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On Properties of Five-dimensional Nonstandard Filiform Lie Algebra

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ABSTRACT

This paper relates the representation of all basis elements of five-dimensional nonstandard Filiform Lie algebra in matrix forms to its Lie bracket computations and its Lie group representation which is constructed by using these matrix forms. The basis elements of its Lie algebra in matrix forms are considered easier to determine the representation of five-dimensional Filiform nonstandard Lie group using the orbit method initiated by Kirillov. The purpose of this paper is to determine the representations of all basis elements of five-dimensional nonstandard Filiform Lie algebras in 5×5 real matrix forms. The computation algorithms of this representation were constructed by following Ceballos, Núñez, and Tenorio's work. The study shows that the basis elements of the five-dimensional nonstandard Filiform Lie algebra can be represented in five 5×5 real matrix forms. This implies that each element of five-dimensional Filiform Lie algebra can be also representated in this matrix forms. The result is in line with the Lie bracket calculations and is significant in representing its Lie group. This research is significant in light of represntation theory of a Lie algebra and its Lie group. Better construction of Lie group representations in the term of matrices can be considered as an easier model in the orbit method. For further research, it can be extended to case of six-dimensional or higher dimensional Filiform Lie algebras both standard and nonstandard types.

Keywords: Ceballos' Algorithm; Filiform Lie algebra; Matrix representation; Lie bracket.

INTRODUCTION

They are saveral type of Lie algebras such as Frobenius Lie algebras, contact Lie algebras, nilpotent Lie algebras, solvable Lie algebras, and Filiform Lie algebras. The notion of a Filiform Lie algebra corresponds to its *nilindex*. Roughly speaking, a nilindex of a Lie algebras $\mathfrak{g}$ is the integer k such that $\mathfrak{g}^k = \{0\}$ but $\mathfrak{g}^{k-1} \neq \{0\}$. A Lie algebra $\mathfrak{g}$ is called Filiform if its nilindex is equal to $\dim \mathfrak{g} - 1$. The familiar example of Filiform Lie algebra is 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ with its nilindex equals 2. The best the best model to understand a representation theory using the orbit method. All elements of $\mathfrak{h}_3$ are represented in matrix forms. This makes easier computations in representations of its Heisenberg Lie group. The popular representation of its Lie group is known as Schrödinger representation. Initiated from this results, we extend to case of

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five-dimensional Filiform Lie algebras. The obtained matrix forms as realization of elements basis will be applied for representation of its Lie group. That is why this research is important to represent all elements of basis of five-dimensional Filiform Lie algebras in matrix forms.

Some researchers classified the Filiform Lie algebra up to the dimension 7([1],[2]). The 3-dimensional Filiform Lie algebras are known as the Heisenberg Lie algebra $\mathfrak{h}_3$ as mentioned above and it has been widely studied as the best model or the best example of representation theory of Lie algebras and their Lie groups, especially the Schrödinger representation ([3]–[8]). Better computations are realized all elements of basis in matrix forms. In the case of four-dimensional standard Filiform Lie algebras, the representation of their Lie groups have been obtained. It is also proved that its representation of 4-dimensional Filiform Lie group is a square-integrable representation. Therefore, it is obtained the Duflo-Moore operator [9]. Furthermore, the unitary irreducible representation of Lie groups of the five-dimensional standard Filiform Lie algebra has also been obtained by means of the orbit method [10]. We emphasize that the classification of Filiform Lie algebra up to dimension six has been classified by [2] and classification of Filiform Lie algebra up to dimension seven has been obtained by ([1], [11], [12]). To the best of author's knowledge, there is none of researcher has yet classified the Filiform Lie algebra with dimension more than seven.

The study of Filiform Lie algebra is very important since it corresponds to another type of Lie algebras. For example, Filiform Lie algebras correspond to nilpotent Lie algebras [13]. Some properties of Filiform Lie algebras are still open to be explored. For example, there is notion of invariant of Filiform Lie algebra [14]. Some Filiform Lie algebras can be also extended to Frobenius Lie algebras by computing their split torus [15].

Different from previous results, this study aims to realize the basis elements of five-dimensional Filiform Lie algebra in real matrices 5×5 forms. We sharp the result in [2] with more direct computations by following the algorithm's Ceballos in our computations. The result is the same as the calculation obtained by Ceballos [2]. Moreover, the obtained matrix forms will be useful when one computes its five-dimensional Filiform Lie group representation via orbit method.

METHODS

We apply literature study for the research method, especially the study of Filiform Lie algebra and matrix representation. For a given the nonstandard filiform Lie algebra, we follow the algorithm's Cheballos to compute representations of its all basis elements. We also proved that the Lie algebra with certain Lie brackets in this paper is nilpotent and nonstandard Filiform. Before going into the discussion, we introduce the theoretical foundations used in this study as follows. We introduce some notions of Lie algebras, nilpotent Lie algebras, and Filiform Lie algebras.

Definition 1 [1] *Let $\mathfrak{g}$ be a vector space over a field $\mathbb{F}$. The vector space $\mathfrak{g}$ is called a Lie algebra if $\mathfrak{g}$ equip the bracket product $[\cdot,\cdot]$ and it satisfies the following three properties*

1. *Bilinearity:* $[u+v,w]=[u,w]+[v,w]$; $[u,v+w]=[u,v]+[u,w]$; $[\alpha u,\beta v]=\alpha\beta[u,v]$, $\forall\alpha,\beta\in\mathbb{F}$, $\forall u,v,w\in\mathfrak{g}$.
2. *Anti-commutative:* $[v,u]=-[u,v]$, $\forall u,v\in\mathfrak{g}$.
3. *Jacobi Identity:* $[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$, $\forall u,v,w\in\mathfrak{g}$.

Definition 2 [1] *Let $\mathfrak{g}$ be an n-dimensional Lie algebra. The lower central series on $\mathfrak{g}$ is* *defined as follows*

$$\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}, \mathcal{C}^2(\mathfrak{g}) = [\mathcal{C}^1(\mathfrak{g}), \mathfrak{g}], \mathcal{C}^3(\mathfrak{g}) = [\mathcal{C}^2(\mathfrak{g}), \mathfrak{g}], \dots, \mathcal{C}^k(g) = [\mathcal{C}^{k-1}(\mathfrak{g}), \mathfrak{g}], \dots \quad (1)$$

and $\mathfrak{g}$ *is called nilpotent if there exist* $k \in \mathbb{N}$ *such that* $\mathcal{C}^k(\mathfrak{g}) = \{0\}$.

Definition 3 [2] *An* n*-dimensional nilpotent Lie algebra* $\mathfrak{g}$ *is called Filiform if its lower central series* (1) *satisfy the following*

$$dim\, \mathcal{C}^1(\mathfrak{g}) = n, dim\, \mathcal{C}^2(\mathfrak{g}) = n-2, dim\, \mathcal{C}^3(\mathfrak{g}) = n-3, \dots, dim\, \mathcal{C}^n(\mathfrak{g}) = 0 \quad (2)$$

Definition 4 [2] *A basis* $\{X_i\}_{i=1}^n$ *of an* n*-dimensional Filiform Lie algebra* $\mathfrak{g}$ *is called adapted basis if it verifies the following relations*

$$\begin{aligned} &[X_1, X_h] = X_{h-1}, && \text{for } 3 \le h \le n; \\ &[X_2, X_h] = 0, && \text{for } 1 \le h \le n; \\ &[X_3, X_h] = 0, && \text{for } 2 \le h \le n. \end{aligned}$$

Definition 5 [2] *A Filiform Lie algebra called standard Filiform Lie algebra if the bracket product only* $[X_1, X_h] = X_{h-1}$, *for* $3 \le h \le n$. *Otherwise, it is called nonstandard Filiform Lie algebra.*

RESULTS AND DISCUSSION

In this section, let $\mathfrak{g}$ be a five-dimensional Lie algebra and let $\mathfrak{B} = \{X_1, X_2, X_3, X_4, X_5\}$ be a basis for $\mathfrak{g}$. The Lie bracket of $\mathfrak{g}$ is defined as usual by multiplication matrices $[X_i, X_j] = X_iX_j - X_jX_i$, $\forall X_i, X_j \in \mathfrak{g}$. In this case, we have the nonzero Lie bracket of $\mathfrak{g}$ written as the following forms

$$\begin{aligned} [X_1, X_3] &= X_2, \\ [X_1, X_4] &= X_3, \quad (3) \\ [X_1, X_5] &= X_4, \\ [X_4, X_5] &= X_2. \quad (4) \end{aligned}$$

Property 1. *Let* $\mathfrak{g} = \text{span}\{X_1, X_2, X_3, X_4, X_5\}$ *be a five-dimensional Lie algebra whose brackets were given by the equations* (3) *and* (4). *Then* $\mathfrak{g}$ *is a five-dimensional nilpotent Lie algebra.*

Proof. To prove this statement, we apply the lower central series of $\mathfrak{g}$ as given in equation (1). To prove the nilpotency of $\mathfrak{g}$, we shall show there exists $k_0 \in \mathbb{N}$ such that $\mathcal{C}^{k_0}(\mathfrak{g}) = \{\mathbf{0}\}$. By direct computation we have

For $k = 1$, then $\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}$.
For $k = 2$, then $\mathcal{C}^2(\mathfrak{g}) = [\,\mathcal{C}^1(\mathfrak{g}), \mathfrak{g}] = \text{span}\{[u, v] | u \in \mathfrak{g}, v \in \mathfrak{g}\} = \text{span}\{X_2, X_3, X_4\}$.
For $k = 3$, then $\mathcal{C}^3(\mathfrak{g}) = [\,\mathcal{C}^2(\mathfrak{g}), \mathfrak{g}] = \text{span}\{[u, v] | u \in \mathfrak{g}^2, v \in \mathfrak{g}\} = \text{span}\{X_2, X_3\}$.
For $k = 4$, then $\mathcal{C}^4(\mathfrak{g}) = [\,\mathcal{C}^3(\mathfrak{g}), \mathfrak{g}] = \text{span}\{[u, v] | u \in \mathfrak{g}^3, v \in \mathfrak{g}\} = \text{span}\{X_2\}$.

For $k = 5$, then $\mathcal{C}^5(\mathfrak{g}) = [\,\mathcal{C}^4(\mathfrak{g}), \mathfrak{g}] = \text{span}\{[u, v] | u \in \mathfrak{g}^4, v \in \mathfrak{g}\} = \text{span}\{\mathbf{0}\} = \{\mathbf{0}\}$. Thus, we find $k_0 = 5 \in \mathbb{N}$ such that $\mathcal{C}^5(\mathfrak{g}) = \{\mathbf{0}\}$. In other words, $\mathfrak{g}$ is five-dimensional nilpotent Lie algebra.

■

Moreover, the five-dimensional Lie algebra defined by the Lie brackets in equation (1) is a five-dimensional nonstandard Filiform Lie algebra. Formally, we have the following property.

Property 2. *Let* $\mathfrak{g} = \text{span}\{X_1, X_2, X_3, X_4, X_5\}$ *be a five-dimensional nilpotent Lie algebra whose brackets were given by the equation* (3) *and* (4). *Then* $\mathfrak{g}$ *is a five-dimensional nonstandard Filiform Lie algebra.*

Proof. To prove the above statement, we cite the Definition 3 to show that the dimension of nilpotent Lie algebra in their lower central series satisfy the equation given in equation (2). It follows from the Property 1 that:

- For $k = 1$, $\dim \mathcal{C}^1(\mathfrak{g}) = \dim \mathfrak{g} = 5$.
- For $k = 2$, $\dim \mathcal{C}^2(\mathfrak{g}) = \dim g^2 = \dim \text{span}\{X_2, X_3, X_4\} = 5 - 2 = 3$.
- For $k = 3$, $\dim \mathcal{C}^3(\mathfrak{g}) = \dim g^3 = \dim \text{span}\{X_2, X_3\} = 5 - 3 = 2$.
- For $k = 4$, $\dim \mathcal{C}^4(\mathfrak{g}) = \dim g^4 = \dim \text{span}\{X_2\} = 5 - 4 = 1$.
- For $k = 5$, $\dim \mathcal{C}^5(\mathfrak{g}) = \dim g^5 = \dim \text{span}\{\mathbf{0}\} = 5 - 5 = 0$.

Therefore, $\mathfrak{g}$ is five-dimensional Filiform Lie algebra. But since we have $[X_4, X_5] = X_2$, then $\mathfrak{g}$ is five-dimensional nonstandard Filiform Lie algebra.

■

From now we denote the five-dimensional nonstandard Filiform Lie algebra by $\mathfrak{g}_{fil}$. Furthermore, we shall discuss representation of basis elements $\mathfrak{B} = \{X_1, X_2, X_3, X_4, X_5\}$ of $\mathfrak{g}_{fil}$ realized in the form of 5×5 real matrices. We follow the result in [2]. Although the result was obtained by [2], but we provide clearer proof in our own way. We compute directly the realization of elements of $\mathfrak{B}$ in the form of 5×5 real matrices. Formally we have the result written in Proposition 1.

Definition 3 [2]. *Let* $\mathfrak{g}_5$ *be the set of all* 5×5 *strictly upper triangular matrices given as follow*:

$$\mathfrak{g}_5 = Span\left\{ g_5(e_{r,s}) = \begin{bmatrix} 0 & e_{1,2} & e_{1,3} & e_{1,4} & e_{1,5} \\ 0 & 0 & e_{2,3} & e_{2,4} & e_{2,5} \\ 0 & 0 & 0 & e_{3,4} & e_{3,5} \\ 0 & 0 & 0 & 0 & e_{4,5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, e_{r,s} \in \mathbb{R}, 1 \le r < s \le 5 \right\}$$

and $\mathcal{B}_5$ *be a basis of* $\mathfrak{g}_5$ *given as follow*:

$$\mathcal{B}_5 = \left\{ E_{i,j} = g_5(e_{r,s}) \in \mathfrak{g}_5 \middle| e_{i,j} = \begin{cases} 1, (i,j) = (r,s) \\ 0, (i,j) \ne (r,s) \end{cases} \right\}.$$

Thus, basis elements of $\mathfrak{B} = \{X_1, X_2, X_3, X_4, X_5\}$ of $\mathfrak{g}_{fil}$ represents as linear combination of matrices $E_{i,j}$ in $\mathcal{B}_5$. We construct matrix representation of elements basis of five-dimensional nonstandard Filiform Lie algebra $\mathfrak{g}_{fil}$ using algorithm introduced in [2] and the computing assisted by the program Maple 18. The algorithm shown as follows:

1. Define vectors basis of $\mathfrak{g}_{fil}$ as linear combination of vectors in $\mathcal{B}_5$
$$X_h = \sum_{1 \le i \le j \le 5} \lambda_{i,j}^h E_{i,j}\,, \quad \text{untuk } 1 \le h \le 5. \qquad (5)$$
2. Next, impose the bracket given in (3), then resulting following relations.
3. After solving the result system of equations in step 2, solve the new system of equations obtained when imposing the rest nonzero bracket in the law of $\mathfrak{g}_{fil}$, given by (3) and (4).

Next, for the matrices representations of basis elements of $\mathfrak{g}_{fil}$, the main result of our discussion in this section, we will prove the Proposition 1 as follows.

Proposition 1[2]. *Let $\mathfrak{g}_{fil}$ be the five-dimensional nonstandard Filiform Lie algebra with basis $\mathfrak{B} = \{X_1, X_2, X_3, X_4, X_5\}$ and nonzero bracket of $\mathfrak{g}_{fil}$ be given by* (1) *and* (2). *Then the representation of all basis elements of $\mathfrak{g}_{fil}$ is realized in the following* 5×5 *real matrix forms.*

$$\left\{ X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Proof. We give more computations to prove this propostion. However, we follow the algorithm of Chebalos [2] to compute this representation directly**.** To represent basis elements of $\mathfrak{g}_{fil}$, we use the algorithm given above as follows. Let $\mathfrak{g}_5$ five-dimensional Filiform Lie algebra, with $\mathcal{B}_5 = \left\{ E_{i,j} = g_5(e_{r,s}) \in \mathfrak{g}_5 \,\middle|\, e_{i,j} = \begin{cases} 1, (i,j) = (r,s) \\ 0, (i,j) \ne (r,s) \end{cases} \right\}$ as basis of $\mathfrak{g}_5$, and $\mathfrak{g}_{fil} = \text{span}\{X_1, X_2, X_3, X_4, X_5\}$ with nonzero bracket given in (3) and (4). The lower central series of $\mathfrak{g}_5$ and $\mathfrak{g}_{fil}$ are compatible shown as follows.

$$\mathcal{C}^1(\mathfrak{g}_{fil}) = \langle X_1, X_2, X_3, X_4, X_5 \rangle \subseteq \mathcal{C}^1(\mathfrak{g}_5) = \langle e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{2,3}, e_{2,4}, e_{2,5}, e_{3,4}, e_{3,5}, e_{4,5} \rangle$$
$$\mathcal{C}^2(\mathfrak{g}_{fil}) = \langle X_2, X_3, X_4 \rangle \subseteq \mathcal{C}^2(\mathfrak{g}_5) = \langle e_{1,3}, e_{1,4}, e_{1,5}, e_{2,4}, e_{2,5}, e_{3,5} \rangle$$
$$\mathcal{C}^3(\mathfrak{g}_{fil}) = \langle X_2, X_3 \rangle \subseteq \mathcal{C}^3(\mathfrak{g}_5) = \langle e_{1,4}, e_{1,5}, e_{2,5} \rangle$$
$$\mathcal{C}^4(\mathfrak{g}_{fil}) = \langle X_2 \rangle \subseteq \mathcal{C}^4(\mathfrak{g}_5) = \langle e_{1,5} \rangle$$
$$\mathcal{C}^5(\mathfrak{g}_{fil}) = \langle \mathbf{0} \rangle \subseteq \mathcal{C}^5(\mathfrak{g}_5) = \langle \mathbf{0} \rangle$$

When applying step 1 and step 2, the following vectors obtained.

$$\begin{aligned} X_1 &= \lambda_{1,2}^1 e_{1,2} + \lambda_{1,3}^1 e_{1,3} + \cdots + \lambda_{3,5}^1 e_{3,5} + \lambda_{4,5}^1 e_{4,5} \\ X_2 &= \lambda_{1,5}^2 e_{1,5} \\ X_3 &= \lambda_{1,4}^3 e_{1,4} + \lambda_{1,5}^3 e_{1,5} + \lambda_{2,5}^3 e_{2,5} \\ X_4 &= \lambda_{1,3}^4 e_{1,3} + \lambda_{1,4}^4 e_{1,4} + \lambda_{1,5}^4 e_{1,5} + \lambda_{2,4}^4 e_{2,4} + \lambda_{2,5}^4 e_{2,5} + \lambda_{3,5}^4 e_{3,5} \\ X_5 &= \lambda_{1,2}^5 e_{1,2} + \lambda_{1,3}^5 e_{1,3} + \cdots + \lambda_{3,5}^5 e_{3,5} + \lambda_{4,5}^5 e_{4,5} \end{aligned} \qquad (6)$$

under the following relations.

$$\begin{aligned}&\{\lambda^1_{1,2}=\lambda^1_{1,2}, \lambda^1_{1,3}=\lambda^1_{1,3}, \lambda^1_{1,4}=\lambda^1_{1,4}, \lambda^1_{1,5}=\lambda^1_{1,5}, \lambda^1_{2,3}=\lambda^1_{2,3}, \lambda^1_{2,4}=\\&\lambda^1_{2,4}, \lambda^1_{2,5}=\lambda^1_{2,5}, \lambda^1_{3,4}=\lambda^1_{3,4}, \lambda^1_{3,5}=\lambda^1_{3,5}, \lambda^1_{4,5}=\lambda^1_{4,5}, \lambda^2_{1,5}=\\&\lambda^1_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}\lambda^5_{4,5}-3\lambda^1_{1,2}\lambda^1_{2,3}\lambda^5_{3,4}\lambda^1_{4,5}+3\lambda^1_{1,2}\lambda^5_{2,3}\lambda^1_{3,4}\lambda^1_{4,5}-\\&\lambda^5_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}\lambda^1_{4,5}, \lambda^3_{1,4}=\lambda^1_{1,2}\lambda^1_{2,3}\lambda^5_{3,4}-2\lambda^1_{1,2}\lambda^5_{2,3}\lambda^1_{3,4}+\lambda^5_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}, \lambda^3_{1,5}=\\&\lambda^1_{1,2}\lambda^1_{2,3}\lambda^5_{3,5}+\lambda^1_{1,2}\lambda^1_{2,4}\lambda^5_{4,5}+\lambda^1_{1,3}\lambda^1_{3,4}\lambda^5_{4,5}-2\big(\lambda^1_{1,2}\lambda^5_{2,3}\lambda^1_{3,5}+\lambda^1_{1,2}\lambda^5_{2,4}\lambda^1_{4,5}+\\&\lambda^1_{1,3}\lambda^5_{3,4}\lambda^1_{4,5}\big)+\lambda^5_{1,2}\lambda^1_{2,3}\lambda^1_{3,5}+\lambda^5_{1,2}\lambda^1_{2,4}\lambda^1_{4,5}+\lambda^5_{1,3}\lambda^1_{3,4}\lambda^1_{4,5}, \lambda^3_{2,5}=\\&\lambda^1_{2,3}\lambda^1_{3,4}\lambda^5_{4,5}-2\lambda^1_{2,3}\lambda^5_{3,4}\lambda^1_{4,5}+\lambda^5_{2,3}\lambda^1_{3,4}\lambda^1_{4,5}, \lambda^4_{1,3}=\lambda^1_{1,2}\lambda^5_{2,3}-\\&\lambda^5_{1,2}\lambda^1_{2,3}, \lambda^4_{1,4}=\lambda^1_{1,2}\lambda^5_{2,4}+\lambda^1_{1,3}\lambda^5_{3,4}-\lambda^5_{1,2}\lambda^1_{2,4}-\lambda^5_{1,3}\lambda^1_{3,4}, \lambda^4_{1,5}=\lambda^1_{1,2}\lambda^5_{2,5}+\\&\lambda^1_{1,3}\lambda^5_{3,5}+\lambda^1_{1,4}\lambda^5_{4,5}-\lambda^5_{1,2}\lambda^1_{2,5}-\lambda^5_{1,3}\lambda^1_{3,5}-\lambda^5_{1,4}\lambda^1_{4,5}, \lambda^4_{2,4}=\lambda^1_{2,3}\lambda^5_{3,4}-\\&\lambda^5_{2,3}\lambda^1_{3,4}, \lambda^4_{2,5}=\lambda^1_{2,3}\lambda^5_{3,5}+\lambda^1_{2,4}\lambda^5_{4,5}-\lambda^5_{2,3}\lambda^1_{3,5}-\lambda^1_{4,5}\lambda^5_{4,5}, \lambda^4_{3,5}=\lambda^1_{3,4}\lambda^5_{4,5}-\\&\lambda^5_{3,4}\lambda^1_{4,5}, \lambda^5_{1,2}=\lambda^5_{1,2}, \lambda^5_{1,3}=\lambda^5_{1,3}, \lambda^5_{1,4}=\lambda^5_{1,4}, \lambda^5_{2,4}=\lambda^5_{2,4}, \lambda^5_{2,5}=\lambda^5_{2,5}, \lambda^5_{3,4}=\\&\lambda^5_{3,4}, \lambda^5_{3,5}=\lambda^5_{3,5}, \lambda^5_{4,5}=\lambda^5_{4,5}\}\end{aligned} \tag{7}$$

Thus, we get the new vectors as follows:

$$\begin{aligned}X_1&=\lambda^1_{1,2}e_{1,2}+\lambda^1_{1,3}e_{1,3}+\cdots+\lambda^1_{3,5}e_{3,5}+\lambda^1_{4,5}e_{4,5}\\X_2&=\lambda^2_{1,5}e_{1,5}\\X_3&=\lambda^3_{1,4}e_{1,4}+\lambda^3_{1,5}e_{1,5}+\lambda^3_{2,5}e_{2,5}\\X_4&=\lambda^4_{1,3}e_{1,3}+\lambda^4_{1,4}e_{1,4}+\lambda^4_{1,5}e_{1,5}+\lambda^4_{2,5}e_{2,5}+\lambda^4_{3,5}e_{3,5}\\X_5&=\lambda^5_{1,2}e_{1,2}+\lambda^5_{1,3}e_{1,3}+\lambda^5_{1,4}e_{1,4}+\lambda^5_{1,5}e_{1,5}+\lambda^5_{2,4}e_{2,4}+\lambda^5_{2,5}e_{2,5}\\&\qquad+\lambda^5_{3,5}e_{3,5}+\lambda^5_{4,5}e_{4,5}\end{aligned} \tag{8}$$

Next, for step 3 we impose the nonzero bracket given in (3) and (4) to vectors in (8), then solve the new obtained system of equations with Maple 18, resulting many sets of solutions. We choose one set of solution, shown as follows.

$$\begin{aligned}&\Big\{\lambda^1_{1,2}=\lambda^1_{1,2}, \lambda^1_{1,3}=\lambda^1_{1,3}, \lambda^1_{1,4}=\lambda^1_{1,4}, \lambda^1_{1,5}=\lambda^1_{1,5}, \lambda^1_{2,3}=\lambda^1_{2,3}, \lambda^1_{2,4}=\\&\lambda^1_{2,4}, \lambda^1_{2,5}=\lambda^1_{2,5}, \lambda^1_{3,4}=\lambda^1_{3,4}, \lambda^1_{3,5}=\lambda^1_{3,5}, \lambda^1_{4,5}=\lambda^1_{4,5}, \lambda^2_{1,5}=\\&\lambda^1_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}\lambda^5_{4,5}, \lambda^3_{1,4}=0, \lambda^3_{1,5}=\lambda^1_{1,2}\lambda^1_{2,3}\lambda^5_{3,5}+\lambda^1_{1,2}\lambda^1_{2,4}\lambda^5_{4,5}-\\&\tfrac{3}{2}\lambda^1_{4,5}\lambda^1_{1,2}\lambda^5_{2,4}+\lambda^1_{1,3}\lambda^1_{3,4}\lambda^5_{4,5}-\tfrac{1}{2}\lambda^1_{4,5}\lambda^1_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}, \lambda^3_{2,5}=\lambda^1_{2,3}\lambda^1_{3,4}\lambda^5_{4,5}, \lambda^4_{1,3}=\\&0, \lambda^4_{1,4}=\tfrac{1}{2}\lambda^1_{1,2}\lambda^5_{2,4}+\tfrac{1}{2}\lambda^1_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}, \lambda^4_{1,5}=\Big(\tfrac{1}{2\lambda^1_{3,4}}\lambda^1_{1,2}\lambda^1_{2,3}\lambda^1_{3,4}\lambda^1_{3,5}+\\&2\lambda^1_{1,2}\lambda^1_{3,4}\lambda^5_{2,5}-\lambda^1_{1,2}\lambda^1_{3,5}\lambda^5_{2,4}+2\lambda^1_{1,3}\lambda^1_{3,4}\lambda^5_{3,5}+2\lambda^1_{1,4}\lambda^1_{3,4}\lambda^5_{4,5}-\\&2\lambda^1_{3,4}\lambda^1_{4,5}\lambda^5_{1,4}\Big), \lambda^4_{2,5}=\lambda^1_{2,3}\lambda^5_{3,5}+\lambda^1_{2,4}\lambda^5_{4,5}-\lambda^1_{4,5}\lambda^5_{4,5}, \lambda^4_{3,5}=\lambda^1_{3,4}\lambda^5_{4,5}, \lambda^5_{1,2}=\\&0, \lambda^5_{1,3}=-\frac{1}{2}\frac{\lambda^1_{1,2}\left(\lambda^1_{2,3}\lambda^1_{3,4}-\lambda^5_{2,4}\right)}{\lambda^3_{1,4}}, \lambda^5_{1,4}=\lambda^5_{1,4}, \lambda^5_{2,4}=\lambda^5_{2,4}, \lambda^5_{2,5}=\lambda^5_{2,5}, \lambda^5_{3,4}=\\&0, \lambda^5_{3,5}=\lambda^5_{3,5}, \lambda^5_{4,5}=\lambda^5_{4,5}\Big\}\end{aligned} \tag{9}$$

Furthermore, we choose the values of λ as follows,
$\lambda^1_{1,2}=\lambda^1_{2,3}=\lambda^1_{3,4}=\lambda^5_{2,4}=\lambda^5_{4,5}=1$ dan $\lambda^1_{1,3}=\lambda^1_{1,4}=\lambda^1_{1,5}=\lambda^1_{2,4}=\lambda^1_{2,5}=\lambda^1_{3,5}=\lambda^1_{4,5}=\lambda^5_{1,4}=\lambda^5_{1,5}=\lambda^5_{2,5}=\lambda^5_{3,5}=0$.

Moreover we substitute the values of λ to the equation (8) such that resulting X_i matrices as follows :

$$\left\{X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

as desired.

■

Thus, matrices representations of basis elements of five-dimensional nonstandard Filiform Lie algebras $\mathfrak{g}_{fil}$ can be written as linear combination given by the following matrix :

$$X = \begin{pmatrix} 0 & \alpha_1 & 0 & \alpha_4 & \alpha_2 \\ 0 & 0 & \alpha_1 & \alpha_5 & \alpha_3 \\ 0 & 0 & 0 & \alpha_1 & \alpha_4 \\ 0 & 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{fil}, \text{ with } \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{R}.$$

As discussion, for future research there is interesting to investigate another property of a Filiform Lie algebra. Specially for case even dimensional Filiform Lie algebra, one can study the notion of symplectic structure and non-degenerate ([16], [17]). In case, a Filiform Lie algebra can be extended to a Frobenius Lie algebra, one can observe the notion of principal element ([18]-[20]).

CONCLUSIONS

It has been proven that the five-dimensional Filiform Lie algebras is nilpotent. Moreover, we obtain the representation of basis elements $\mathfrak{B} = \{X_1, X_2, X_3, X_4, X_5\}$ of the five-dimensional nonstandard Filiform Lie algebra $\mathfrak{g}_{fil}$ in the form of 5×5 real matrices which is shown as follows :

$$\left\{X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right\}.$$

As discussion, other researchers can study more about $\mathfrak{g}_{fil}$ five-dimensional nonstandard Filiform Lie algebra especially in case of unitary irreducible representations of its Lie group. Namely, the obtained five real matrices can be applied to find the unitary irreducible representation of the Lie group of $\mathfrak{g}_{fil}$. The other cases, this research can be extended to a higher dimension of Filiform Lie algebras and their constructions of its unitary irreducible representations of Lie groups.

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