CAUCHY: Jurnal Matematika Murni dan Aplikasi
Volume 9(1) (2024), Pages 1-13
p-ISSN: 2086-0382; e-ISSN: 2477-3344

Inclusive Local Irregularity Vertex Coloring in Grid Graph Family

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ABSTRACT

Let $G(V,E)$ is a simple graph and connected where $V(G)$ is vertex set and $E(G)$ is edge set. A mapping $l: V(G) \longrightarrow \{1,2,\ldots k\}$ as vertex k- labeling and function : $w^i: V(G) \longrightarrow N$ is inclusive local irregularity vertex coloring, with $w^i(v) = l(v) + \sum_{u \in N(v)} l(u)$. The minimum number of colors produced from inclusive local irregularity vertex coloring of graph G is called inclusive chromatic number local irregularity, denoted by $\chi^i_{lis}(G)$. There are two research methods in this study, namely the axiomatic deductive method and the pattern detection method. The axiomatic deductive method is a research method using the principles of deductive proof that apply in mathematical logic by using existing theorems, axioms, and lemmas and then applying them to the coloring of inclusive local irregularity vertices in the grid graph family. On this paper, we learn about the inclusive local irregularity vertex coloring and determine the chromatic number on grid graph family.

Keywords: Inclusive Local Irregularity Vertex Coloring; Grid Graph

INTRODUCTION

A graph G is the set of pairs (V,E) where V is the non-empty set of elements called vertices, and E is the (possibly empty) set of unordered pairs of distinct vertices called edges [1]. Graph theory is a labeling of each element of a graph G, namely at a vertices or an edge or both [2]. Graph coloring is one of the labeling in a graph to give color to neighboring vertices, edges or regions [3]. According to the definition of a graph G for V is a non-empty set, it must have at least one vertices. Whereas for E is a set (possibly empty), it is possible that it does not have any sides. So a graph G has at least one vertex and it is possible that it has no edges [4]. Can also learn the complete types of graphs and research similar to this research [5], [6], [7], [8], [9], [10].

Vertices coloring is color the dots neighbors with different colors [11]. In coloring there is also a chromatic number. Chromatic number is the minimum number k-coloring in graph, denoted by $\chi(G)$ [12]. vertices labeling is a mapping that pairs each graph element, namely the set of vertices (vertex) to positive integers [13]. In this case it is done to minimize the vertices labels on the graph. Inclusive local irregularity spot

Submitted: April 12, 2023 Reviewed: March 23, 2024 Accepted: April 04, 2024
DOI: http://dx.doi.org/10.18860/ca.v9i1.20947

coloring is the development of the concept of local irregularity spot coloring by adding neighboring dots and themselves to produce a vertices weight. In coloring it is done to a minimum so that neighboring vertices have different colors [14].

Definition 1. [15] Let be $l : V(G) \longrightarrow \{1,2,\dots k\}$ is the label function and $w^i: V(G) \longrightarrow N$ is the weight function of the inclusive local irregularity spot coloring with $w^i(v) = l(v) + \sum_{u \in N(v)} l(u)$. Labeling l is said to be spot staining local irregularities inclusive if.

(i) $Opt(l) = min\{max\{l_a\}\}; l_a$ is the inclusive local irregularity vertices labeling For each vertices.$uv \in E(G), w^i(u) \neq w^i(v)$

Definition 2. [15] The minimum number of colors resulting from coloring of local irregularities inclusive of a graph G is called the inclusive local irregularity point chromatic number, denoted by $\chi^i_{lis}(G)$.

Lemma 1. [15] For graph G $\chi^i_{lis}(G) \geq \chi_{lis}(G)$

In addition, [16-20] has also conducted research on the coloring of local irregularity vertices on grid graphs. This research produced a theorem regarding the coloring of local irregularities on a grid graph. The theorem will be used as a prepositional form. The prepositional form of the theorem will be used by researchers to support the lower bound of dot coloring of inclusive local irregularities in the grid graph family.

Preposition 1. Local irregular chromatic number *on a grid graph with is* $m \geq 3, n \geq 3$

$$\chi_{lis}(G_{m,n}) = \begin{cases} 3, & \text{for } m = 3 \text{ dan } n = 3 \\ 4, & \text{for } m \text{ odd dan } n \text{ odd} \\ 5, & \text{for } m \text{ even dan } n \text{ even}, m \text{ odd dan } n \text{ even, and m even and n odd} \end{cases}$$

Preposition 2. Local irregular chromatic number*on a ladder graph with is* $n \geq 3$

$$\chi_{lis}(L_n) = 4$$

Preposition 3. Local irregular chromatic number*on a graph with is* $n \geq 2$

$$\chi_{lis}(H_n) = 4$$

Observation 1. For example graph G *is a centipede graph* Cp_n *with* $n \geq 3$*, then the local irregularity chromatic number* Cp_n

$$\chi_{lis}(Cp_n) = \begin{cases} 3, & n = 3 \\ 4, & n \geq 4 \end{cases}$$

Proof. The local irregular chromatic number $\chi_{lis}(Cp_n)$ for $n \geq 3$. For example $\chi_{lis}(Cp_n) \leq 3$, take it $\chi_{lis}(Cp_n) = 2$ that there will be two neighboring vertices with the same color, this will conflict with the definition of vertex coloring on a graph, so that $\chi_{lis}(Cp_n) \geq 3$. Then prove the upper bound of the local irregularity chromatic numbers Cp_n. If each vertex on the graph $n = 3$ is colored with 3 colors and $n \geq 4$ colored with 4 colors, then it will meet the definition of vertex coloring on a graph. Based on the upper and lower bounds of the chromatic numbers, the local irregularities in the graph Cp_n :

$$\chi_{lis}(Cp_n) = \begin{cases} 3, & n = 3 \\ 4, & n \geq 4 \end{cases}$$

METHODS

There are two research methods in this study, namely the axiomatic deductive method and the pattern detection method. The axiomatic deductive method is a research method using the principles of deductive proof that apply in mathematical logic by using existing theorems, axioms, and lemmas and then applying them to the coloring of inclusive local irregularity vertices in the grid graph family. Meanwhile, the pattern detection method is one of the research methods used to formulate patterns and chromatic numbers for inclusive local irregularity spot coloring in grid graph families.

RESULTS AND DISCUSSION

In this research, several theorems of chromatic number coloring of inclusive local irregularity dots on the grid graph family, namely the grid graph, ladder graph, centipede graph, and graph H.

Theorem 1. *Let $G_{m,n}$ be a grid graph with,* $m \geq 3, n \geq 3$

$$\chi_{lis}^{i}(G_{m,n}) = \begin{cases} 3, & \text{for } m = 3 \text{ dan } n = 3 \\ 5, & \text{for } m \text{ odd dan } n \text{ odd with } m > 3, n > 3, \\ & m \geq 4 \text{ dan } n = 3, \\ & m = 3 \text{ dan } n \geq 4 \\ 6, & \text{for } m \text{ even dan } n \text{ even}, \\ & m \text{ odd dan } n \text{ even with } m > 3, \\ & m \text{ even dan } n \text{ odd with } n > 3 \end{cases}$$

Proof. The set of vertices on a grid graph $(G_{m,n})$ is $V(G_{m,n}) = \{x_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}$. So the vertices cardinality of the grid graph $(G_{m,n})$ is $|V(G_{m,n})| = mn$. A grid graph $(G_{m,n})$ has a set of edges, namely $E(G_{m,n}) = \{x_{i,j}x_{i+1,j}; x_{i,j}x_{i,j+1}; 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$. Thus the edge cardinality of the grid graph $(G_{m,n})$ is $|E(G_{m,n})| = 2mn - m - n$. There are 3 cases of inclusive local irregular chromatic numbers on a grid graph $(G_{m,n})$, namely when case 1 $m = 3$ dan $n = 3$ is , case 2 is m odd and n odd and case 3 is m even and n even. The following describes the five cases.

Case 1. For $m = 3$ and $n = 3$

$$l(x_{i,j}) = 1, \text{for } i = 1, 1 \leq j \leq n; i = 2, 1 \leq j \leq n; i = 3, 1 \leq j \leq n$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^{i}(x_{i,j}) = \begin{cases} 3, & \text{for } i = 1, j = 1, j = n, \\ & \text{for } i = 3, j = 1, j = n \\ 4, & \text{for } i = 1, j = 2, \\ & \text{for } i = 2, j = 1, j = n, \\ & \text{for } i = 3, j = 2 \\ 5, & \text{for } i = 2, j = 2 \end{cases}$$

From the weight calculation above we get $|w(G_{m,n})| = 3$ for $m = 3$ dan $n = 3$. Based Preposition 1 and Definition 1 we can conclude $\chi_{lis}^{i} \geq \chi_{lis}(G_{m,n}) = 3$ that the local irregularity chromatic number inclusive of a grid graph $(G_{m,n})$ is 3 for .$m = 3$ dan $n = 3$ or $\chi_{lis}^{i}(G_{m,n}) = 3$.

■

Case 2. For m odd and n odd

$$l(x_{i,j}) = \begin{cases} 1, & \text{for } j \text{ odd}, 1 \le i \le m, \\ & \text{for } j \text{ even}, i \text{ odd}, \\ 2, & \text{for } j \text{ even}, i \text{ even} \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_{i,j}) = \begin{cases} 3, & \text{for } j = 1, i = 1, i = m, \\ & \text{for } j = n, i = 1, i = m \\ 4, & \text{for } j = 1, i \text{ odd with } 3 \le i \le m-2, \\ & \text{for } j \text{ odd with } 3 \le j \le n, i = 1 \text{ and } i = m, \\ & \text{fork } j = n, i \text{ odd with } 3 \le i \le m-2 \\ 5, & \text{for } j = 1, i \text{ even}, \\ & \text{for } j \text{ even}, i = 1 \text{ and } i = m, \\ & \text{for } j \text{ odd with } 3 \le j \le n-2, i \text{ odd} \\ & \text{with } 3 \le i \le m-2, \\ & \text{for } j = n, i \text{ even} \\ 6, & \text{for } j \text{ even}, i \text{ even} \\ 7, & \text{for } j \text{ even}, i \text{ odd with } 3 \le i \le m-2, \\ & \text{for } j \text{ odd with } 3 \le j \le n-2, i \text{ even} \end{cases}$$

From the weight calculation above we get $|w(G_{m,n})| = 5$ for m odd and n odd. Previously, we obtained the lower bound of Preposition 1 and the upper bound of the weight function $\chi^i_{lis} \ge \chi_{lis}(G_{m,n}) = 5$, so that based on Definition 1 we can conclude that the local irregularity chromatic number inclusive of a grid graph $(G_{m,n})$ is 5 for or m ganjil dan n ganjil or $\chi^i_{lis}(G_{m,n}) = 5$.

■

Case 3. For m even and n even

$$l(x_{i,j}) = \begin{cases} 1, & \text{for } j = 1, 1 \le i \le m, \\ & \text{for } j \text{ even with } 2 \le j \le n-2, i \text{ odd and } i = m, \\ & \text{for } j \text{ odd with } 3 \le j \le n-3, 1 \le i \le m-2, i = m, \\ & \text{for } j = n-1, i = 1 \text{ and } i \text{ even}, \\ & \text{for } j = n, 1 \le i \le m \\ 2, & \text{for } j \text{ even with } 2 \le j \le n-2, i \text{ even} \\ & \text{with } 2 \le i \le m-2, \\ & \text{for } j \text{ odd with } 3 \le j \le n-3, i = m-1, \\ & \text{for } j = n-1, i \text{ odd with } 3 \le i \le m \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_{i,j}) = \begin{cases} 3, & \text{for } j = 1, i = 1, i = m, \\ & \text{for } j = n, i = 1, i = m \\ 4, & \text{for } j = 1, i \text{ odd with } 3 \le i \le m, \\ & \text{for } j \text{ even } 2 \le j \le n-2, i = n, \\ & \text{for } j \text{ odd } 3 \le j \le n, i = 1, \\ & \text{for } j = n, i \text{ even with } 2 \le i \le m-2 \\ 5, & \text{for } j = 1, i \text{ even with } 2 \le i \le m-2, \\ & \text{for } j \text{ even } 2 \le j \le n-2, i = 1, \\ & \text{for } j \text{ odd } 3 \le j \le n, i = n \text{ and } i \text{ odd} \\ & \text{with } 3 \le i \le m-3, \\ & \text{for } j = n, i \text{ odd with } 3 \le i \le m \\ 6, & j \text{ even with } 2 \le j \le n-2, i \text{ even} \\ & \text{with } 2 \le i \le m-2, \\ & \text{for } j \text{ odd with } 3 \le j \le n, i = m-1, \\ & \text{for } j = n-1, i \text{ odd with } 3 \le i \le m \\ 7, & \text{for } j \text{ even with } 2 \le j \le n-4, i \text{ odd} \\ & \text{with } 3 \le i \le m, \\ & \text{for } j \text{ odd with } 3 \le j \le n-3, i \text{ even} \\ & \text{with } 2 \le i \le m-4, \\ 8, & \text{for } j = n-1, i = 1, \\ & \text{for } j \text{ odd with } 3 \le j \le n-3, i = m-2, \\ & \text{for } j = n-2, i \text{ odd with } 3 \le i \le m, \\ & \text{for } j = n-1, i \text{ even } 4 \le i \le m-2 \end{cases}$$

From the weight calculation above we get $|w(G_{m,n})| = 6$ for m even and n even. In the lower bound statement above and the upper bound we have looked for above we can write it .$6 \le \chi_{lis}^i \le 6$ in such a way that we can conclude that the chromatic number of local irregularities inclusive of the grid graph $(G_{m,n})$ is 6 for m genap dan n genap or $\chi_{lis}^i(G_{m,n}) = 6$.

Theorem 2. *The inclusive local irregularity chromatic number on a ladder graph L_n with $n \ge 3$*

$$\chi_{lis}^i(L_n) = \begin{cases} 4, & n = 3,\ n \equiv 1 \bmod 3 \\ 5, & n \equiv 0 \bmod 3; 2 \bmod 3 \end{cases}$$

Proof. The set of vertices on a ladder graph (L_n) is $V(L_n) = \{x_i, y_i; 1 \le i \le n\}$. So the vertex cardinality of the ladder graph (L_n) is $|V(L_n)| = 2n$. A ladder graph (L_n) has a set of edges, namely $E(L_n) = \{x_i x_{i+1,} y_i y_{i+1}; 1 \le i \le n-1\} \cup \{x_i y_i; 1 \le i \le n\}$. Thus the edge cardinality of the ladder graph (L_n) is $|E(L_n)| = 3n - 2$. There are 2 cases of inclusive local chromatic irregularity on a ladder graph (L_n), namely when case 1 is $n = 3,\ n \equiv 1 \bmod 3$ and case 2 is $n \equiv 0 \bmod 3; 2 \bmod 3$. The following describes the two cases.

Case 1. $n = 3$ and $n \equiv 1 \bmod 3$
For $n = 3$

$$l(x_i) = 1, \text{for } 1 \leq i \leq 3$$
$$l(y_i) = \begin{cases} 1, & \text{untuk } i = 1 \\ 2, & \text{untuk } 2 \leq i \leq 3 \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1 \\ 5, & \text{for } i = 2 \\ 4, & \text{for } i = 3 \end{cases}$$
$$w^i(y_i) = \begin{cases} 4, & \text{for } i = 1 \\ 6, & \text{for } i = 2 \\ 5, & \text{for } i = 3 \end{cases}$$

Following is the labeling of the inclusive local irregularity vertices coloring on a ladder graph:(L_n)

For $n \equiv 1 \bmod 3$

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, i \text{ even}, i \equiv 1 \bmod 3 \text{ with } i \text{ even} \\ 2, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd} \end{cases}$$
$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ odd}, i \equiv 1 \bmod 3 \text{ with } i \text{ odd} \\ 2, & \text{for } i = 2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even} \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1, i = n \text{ with } n \text{ odd} \\ 4, & \text{for } i = 7 \text{ with } n \geq 10, i = n \text{ with } n \text{ even} \\ 5, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd} \\ 6, & \text{for } i \text{ even} \end{cases}$$
$$w^i(y_i) = \begin{cases} 3, & \text{for } i = n \text{ with } n \text{ even} \\ 4, & \text{for } i = 1, i = n \text{ with } n \text{ odd}, i = 4 \text{ with } n \geq 7 \\ 5, & \text{for } i = 2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even} \\ 6, & \text{for } i \text{ odd} \end{cases}$$

From the weight calculation above we get $|w(L_n)| = 4$ for $n = 3$ and $3n \equiv 1 \bmod 3$. Previously, we obtained the lower bound of Preposition 2 and the upper bound of the weight function $4 = \chi_{lis}^i \geq \chi_{lis} = 4$, so that based on Definition 1 we can conclude that the local irregularity chromatic number inclusive of a ladder graph (L_n) is 4 for $n = 3$ and 1 mod 3 or $n \equiv \chi_{lis}^i(L_n) = 4$.

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Case 2. $n \equiv 0 \bmod 3$ and $n \equiv 2 \bmod 3$
For $n = \text{even}, n \equiv 0 \bmod 3$

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, i = n, i \text{ even}, i \equiv 1 \bmod 3 \text{ with } i \text{ even} \\ 2, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd} \end{cases}$$
$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ odd}, i = n, i \equiv 1 \bmod 3 \text{ with } i \text{ odd except } n-2 \\ 2, & \text{for } i = 2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even}, \mathrm{i} = n-2 \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1 \\ 4, & \text{for } i \equiv 1 \bmod 3 \text{ with } i \text{ even}, i = n \\ 5, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd} \\ 6, & \text{for } i \text{ even except } i = n-2 \\ 7. & \text{for } i = n-2 \end{cases}$$

$$w^i(y_i) = \begin{cases} 3, & \text{for } i = n \\ 4, & \text{for } i \equiv 1 \bmod 3 \text{ with } i \text{ odd}, i = 1 \\ 5, & \text{for } i = 2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even}, i = n-2 \\ 6, & \text{for } i \text{ odd except } i = n-3 \\ 7. & \text{for } i = n-3 \end{cases}$$

Following is the labeling of the inclusive local irregularity vertices coloring on a ladder graph:(L_n)

For $n =$ odd, $n \equiv 0 \bmod 3$

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, i = n, i \text{ even}, i \equiv 1 \bmod 3 \text{ with } i \text{ even} \\ & \text{except } i = n-2 \\ 2, & \text{for } i = n-2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd} \\ & \text{except } i = n \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ odd}, i \equiv 1 \bmod 3 \text{ with } i \text{ odd} \\ 2, & \text{for } i = 2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even} \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1, i = n \\ 4, & \text{for } i \equiv 1 \bmod 3 \text{ with } i \text{ even and } n \geq 15 \\ 5, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd}, i = n-2 \\ 6, & \text{for } i \text{ even except } n-3 \\ 7. & \text{for } i = n-3 \end{cases}$$

$$w^i(y_i) = \begin{cases} 4, & \text{for } i = 1, i = n, i \equiv 1 \bmod 3 \text{ with } i \text{ odd} \\ 5, & \text{for } i = 2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even} \\ 6, & \text{for } i \text{ odd except } n-2 \\ 7, & \text{for } i = n-2 \end{cases}$$

Following is the labeling of the inclusive local irregularity vertices coloring on a ladder graph:(L_n)

For $n =$ even, $n \equiv 2 \bmod 3$

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, i \text{ even}, i \equiv 1 \bmod 3 \text{ with } i \text{ even} \\ & \text{except } i = n-1 \\ 2, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd}, i = n-1 \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i = n, i \text{ odd}, i \equiv 1 \bmod 3 \text{ with } i \text{ odd} \\ & \text{except } i = n-2 \\ 2, & \text{for } i = 2, i = n-2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with} \\ & i \text{ even except } i = n \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1 \\ 4, & \text{for } i = n, i \equiv 1 \bmod 3 \text{ with } i \text{ even except } i = n-1 \\ 5, & \text{for } i = n-1, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd} \\ 6, & \text{for } i \text{ even with } 2 \leq i \leq n-4 \\ 7. & \text{for } i = n-2 \end{cases}$$

$$w^i(y_i) = \begin{cases} 3, & \text{for } i = n \\ 4, & \text{for } i = 1, i \equiv 1 \bmod 3 \text{ with } i \text{ odd} \\ 5, & \text{for } i = 2, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even} \\ & \text{except } i = n \\ 6, & \text{for } i \text{ odd with } i \geq 3 \end{cases}$$

Following is the labeling of the inclusive local irregularity vertices coloring on a ladder graph:(L_n)

For $n =$ odd, $n \equiv 2 \bmod 3$

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, i \text{ even}, i = n, i \equiv 1 \bmod 3 \text{ with } i \text{ even} \\ 2, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd except } i = n \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ odd}, i \equiv 1 \bmod 3 \text{ with } i \text{ odd except } i = n-1 \\ 2, & \text{for } i = 2, i = n-1, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even} \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1, i = n \\ 4, & \text{for } i \equiv 1 \bmod 3 \text{ with } i \text{ even} \\ 5, & \text{for } i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ odd except } i = n \\ 6, & \text{for } i \text{ even} \end{cases}$$

$$w^i(y_i) = \begin{cases} 4, & \text{for } i = 1, i = n, i \equiv 1 \bmod 3 \text{ with } i \text{ odd} \\ 5, & \text{for } i = 2, i = n-1, i \equiv 0 \bmod 3; 2 \bmod 3 \text{ with } i \text{ even} \\ 6, & \text{for } i \text{ odd with } 3 \leq i \leq n-4 \\ 7, & \text{for } i = n-2 \end{cases}$$

From the weight calculation above we get $|w(L_n)| = 5$ for $n \equiv 0 \bmod 3$ and $n \equiv 2 \bmod 3$. In the lower bound statement above and the upper bound that we have looked for above, we can write it $5 \leq \chi_{lis}^i \leq 5$ in such a way that we can conclude that the chromatic number of local irregularities inclusive of a ladder graph (L_n) is 5 for $\equiv 0 \bmod 3$ and $n \equiv 2$ mod or $n \equiv \chi_{lis}^i(L_n) = 5$. ■

Theorem 3. *The inclusive local disorder chromatic number on a graph Cp_n with $n \geq 3$*

$$\chi_{lis}^{i}(Cp_n) = \begin{cases} 3, & n = 3 \\ 4, & n \geq 4 \end{cases}$$

Proof. The set of vertices on a centipede (Cp_n) graph is $V(Cp_n) = \{x_i, y_i; 1 \leq i \leq n\}$. So the vertex cardinality of the centipede graph is $|V(Cp_n)| = 2n$. A centipede graph (Cp_n) has a set of edges, namely $E(Cp_n) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i y_i; 1 \leq i \leq n\}$. Thus the edge cardinality of a centipede graph (Cp_n) is $|E(Cp_n)| = 2n - 1$. There are 2 cases of inclusive local irregular chromatic numbers on a centipede graph (Cp_n), namely when case 1 is $n = 3$ and case 2 is $n \geq 4$. The following describes the two cases.

Case 1. $n = 3$
For $n = 3$

$$l(x_i) = 1, \text{for } 1 \leq i \leq 3$$
$$l(y_i) = 1, \text{for } 1 \leq i \leq 3$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1, i = 3 \\ 4, & \text{for } i = 2 \end{cases}$$
$$w^i(y_i) = 2, \text{for } 1 \leq i \leq 3$$

From the weight calculation above we get $|w(Cp_n)| = 3$ for $.n = 3$. So much $4 = \chi_{lis}^{i} \geq \chi_{lis} = 4$ so that based on Definition 1 we can conclude that the chromatic number of local irregularities inclusive of a centipede graph (Cp_n) is 3 for $n = 3$ or $\chi_{lis}^{i}(Cp_n) = 3$ ■

Case 2. $n \geq 4$
For $n =$ even, $n \geq 4$

$$l(x_i) = 1, \text{for } 1 \leq i \leq n$$
$$l(y_i) = \begin{cases} 1, & \text{for } i = 1, i \text{ even} \\ 2, & \text{for } i \text{ odd with } i \geq 3 \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1, i = n \\ 4, & \text{for } i \text{ even with } i \leq n-2 \\ 5, & \text{for } i \text{ odd with } i \geq 3 \end{cases}$$
$$w^i(y_i) = \begin{cases} 2, & \text{for } i = 1, i \text{ even} \\ 3, & \text{for } i \text{ odd with } i \geq 3 \end{cases}$$

Following is the labeling of the inclusive local irregularity vertices coloring on a centipede graph:(Cp_n)

For $n =$ odd, $n \geq 4$

$$l(x_i) = 1, \text{for } 1 \leq i \leq n$$
$$l(y_i) = \begin{cases} 1, & \text{for } i = 1, i = n, i \text{ even} \\ 2, & \text{for } i \text{ odd with } 3 \leq i \leq n-2 \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 1, i = n \\ 4, & \text{for } i \text{ even} \\ 5, & \text{for } i \text{ odd with } 3 \leq i \leq n-2 \end{cases}$$
$$w^i(y_i) = \begin{cases} 2, & \text{for } i = 1, i = n, i \text{ even} \\ 3, & \text{for } i \text{ odd with } 3 \leq i \leq n-2 \end{cases}$$

From the weight calculation above we get $|w(Cp_n)| = 4$ for $n \geq 4$. So that $4 = \chi_{lis}^i \geq \chi_{lis} = 4$ based on Definition 1 we can conclude that the chromatic number of local irregularities inclusive of a centipede graph (Cp_n) is 4 for $n \geq 4$ or $\chi_{lis}^i(Cp_n) = 4$.

■

Theorem 4. *The inclusive local disorder chromatic number on a graph H_n with $n \geq 2$*

$$\chi_{lis}^i(H_n) = \begin{cases} 4, & n = 2 \\ 5, & n \geq 3 \end{cases}$$

Proof. The set of vertices on a graph (H_n) is $V(H_n) = \{x_i, y_i; 1 \leq i \leq 2n\} \cup \{z_j; 1 \leq j \leq 2n\}$. So the cardinality of the vertices of the graph (H_n) is $|V(H_n)| = 6n$. A graph (H_n) has a set of edges, namely $E(H_n) = \{x_i x_{i+1}, y_i y_{i+1}; 1 \leq i \leq 2n-1\} \cup \{z_j z_{j+1}; 1 \leq j \leq n\} \cup \{x_i z_j; 1 \leq i = j \leq 2n\} \cup \{y_i z_j; 1 \leq i = j \leq 2n\}$. Thus the edge cardinality of the graph (H_n) is $|E(H_n)| = 9n - 2$. There are 2 cases of inclusive local chromatic irregularity on a graph (H_n), namely when case 1 is $n = 2$ and case 2 is $n \geq 3$. The following describes the two cases.

Case 1. $n = 2$
For $n = 2$

$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ even} \\ 2, & \text{for } i \text{ odd} \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ even} \\ 2, & \text{for } i \text{ odd} \end{cases}$$
$$l(z_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq 2n-2, i = 2n \\ 3, & \text{for } i = 2n-1 \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

$$w^i(x_i) = \begin{cases} 4, & \text{for } i = 1, i = 2n \\ 6, & \text{for } i = 2 \\ 7, & \text{for } i = 3 \end{cases}$$

$$w^i(y_i) = \begin{cases} 4, & \text{for } i = 1, i = 2n \\ 6, & \text{for } i = 2 \\ 7, & \text{for } i = 3 \end{cases}$$

$$w^i(z_i) = \begin{cases} 4, & \text{for } i = 2 \\ 6, & \text{for } i = 1, i = 2n \\ 8, & \text{for } i = 3 \end{cases}$$

From the weight calculation above we get $|w(H_n)| = 4$ for $n = 2$. Previously, we obtained the lower bound of Preposition 3 and the upper bound of the weight function $4 = \chi_{lis}^i \geq \chi_{lis} = 4$, so that based on Definition 1 we can conclude that the local irregularity chromatic number inclusive of a graph (H_n) is 4 for $n = 2$ or $\chi_{lis}^i(Cp_n) = 4$.

■

Case 2. $n \geq 3$

For $n \geq 3$

$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ even} \\ 2, & \text{for } i \text{ odd} \end{cases}$$

$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ even} \\ 2, & \text{for } i \text{ odd} \end{cases}$$

$$l(z_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq 2n-2, i = 2n \\ 3, & \text{for } i = 2n-1 \end{cases}$$

So that the vertices weight obtained from the sum of the neighboring labels with the label itself is as follows.

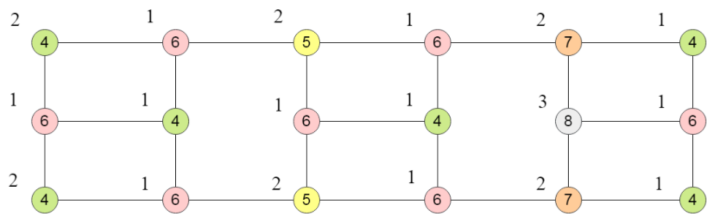
$$w^i(x_i) = \begin{cases} 4, & \text{for } i = 1, i = 2n \\ 5, & \text{for } i \text{ odd with } 3 \leq i \leq 2n-3 \\ 6, & \text{for } i \text{ even with } 1 \leq i \leq 2n-2 \\ 7, & \text{for } i = 2n-1 \end{cases}$$

$$w^i(y_i) = \begin{cases} 4, & \text{for } i = 1, i = 2n \\ 5, & \text{for } i \text{ odd with } 3 \leq i \leq 2n-3 \\ 6, & \text{for } i \text{ even with } 1 \leq i \leq 2n-2 \\ 7, & \text{for } i = 2n-1 \end{cases}$$

$$w^i(z_i) = \begin{cases} 4, & \text{for } i \text{ even with } 2 \leq i \leq 2n-2 \\ 6, & \text{for } i \text{ odd with } 1 \leq i \leq 2n-3, i = 2n \\ 8, & \text{for } i = 2n-1 \end{cases}$$

From the weight calculation above we get $|w(H_n)| = 5$ for $n \geq 3$. In the lower bound statement above and the upper bound that we have looked for above we can write it $5 \leq \chi_{lis}^i \leq 5$ in such a way that we can conclude that the chromatic number of the local irregularity inclusive of the graph (H_n) is 5 for $n \geq 3$ or $\chi_{lis}^i(H_n) = 5$.

■



Picture 1. Inclusive Local Irregularity Vertex Coloring, $\chi^{i}_{lis}(H_3) = 5$

CONCLUSIONS

Based on the research results from the previous chapter, four new theorems have been obtained regarding the topic of spot coloring of inclusive local irregularities in the grid graph family, namely graph $G_{m,n}$, graph L_n, graph Cp_n, and graph H_n. The four graphs get the number of each chromatic number. Following are the chromatic numbers of the four graphs.

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