If it is assumed that $(\pi_i \circ \delta_A)(0) < (\pi_i \circ \delta_A)(u)$, then there exists $p \in X$ and $r_i \in (0,1]$ for $i \in \{1,2, \dots, m\}$ such that $(\pi_i \circ \delta_A)(0) < r_i \leq (\pi_i \circ \delta_A)(p)$. It follows that $p_{r_i} \in A$, but $0_{r_i} \notin A$. This statement contradicts condition (4). Thus, $(\pi_i \circ \delta_A)(0) \geq (\pi_i \circ \delta_A)(u)$, for all $u \in X$ and $i = 1,2,3,4, \dots., m$.
Otherwise, it is assumed that $A$ satisfies condition (6). Suppose $u \in X$ and $\hat{r} \in [0,1]^m$ such that $u_{\hat{r}} \in A$ which means $\tilde{A}(u) \geq \hat{r}$ (by Definition 2.12), which can also be written $(\pi_i \circ \delta_A)(u) \geq r_i$ (by Definition 2.10), for every $i = 1,2,3,4, \dots, m$. According to (7), we have the condition below.

$$(\pi_i \circ \delta_A)(0) \geq (\pi_i \circ \delta_A)(u) \geq r_i$$

So, it can be concluded that $(\pi_i \circ \delta_A)(0) \geq r_i$, which means $0_{\hat{r}} \in A$ (by Definition 2.12). ■

**Propotition 3.5** For an m-polar fuzzy set $A$, axiom (5) is equivalent to the inequality below.

$$\tilde{A}(v * w) \geq \inf\{\tilde{A}(u * v), \tilde{A}(w * u)\} \quad (8)$$

for all $u, v, w \in X$. Based on Definition 2.10, inequality (8) is the same as the inequality below.

$$(\pi_i \circ \delta_A)(v * w) \geq \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(w * u)\} \quad (9)$$

for all $u, v, w \in X$and $i = 1,2,3,4, \dots, m$.

***Proof.*** Suppose $A$ satisfies axiom (5). If we assume that $(\pi_i \circ \delta_A)(v * w) < \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(w * u)\}$, then there exists $p, q, s \in X$ and $r_i \in (0,1]$ for $i \in \{1,2, \dots, m\}$, so we obtain the condition below.

$$(\pi_i \circ \delta_A)(q * s) < t_i \leq \inf\{(\pi_i \circ \delta_A)(p * q), (\pi_i \circ \delta_A)(s * p)\}$$

It follows that $(p * q)_{t_i} \in A$ and $(s * p)_{t_i} \in A$, but $(q * s)_{t_i} \notin A$. This statement contradicts axiom (5).Thus,$(\pi_i \circ \delta_A)(v * w) \geq \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(w * u)\}$, for all $u, v, w \in X$ and $i = 1,2,3,4 \dots., m$.
Otherwise, we assume that $A$ satisfies inequality (8). Suppose $u, v, w \in X$ and $\hat{r}, t \in [0,1]^m$ then we obtain $(u * v)_{\hat{r}} \in A$ and $(w * u)_{\hat{t}} \in A$ which means $\hat{A}(u * v) \geq \hat{r}$ and $\tilde{A}(w * u) \geq \hat{t}$ (by Definition 2.12), which can also be written $(\pi_i \circ \delta_A)(u * v) \geq r_i$ and

$(\pi_i \circ \delta_A)(w * u) \geq t_i$ (by Definition 2.10), for each $i = 1,2,3,4 \ldots, m$. According to (9), we have the condition below.

$$(\pi_i \circ \delta_A)(v * w) \geq \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(w * u)\} \geq \inf\{r_i, t_i\}$$

So, it can be concluded that $(\pi_i \circ \delta_A)(v * w) \geq \inf\{r_i, t_i\}$, which means $(v * w)_{\inf\{r_i, t_i\}} \in A$ (by Definition 2.12). ■

In Example 3.6 below, a modification of the $\pi_i$ formula is included to determine whether the m-polar fuzzy subset $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra $(X, *, 0)$.

**Example 3.6.** Suppose $(X, *, 0)$ with $X = \{0, a, b, c\}$ is a B-algebra with operation $*$ shown in the table below.

**Table 3**: Definition operation $*$ for $X = \{0, a, b, c\}$

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | c |
| $a$ | $a$ | 0 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 0 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 0 |

Suppose $A$ is a 4-polar fuzzy set with $\delta_A: X \to [0,1]^4$ and $\pi_i: [0,1]^4 \to [0,1]$ defined as follows.

$$\delta_A: X \to [0,1]^4$$
$$\delta_A(0) = (0.8, 0.7, 0.8, 0.7)$$
$$\delta_A(a) = (0.6, 0.4, 0.6, 0.4)$$
$$\delta_A(b) = (0.6, 0.4, 0.6, 0.4)$$
$$\delta_A(c) = (0.8, 0.7, 0.8, 0.7)$$

$$\pi_i: [0,1]^4 \to [0,1]$$
$$\pi_1(w_1, w_2, w_3, w_4) = \max(w_1, w_2, w_3, w_4)$$
$$\pi_2(w_1, w_2, w_3, w_4) = \min(w_1, w_2, w_3, w_4)$$
$$\pi_3(w_1, w_2, w_3, w_4) = \max(w_1, w_2, w_3, w_4)$$
$$\pi_4(w_1, w_2, w_3, w_4) = \min(w_1, w_2, w_3, w_4)$$

Then, $A$ is an 4-polar $(\in, \in)$-fuzzy B-ideal of $X$.

In Proposition 3.7 and Proposition 3.8, we discuss the general properties obtained by giving special condition on axiom (9) such that for every m-polar $(\in, \in)$-fuzzy B-ideal a certain inequality holds.

**Propotition 3.7.** If $(X, *, 0)$ is a B-algebra , then every m-polar $(\in, \in)$-fuzzy B-ideal $A$ of $X$ with membership function $\delta_A$ satisfies the following inequality.

$$(\pi_i \circ \delta_A)(v) \geq (\pi_i \circ \delta_A)(0 * v)$$

for all $v \in X$ and $i = 1,2, \ldots, m$.

***Proof.*** In inequality (9), it is asserted that $u = v$ and $w = 0$. By using axioms B1 and B2 (in Definition 2.1) and inequality (6), we have the following condition.

$$\begin{aligned}(\pi_i \circ \delta_A)(v) = (\pi_i \circ \delta_A)(v * 0) &\geq \inf\{(\pi_i \circ \delta_A)(v * v), (\pi_i \circ \delta_A)(0 * v)\} \\ &= \inf\{(\pi_i \circ \delta_A)(0), (\pi_i \circ \delta_A)(0 * v)\} \\ &= (\pi_i \circ \delta_A)(0 * v)\end{aligned}$$

for all $v \in X$ and $i = 1,2,3,4 \dots, m$. ■

**Propotition 3.8** If $(X,*,0)$ is a B-algebra , then every m-polar $(\in, \in)$-fuzzy B-ideal $A$ of $X$ with membership function $\delta_A$ satisfies the following inequality.

$$(\pi_i \circ \delta_A)(0 * u) \geq (\pi_i \circ \delta_A)(u)$$

for all $u \in X$ and $i = 1,2,3,4 \dots, m$.

***Proof.*** In inequality (9), it is asserted that $v = 0$ and $w = u$. By using axioms B1 and B2 (in Definition 2.1) and inequality (6), we have the following condition.

$$\begin{aligned}(\pi_i \circ \delta_A)(0 * u) &\geq \inf\{(\pi_i \circ \delta_A)(u * 0), (\pi_i \circ \delta_A)(u * u)\} \\ &= \inf\{(\pi_i \circ \delta_A)(u), (\pi_i \circ \delta_A)(0)\} \\ &= (\pi_i \circ \delta_A)(u)\end{aligned}$$

for all $u \in X$ and $i = 1,2,3,4 \dots, m$. ■

Based on Propotition 3.7 and Propotition 3.8, we have the corollary below.

**Corollary 3.9** If $(X,*,0)$ is a B-algebra , then every m-polar $(\in, \in)$-fuzzy B-ideal $A$ of $X$ satisfies the following equation.

$$(\pi_i \circ \delta_A)(u) = (\pi_i \circ \delta_A)(0 * u) \tag{10}$$

for all $u \in X$ and $i = 1,2,3,4, \dots, m$.

Futhermore, Theorem 3.10 explains the relationship between m-polar $(\in, \in)$-fuzzy B-ideal with m-polar fuzzy ideal dan m-polar fuzzy subalgebra.

**Theorem 3.10** Any m-polar $(\in, \in)$-fuzzy B-ideal $A$ of $X$ is an m-polar fuzzy ideal and an m-polar fuzzy subalgebra.

***Proof.*** Suppose $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra $(X,*,0)$ with membership function $\delta_A$.
By using equality (9), equation (10), and Propotition 2.3 (viii), we have the following condition.

$$\begin{aligned}(\pi_i \circ \delta_A)(v) = (\pi_i \circ \delta_A)(v * 0) &\geq \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(0 * u)\} \\ &= \inf\{(\pi_i \circ \delta_A)\big(0 * (v * u)\big), (\pi_i \circ \delta_A)\big(0 * (u * 0)\big)\} \\ &= \inf\{(\pi_i \circ \delta_A)(v * u), (\pi_i \circ \delta_A)(u)\}\end{aligned}$$

for all $u, v \in X$ and $i = 1,2,3,4 \dots, m$. So, any m-polar $(\in, \in)$-fuzzy B-ideal $A$ is an m-polar fuzzy ideal.
In equality (9), we assume that $u = 0$. Futhermore, by using axiom B2 (in Definition 2.1) and equation (10), we have the following condition

$$\begin{aligned}(\pi_i \circ \delta_A)(v * w) &\geq \inf\{(\pi_i \circ \delta_A)(0 * v), (\pi_i \circ \delta_A)(w * 0)\} \\ &= \inf\{(\pi_i \circ \delta_A)(v), (\pi_i \circ \delta_A)(w)\}\end{aligned}$$

for all $v, w \in X$ and $i = 1,2,3,4 \dots, m$. So, any m-polar $(\in, \in)$-fuzzy B-ideal $A$ is an m-polar fuzzy subalgebra. ■

In Example 3.11 below, we provide a case where the m-polar fuzzy set $A$ is an m-polar fuzzy ideal and and an m-polar fuzzy subalgebra of $X$, but $A$ is not an m-polar $(\in, \in)$-fuzzy B-ideal of $X$. Moreover, in Example 3.11, the modified $\pi_i$ formula is also included to prove that $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra $(X, *, 0)$ or not.

**Example 3.11** Suppose $(X, *, 0)$ with $X = \{0, 2, a, b\}$ is a B-algebra with operation $*$ shown in the table below.

**Table 4**: Definition operation $*$ for $X = \{0, 2, a, b\}$

| $*$ | 0 | 2 | $a$ | $b$ |
|---|---|---|---|---|
| 0 | 0 | 0 | $a$ | $b$ |
| 2 | 2 | 0 | $b$ | $a$ |
| $a$ | $a$ | $b$ | 0 | 2 |
| $b$ | $b$ | $a$ | 2 | 0 |

Suppose $A$ is a 4-polar fuzzy set with membership function $\delta_A: X \to [0,1]^4$ and $\pi_i: [0,1]^4 \to [0,1]$ defined as follows.

$$\begin{gathered}\delta_A: X \to [0,1]^4 \\ \delta_A(0) = (0.6, 0.7, 0.8, 0.9) \\ \delta_A(2) = (0.5, 0.6, 0.7, 0.8) \\ \delta_A(a) = (0.4, 0.5, 0.6, 0.7) \\ \delta_A(b) = (0.4, 0.5, 0.6, 0.7)\end{gathered}$$

$$\begin{gathered}\pi_i: [0,1]^4 \to [0,1] \\ \pi_1(w_1, w_2, w_3, w_4) = \min(w_1, w_2, w_3, w_4) \\ \pi_2(w_1, w_2, w_3, w_4) = \min(w_1, w_2, w_3, w_4) + 0{,}1 \\ \pi_3(w_1, w_2, w_3, w_4) = \min(w_1, w_2, w_3, w_4) + 0.2 \\ \pi_4(w_1, w_2, w_3, w_4) = \min(w_1, w_2, w_3, w_4) + 0.3\end{gathered}$$

Then, $A$ is an 4-polar fuzzy ideal and 4-polar fuzzy subalgebra of $X$, but $A$ is not 4-polar $(\in, \in)$-fuzzy B-ideal of $X$ because there are $u = 0, v = 2, w = 0$ and $u = 2, v = 2, w = 0$ that do not satisfy condition (9).

Based on Theorem 3.10 and Example 3.11, it can be seen that every m-polar $(\in, \in)$-fuzzy B-ideal is an m-polar fuzzy ideal but every m-polar fuzzy ideal is not necessarily an m-polar $(\in, \in)$-fuzzy B-ideal. Therefore, in Theorem 3.12 and Theorem 3.14, a condition is given for an m-polar fuzzy ideal to be an m-polar $(\in, \in)$-fuzzy B-ideal of $X$.

**Theorem 3.12** Suppose $A$ is an m-polar fuzzy ideal of B-algebra $(X,*,0)$ with membership function $\delta_A$. $A$ can be said as an m-polar $(\in,\in)$-fuzzy B-ideal of $X$ if it fulfills the condition below.

$$(\pi_i \circ \delta_A)((v * w) * (w * u)) \geq (\pi_i \circ \delta_A)(u * v) \tag{11}$$

for all $u, v, w \in X$ and $i = 1,2,3,4, \dots, m$.

***Proof.*** By using the axioms B1 and B2 (in Definition 2.1) and inequality (9), we have the following condition.

$$\begin{aligned}(\pi_i \circ \delta_A)(v) = (\pi_i \circ \delta_A)(v * 0) = (\pi_i \circ \delta_A)((v * 0) * 0)\\ = (\pi_i \circ \delta_A)((v * 0) * (w * w))\\ = (\pi_i \circ \delta_A)((v * w) * (0 * w))\end{aligned}$$

Assume $0 * w = 0$ so that we have the following condition.

$$\begin{aligned}(\pi_i \circ \delta_A)(v * w) &\geq \inf\{(\pi_i \circ \delta_A)\big((v * w) * (w * u)\big), (\pi_i \circ \delta_A)(w * u)\}\\ &\geq \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(w * u)\}\end{aligned}$$

for all $u,, v, w \in X$ and $i = 1,2,3,4 \dots, m$. ■

**Lemma 3.13** If $(X,*,0)$ is a B-algebra, then any m-polar fuzzy ideal dari $X$ fulfills inequality (12) below.

$$(\pi_i \circ \delta_A)(u * v) \geq (\pi_i \circ \delta_A)(v * u) \tag{12}$$

for all $u, v \in X$ and $i = 1,2,3,4, \dots, m$.

***Proof.*** By applying Propotition 2.3 (viii), inequality (6), inequality (9), and equation (10) we have the following condition.

$$\begin{aligned}(\pi_i \circ \delta_A)(u * v) &= (\pi_i \circ \delta_A)(0 * (u * v))\\ &\geq \inf\{(\pi_i \circ \delta_A)\Big(\big(0 * (u * v)\big) * (v * u)\Big), (\pi_i \circ \delta_A)(v * u)\}\\ &= \inf\{(\pi_i \circ \delta_A)\big((v * u) * (v * u)\big), (\pi_i \circ \delta_A)(v * u)\}\\ &= \inf\{(\pi_i \circ \delta_A)(0), (\pi_i \circ \delta_A)(v * u)\} = (\pi_i \circ \delta_A)(v * u)\end{aligned}$$

for all $u, v \in X$ and $i = 1,2,3,4, \dots, m$. ■

**Theorem 3.14** Suppose $A$ is an m-polar fuzzy ideal of B-algebra $(X,*,0)$ with membership function $\delta_A$. $A$ is an m-polar $(\in,\in)$-fuzzy B-ideal of X if it fulfills inequality (13) below.

$$(\pi_i \circ \delta_A)(0 * ((v * w) * (u * w))) \leq (\pi_i \circ \delta_A)\big((v * w) * (w * u)\big) \tag{13}$$

for all $u, v, w \in X$ and $i = 1,2,3,4, \dots, m$.

***Proof.*** Based on Propotition 2.3 (viii), Propotition 2.3 (ix), and Theorem 3.12, we have the following condition.

$$
\begin{aligned}
(\pi_i \circ \delta_A)(u * v) &= (\pi_i \circ \delta_A)((u * w) * (v * w)) \\
&= (\pi_i \circ \delta_A)(0 * ((v * w) * (u * w))) \\
&\leq (\pi_i \circ \delta_A)(0 * ((w * u) * (v * w))) \\
&= (\pi_i \circ \delta_A)((v * w) * (w * u))
\end{aligned}
$$

for all $u, v, w \in X$ and $i = 1,2,3,4, \ldots, m$. So, based on Theorem 3.12, $A$ is m-polar fuzzy B-ideal of X. ■

In the following theorem, we discuss one of the properties of m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra $(X, *, 0)$ involving m-polar level set.

**Theorem 3.15** Suppose $(X, *, 0)$ is a B-algebra. $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of $X$ with membership function $\delta_A$ if and only if the m-polar set $U(\bar{A}, \hat{r})$ of $A$ is a B-ideal of $X$, for each $\hat{r} \in [0,1]^m$.

***Proof.*** $(\Rightarrow)$ We assume $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of X and $\hat{r} = (r_1, r_2, \ldots, r_m) \in (0,1]^m$. It is obvious that $0 \in U(\tilde{A}, \hat{r})$. Suppose $u, v, w \in X$, so we obtain $(u * v) \in U(\tilde{A}, \hat{r})$ and $(w * u) \in U(\tilde{A}, \hat{r})$, hence $(\pi_i \circ \delta_A)(u * v) \geq r_i$ and $(\pi_i \circ \delta_A)(w * u) \geq r_i$. Based on (9), we have the following condition.

$$
(\pi_i \circ \delta_A)(v * w) \geq \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(w * u)\} \geq r_i
$$

for $i = 1,2,3,4, \ldots, m$. Therefore, $(w * u) \in U(\tilde{A}, \hat{r})$ and $U(\tilde{A}, \hat{r})$ are B-ideal of $X$.
$(\Leftarrow)$ We assume the set of polar levels $U(\tilde{A}, \hat{r})$ of $A$ is a B-ideal of $X$, for all $\hat{r} \in [0,1]^m$. If $\tilde{A}(0) < \tilde{A}(p)$, for $p \in X$ and suppose $\hat{r} \in \tilde{A}(p)$, then $p \in U(\tilde{A}, \hat{r})$ and $0 \notin U(\tilde{A}, \hat{r})$. The statement is a contradiction and we get $\tilde{A}(0) \geq \tilde{A}(u)$, for all $u \in X$. Next, we assume that there exists $p, q, s \in X$ such that $\tilde{A}(q * s) < \inf\{\tilde{A}(p * q), \tilde{A}(s * p)\}$. If $\hat{r} = \inf\{\tilde{A}(p * q), \tilde{A}(s * p)\}$, then $(p * q) \in U(\tilde{A}, \hat{r})$ and $(s * p) \in U(\tilde{A}, \hat{r})$. Because $U(\tilde{A}, \hat{r})$ of $A$ is a B-ideal of $X$, then $(q * s) \in U(\tilde{A}, \hat{r})$. Therefore, we get $\tilde{A}(q * s) \geq \hat{r}$, which is a contradiction, so we get $\tilde{A}(v * w) \geq \inf\{\tilde{A}(u * v), \tilde{A}(w * u)\}$,for all $u, v, w \in X$. Thus, $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of $X$.

Next, in Theorem 3.16 and Theorem 3.17, we provide the expansion properties of m-polar fuzzy B-ideal.

**Theorem 3.16** Suppose $A$ is an m-polar fuzzy set of B-algebra $(X, *, 0)$ and $f: X \to X$ is an epimorphism. The composition $A \circ f$ is an m-polar $(\in, \in)$-fuzzy B-ideal of X if $A$ is a m-polar $(\in, \in)$-fuzzy B-ideal of X with membership function $\delta_A$.
***Proof.*** Suppose $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of X with membership function $\delta_A$. Then, for $A \circ f$ we have the following condition.

$$
\begin{aligned}
\big(\pi_i \circ (\delta_A \circ f)\big)\big((v * w) * (w * u)\big) &= \pi_i((\delta_A \circ f)\big((v * w) * (w * u)\big)) \\
&= \pi_i(\delta_A \Big(f\big((v * w) * (w * u)\big)\Big)) \\
&= (\pi_i \circ \delta_A)(f\big((v * w) * (w * u)\big)) \\
&\geq (\pi_i \circ \delta_A)(0 * f\big((v * w) * (u * w)\big)) \\
&= (\pi_i \circ \delta_A)(f(0) * \big(f(v * w) * f(u * w)\big)) \\
&= (\pi_i \circ \delta_A)\big(f(0 * ((v * w) * (u * w)))\big)
\end{aligned}
$$

$$= (\pi_i \circ (\delta_A \circ f))(0 * ((v * w) * (u * w))$$

for all $u, v, w \in X$ and $i = 1,2,3,4, \dots, m$. Thus, by Theorem 3.14, $A \circ f$ is an m-polar $(\in, \in)$-fuzzy B-ideal of X. ■

**Theorem 3.17** Suppose $g: X \to Y$ is an epimorphism of B-algebra. If $B$ is an m-polar $(\in, \in)$-fuzzy B-ideal of $Y$ with membership function $\delta_B$, then the m-polar fuzzy set $A$ of $X$ with membership function $\delta_A$ can be written down as below.

$$\delta_A: X \to [0,1]^m$$
$$u \mapsto \delta_B(g(u))$$

that is , $(\pi_i \circ \delta_A)(u) = (\pi_i \circ \delta_B)(g(u))$, for all $u \in X$ and $i = 1,2,3,4, \dots, m$ is an m-polar $(\in, \in)$-fuzzy B-ideal of $X$.

***Proof.*** We have

$$(\pi_i \circ \delta_A)(u) = (\pi_i \circ \delta_B)\big(g(u)\big) \leq (\pi_i \circ \delta_B)(0) = (\pi_i \circ \delta_B)\big(g(0)\big) = (\pi_i \circ \delta_A)(0)$$

for $u \in X$ and $i = 1,2,3,4, \dots, m$. We also have

$$\begin{aligned}(\pi_i \circ \delta_A)(v * w) &= (\pi_i \circ \delta_B)(g(v * w)) = (\pi_i \circ \delta_B)(g(v) * g(w)) \\ &\geq \inf\{(\pi_i \circ \delta_B)\big(g(u) * g(v)\big), (\pi_i \circ \delta_B)(g(w) * (g(u)) \\ &= \inf\{(\pi_i \circ \delta_B)\big(g(u * v)\big), (\pi_i \circ \delta_B)(g(w * u))\} \\ &= \inf\{(\pi_i \circ \delta_A)(u * v), (\pi_i \circ \delta_A)(w * u)\}\end{aligned}$$

for $u, v, w \in X$ and $i = 1,2,3,4, \dots, m$. So, $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of $X$. ■

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## CONCLUSIONS

In this paper, we contruct a new structure, an m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra which is a combination of the concept of fuzzy B-ideal and m-polar fuzzy set. We elaborate and explain some properties which include the general properties obtained by giving special condition on m-polar $(\in, \in)$-fuzzy B-ideal, the connection among m-polar $(\in, \in)$-fuzzy B-ideal,m-polar fuzzy ideal, and m-polar fuzzy subalgebra. We give a condition that causes an m-polar fuzzy ideal to become an m-polar m-polar $(\in, \in)$-fuzzy B-ideal. Futhermore, we give expansion properties of an m-polar $(\in, \in)$-fuzzy B-ideal. We also add a $\pi_i$ formula in proving whether an m-polar fuzzy subset $A$ is an m-polar $(\in, \in)$-fuzzy B-ideal of B-algebra $(X, *, 0)$ (in Example 3.6 and Example 3.11). For future research, the structure of m-polar fuzzy B-ideal of B-algebra can be developed into m-polar intuitionistic fuzzy B-ideal of B-algebra. In addition, for the next research, the concept of bipolar fuzzy sets and m-polar fuzzy sets can be combined so it can be called m-bipolar fuzzy.

## REFERENCES


[1] L.A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965.

[2] W. R. Zhang, "Bipolar fuzzy sets and relations," *Proceedings of the Fuzzy Information Processing Society Biannua Conference,* San Antonio, USA, pp. 305-309, 1994 .

[3] J. Chen, S. Li, S. Ma, X, Wang, "m-polar fuzzy sets: extension of bipolar fuzzy sets," *Science Worlds Journal*, pp. 41530, 2014.

[4] Y. Imai and K. Iseki, "On axiom system of propositional calculi," *Japan Acad*, no. 42, pp. 19-22, 1966.

[5] Neggers and H. S. Kim, "On B-algebras," *Mathematic International Journal*, no.54, pp. 21-29, 2002.

[6] Y. B. Jun, E. H. Roh, and H. S. Kim, "On fuzzy B-algebras," *Chzechoslovak Mathematical Journal*, vol. 52, no. 2, pp. 375-384, 2002.

[7] S. S. Ahn and K. Bang, "On fuzzy subalgebras in B-algebra," Community *Korean Math,* no. 3, pp. 429-437.

[8] C. Yamini and S. Kailasavalli, "Fuzzy B-ideals on B-algebras," *International Journal Mathematical Archieve,* pp. 227-233, 2014.

[9] A. Al-Marsawah and A. G. Ahmad, "m-polar fuzzy ideal of BCI/BCK-algebras," *Journal of King Sand University-Science*, no. 31, pp. 1220-1226, 2019.

[10] M. M. Takallo, S. S. Ahn, R. A. Borzooei, and Y. B. Jun, "Multipolar fuzzy p-ideals of BCI-algebras," *Mathematics,* no. 7, pp. 1094, 2019.

[11] G. Muhiuddin, M. M. Takallo, R. A. Borzooei, and Y. B. Jun, "m-polar fuzzy q-ideal in BCI-algebras," *Journal of King Saud University-Science*, 2020.

[12] R. A. Borzooei, G. R. Rezaei G. Muhiuddin, and Y. B. Jun, "Multipolar fuzzy a-ideal in BCI-algebras," *Journal of Machine Learning and Cybernetics,* 2021.

[13] Y. Huang "BCI-Algebras," *Science Press*, Beijing, China, 2006.

[14] R. A. Senapati, M. Bhowmik, and M. Pal, "Fuzzy closed ideal of B-alegbras with interval-valued membership function," *International Journal of Fuzzy Mathematical Arc Archive.* vol. 1, pp. 79-91, 2013.

[15] H. K. Abdullah and A. A. Atshan, "Complete ideal and n-ideal of B-algebra," *Applied Mathematical Journal,* vol. 11, no. 35, pp. 1705-1713, 2017.

[16] R. A. Borzooei, H. S. Kim, Y. B. Jun, and S. S. Ahn, "On multipolar intuitionistic fuzzy B-algebras," *Mathematics*, no, 8, pp. 907, 2020.

[17] J. Zhan, Y. B. Jun, B. Davvaz, "On (∈,∈∨q)-fuzzy ideals of BCI-algebras," *Iranian Journal of Fuzzy Systems,* vol. 6, no, 1, pp. 81-94.