



Category of Discrete Dynamical System

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ABSTRACT

Category theory is a very universal theory of mathematical concepts. Category theory is a branch of (meta)mathematics that was invented as a formal language for comparing mathematical constructions. This category theory can be applied to a dynamical system. The dynamical system used in this research is a discrete dynamical system represented as a directed graph with vertices in the graph called states. This discrete dynamical system has a certain height that can be seen on the dynamic map where the number of states at each height is called a profile. Research in this category field is carried out by many other researchers but no one has explained more specifically the proof. Therefore, this research will prove the category of the set of discrete dynamical systems that have the same profile. The method used in this research is a literature review that is focused on proving a category. The results of this study show that the set of discrete dynamical systems with the same profile is a category with its morphism is an evolution in discrete dynamical systems. This research is needed to find out about the properties and algebraic structure of the set of discrete dynamical systems.

Keywords: discrete dynamical system; category; graph; function

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INTRODUCTION

A dynamical systems were first introduced by Henri Poincare by publishing two monographs, namely "New Methods of Celestial Mechanics" in 1892-1899 and "Lectures on Celestial Mechanics" in 1905-1910. In the late 20th century, Ali H. Nayfeh introduced the application of nonlinear dynamics to mechanical and engineering systems. Systems that change over time as a result of the repeated application of underlying dynamical laws are called dynamical systems [1]. The transition rule explains how a state might change from its current state or from a previous one. The two types of dynamical systems that make up dynamical systems are discrete and continuous. A discrete dynamical system is the set of discrete variables and the relations defined by these variables. Meanwhile, the continuous dynamical system is a continuous system characterized by a continuous set of variables, a continuous set of functions, and derivatives of variables and functions. [2]. Dynamical systems are very broad in their definition. There are also various forms of dynamical systems. Simple examples of the application of discrete dynamical systems are

the logistic mapping model, the Beverton-Holt model, and logistic mapping with the Allee effect. While the application of continuous dynamical systems can be seen in the logistic equation. The solutions to a dynamical system can be represented on a map called a dynamical map.

The algebraic structure of dynamical systems can be observed. Category theory is one of the most prevalent algebraic structures. Samuel Eilenberg and Saunders Mac Lane introduced category theory, a broad theory of mathematical structures and interactions, in their seminal work on algebraic topology in the middle of the 20th century. Today, category theory is employed in several branches of computer science as well as mathematics. Categories are formed by objects and morphisms that satisfy the associative rules and the identity rules [3].

Research on the category of discrete dynamical systems has been conducted by several researchers. Research from Takeo Uramoto [4] on the extension of computational discrete dynamical systems is such that can extend Christol's theorem which later the arithmetic analogy of Christol's theorem can be proved based on the category of semigalois. The research of Dikran and Anna [5] on the category of entropy semigroups on discrete dynamical systems and functors between two categories. The research of Madalina Roxana [6] on the isomorphism of the category of all subgroupoids of the trivial groupoid and shows a simple model of the discrete dynamical system can be seen as a subgroupoid, trivial groupoid with group \mathbb{Z} . The research of Jean-Pierre [7] on DEVS (Discrete Event System), uses category theory to define semantic mappings, by defining DEVS categories, trajectory categories, and mappings between them. The research of David [8] on category theory as a mathematical modeling framework, which explained that discrete dynamical systems are the same as functors from the monoid category of natural numbers to the category of sets. The research of Brendan Fong et al [9] on the system class of discrete dynamical systems LTI (linear time-invariant) as a category of matrix correlation and used to present the theory of symmetric monoidal. The research of Davide and Grzegorz [10] on the 0-1 test for chaos which can be used on any dynamical system in which this dynamical system is a category used to characterize its trajectory. The research of J.J. Sanchez-Gabites [11] on attractors having polyhedral form which can be generalized to discrete dynamical systems generated by homeomorphisms where this attractor is a category. The research of Timothy Ngotiaoco [12] on dynamical systems in which inputs and outputs are described using operadic wiring diagrams (objects of a dual category) can be solved by Euler's method.

In previous studies, the category theory topic has been studied with different forms of discrete dynamical systems and only mentioned that the discrete dynamical system is a category. There is no explanation about the proof of the discrete dynamical system becoming a category and the reason for the application of category theory to the discrete dynamical system. The discrete dynamical system to be studied is a discrete dynamical system with the same profile defined as a directed graph. Examining this is not easy from an algebraic perspective. This research is focused on studying whether a discrete dynamical system is a category or not. Moreover, it will also study the role of category theory in the discrete dynamical system.

METHODS

In this section, we will discuss the materials and methods used and discuss about the definitions used in this study related to category theory of discrete dynamical systems. The research used in this study is a literature study on category theory its properties. The

following are definitions and reference theories used in this study to answer the existing problems.

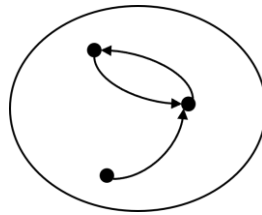


Figure 1. Internal diagram of a set

Figure 1 is called the internal diagram of a set. A point inside the set represents an element whose domain and codomain are the same. The internal diagram above is called an endomap [13].

Definition 1. (See [14]) **(Graph)**

A graph G is made up of two sets of objects: vertices (also known as points or nodes) $V = \{v_1, v_2, v_3, \dots\}$, and edges (also known as lines or arcs) $E = \{e_1, e_2, e_3, \dots\}$.

Definition 2. (See [15]) **(Directed Graph)**

A graph with a set of directed arcs E and a set of vertices V is said to be a directed graph (V, E) . An ordered pair of vertices is connected to each directed edge.

Definition 3. (See [3]) **(Composition Function)**

Let f be a function from set A to set B denoted by $f: A \rightarrow B$ and let g be a function from set B to set C denoted by $g: B \rightarrow C$. Then there is a composition of functions $g \circ f: A \rightarrow C$ defined by $(g \circ f)(a) = g(f(a))$, $a \in A$. An illustration of such a composition of functions can be seen in Figure 2.

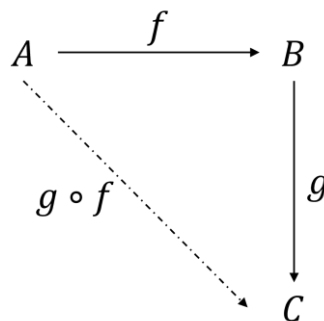


Figure 2. Composition of function $g \circ f: A \rightarrow C$

The composite operation " \circ " is associative. Let h be a function from C to D denoted by $h: C \rightarrow D$ which is illustrated as follows

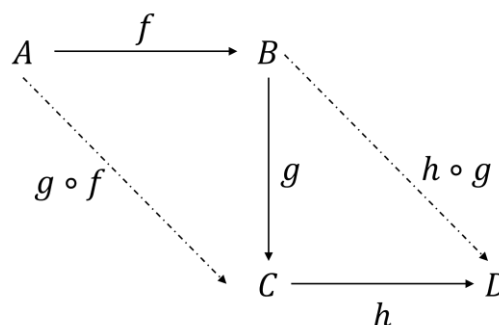


Figure 3. Function $h: C \rightarrow D$

Based on Figure 3, there are forms $g \circ f$ and $h \circ g$, therefore $(h \circ g) \circ f = h \circ (g \circ f)$.

Definition 4. (See [3]) **(Category)**

Given objects A, B, C, \dots and arrows f, g, h, \dots which satisfy

1. For each arrow f there are domain $\text{dom}(f)$ and codomain $\text{cod}(f)$ objects and a function $f: A \rightarrow B$ to indicate that $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
2. If given arrows $f: A \rightarrow B$ and $g: B \rightarrow C$ with $\text{cod}(f) = \text{dom}(g)$, then there exists an arrow $g \circ f: A \rightarrow C$ called the composite of f and g .
3. For every object A , there is an arrow $1_A: A \rightarrow A$ called the identity arrow.

The objects A, B, C, D, \dots and the arrows f, g, h, \dots are called categories if they satisfy :

1. Associative, i.e. $(h \circ g) \circ f = h \circ (g \circ f)$ for every arrow $f: A \rightarrow B, g: B \rightarrow C$ and $h: C \rightarrow D$.
2. Unit existence $f \circ 1_A = f = 1_B \circ f$ for every arrow $f: A \rightarrow B$.

Definition 5. (See [16]) **(Morphisms on Dynamical Systems)**

A map $\phi: X \rightarrow Y$ that connects T and S with $S \circ \phi = \phi \circ T$ is called as a morphism $\phi: (X, T) \rightarrow (Y, S)$ between two dynamical systems.

The definition written above will be used in the result and discussion section to prove whether the set of discrete dynamical systems is a category.

The steps of this research are as follows:

1. Literature study by searching for appropriate articles on category theory with the help of Publish or Perish software and several databases (scopus, science direct and dimension).
2. The literature that has been obtained will be selected by eliminating with the help of Jabref software.
3. The literature that has been selected will be carried out in-depth understanding to prove that the set of discrete dynamical systems is a category.

The process of screening the literature review in this study used prism analysis. The purpose of the prism method is to assist authors in improving systematic reviews and meta-analysis reports [17]. The prism analysis in this study can be seen in Figure 4.

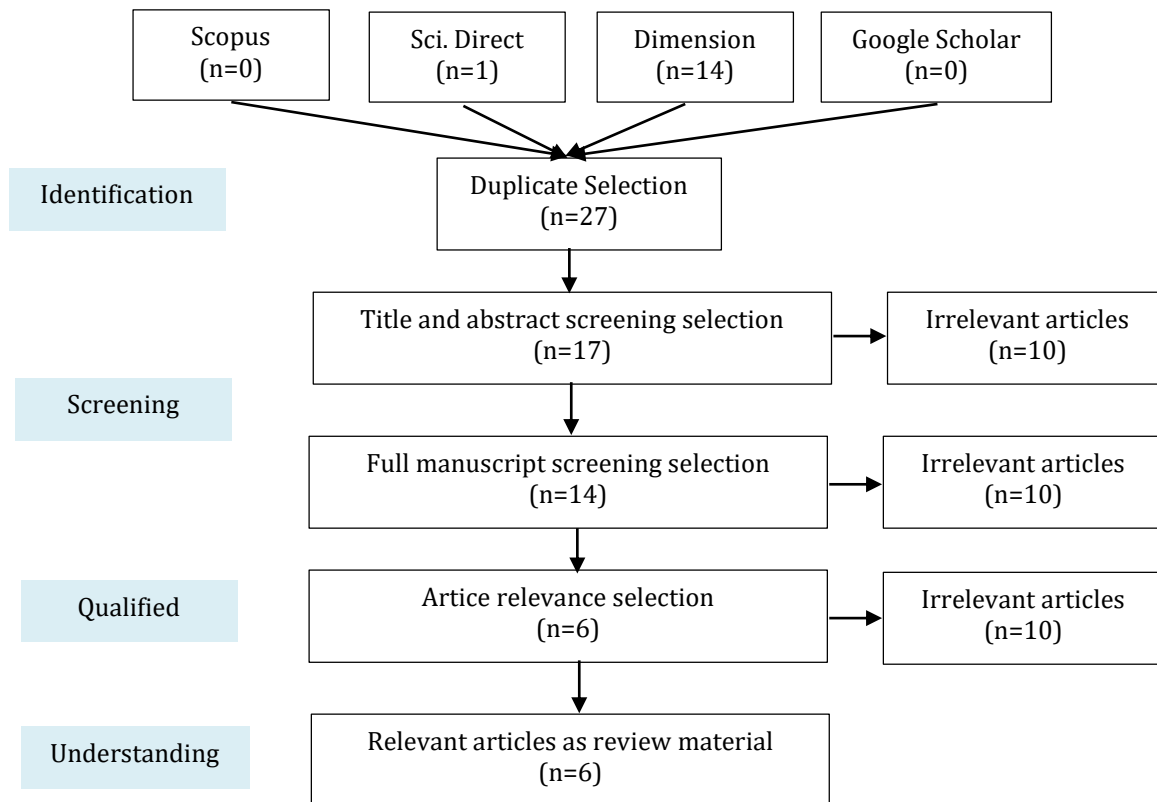


Figure 4. Prism analysis

RESULTS AND DISCUSSION

Discrete Dynamical System

A discrete dynamical system is represented as a directed graph with the vertices of the graph called states that can be seen in the dynamical map. This discrete dynamical system has special characteristics, such as:

- Depicted on contour lines that have heights,
- The deepest height (height 0) contains cycles,
- The direction of the graph is from the largest height to the smallest height,
- A discrete dynamic systems have at least 1 cycle,
- From one state to another state has only one arrow,
- The distance from one state to another must be close in order to be connected by an arrow.

(A, f) represents a discrete dynamical system, where A is a finite set of states and $f: A \rightarrow A$ is a function from A to A . Figure 5 shows the dynamical map of a set of discrete dynamical systems (A, f) .

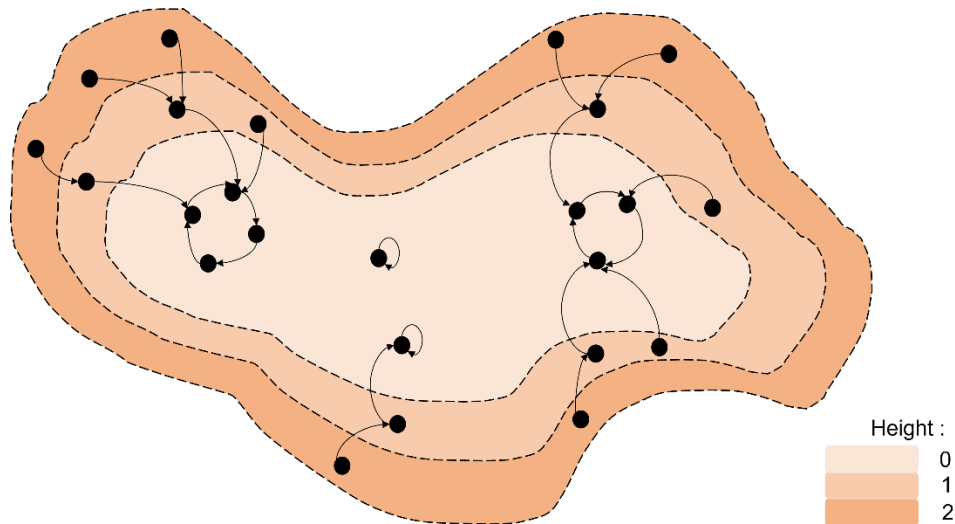


Figure 5. Dynamical map of the set of discrete dynamical systems (A, f)

In a discrete dynamical system, we can know the profile of the dynamical system. Profile is the number of states at each height. The profile of the discrete dynamic system (A, f) that can be seen in figure 5 is $\text{prof}(9,8,7)$, that is, there are 9 states at height 0, 8 states at height 1, and 7 states at height 2. Furthermore, research on profiles can be seen in the research of Ananda Ayu Permatasari et al [18].

Category

Previously we have known the definition of category which can be seen in definition 4. Here is an example of category.

1. Category of Group

Let A be a set with binary operations and $a, b \in A$. Each ordered pair $(a, b) \in A \times A$ is an element of A denoted by ab (can be written as $(a, b) \mapsto ab$).

The set A is called a group with binary operation if it satisfies :

- a. Associative : $(a \times b) \times c = a \times (b \times c) \forall a, b, c \in A$,
- b. Has an identity in A , that is $a \times e = e \times a = a \forall a \in A$,
- c. For each element $a \in A$ there is an element $a^{-1} \in A$, such that $a \times a^{-1} = a^{-1} \times a = e$.

Let set A and set B be groups. A group homomorphism f from group A to group B is a map from A to B with $f(a \times a') = f(a) \times f(a') \forall a \in A, a' \in B$. In category theory, groups are objects and group homomorphisms are morphisms on the category of groups.

2. Category of Unitary and Associative Ring

An associative and unitary ring consists of an abelian group A (i.e. a group whose operation can be written as $(a, b) \mapsto a + b$) with the operation that every ordered pair $(a, b) \in A \times A$ is mapped to ab with $ab \in A$ (can be written as $(a, b) \mapsto ab$). The set A is called associative and unitary ring if it satisfies the following axioms:

- a. Left Distributive : $a(b + c) = ab + ac \forall a, b, c \in A$,
- b. Right Distributive : $(a + b)c = ac + bc \forall a, b, c \in A$,
- c. Associative : $(ab)c = a(bc) \forall a, b, c \in A$,
- d. Has an identity in A i.e. there exists $1 \in A$ such that $1a = a1 = a \forall a \in A$.

Let set A and set B be associative and unitary rings. A unitary ring

homomorphism f from set A to B is a map from A to B with

- a. $f(a + a') = f(a) + f(a') \quad \forall a \in A, a' \in B,$
- b. $f(a \times a') = f(a) \times f(a') \quad \forall a \in A, a' \in B,$
- c. $f(1) = 1.$

In category theory, associative and unitary rings are objects and unitary ring homomorphisms are morphisms on categories.

3. Category of Semiring

Let A be a nonempty set with two binary operations. The set A is semiring if it satisfies the following axioms:

- a. $(A, +)$ commutative monoid,
- b. (A, \times) monoid,
- c. Distributive : $a(b + c) = ab + ac$ and $(a + b)c = ac + bc \quad \forall a, b, c \in A,$
- d. $0a = a0 = 0 \quad \forall a \in A.$

Let set A and set B be a semiring. A semiring homomorphism f from set A to B is a map from A to B with

- a. $f(a + a') = f(a) + f(a') \quad \forall a \in A, a' \in B,$
- b. $f(a \times a') = f(a) \times f(a') \quad \forall a \in A, a' \in B.$

In category theory, semiring is an object and semiring homomorphisms are morphisms on the category of semiring.

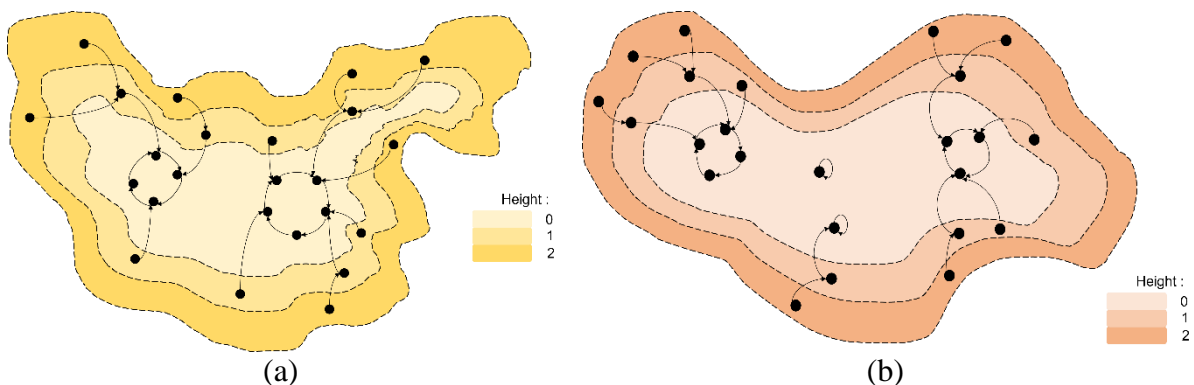
Category of Discrete Dynamical System

In this research, we will investigate the category of discrete dynamical systems where the discrete dynamical systems used have the same profile.

Theorem 1. A discrete dynamical system with the same profile is a category with morphisms consisting of the evolution of the discrete dynamical system.

Proof:

The discrete dynamic system used is a discrete dynamic system with the same profile. Furthermore, profile will be shortened as prof. An example of a discrete dynamical system with the same profile is a discrete dynamical system with $\text{prof}(9,8,7)$, which has 9 states at height 0, 8 states at height 1, and 7 states at height 2. The following is an overview of three dynamical systems with the same profile, $\text{prof}(9,8,7)$, which can be seen in Figure 6.



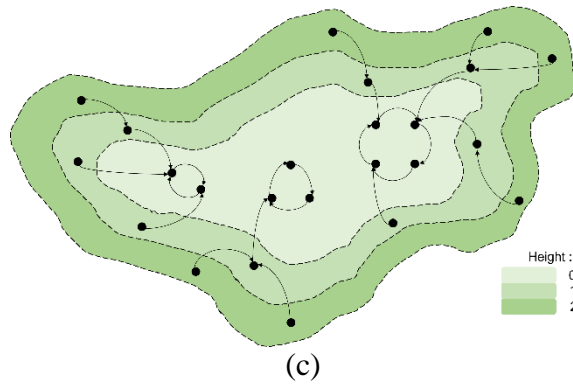


Figure 6. (a) Discrete dynamical system (A, f) with $\text{prof}(9,8,7)$, (b) Discrete dynamical system (B, g) with $\text{prof}(9,8,7)$, (c) Discrete dynamical system (C, h) with $\text{prof}(9,8,7)$.

To prove a category, the objects and arrows will be determined first. The objects of this category of discrete dynamical systems are dynamical systems $(A, f), (B, g), (C, h), \dots$ with $f: A \rightarrow A, g: B \rightarrow B, h: C \rightarrow C, \dots$

1. For all objects $(A, f), (B, g), (C, h), \dots$ have morphisms. According to Definition 5, the arrow $(A, f) \rightarrow (B, g)$ is defined as the function $\text{Mor}((A, f), (B, g)) = \{\varphi: A \rightarrow B\}$ which is compatible with two dynamics, namely $g \circ \varphi = \varphi \circ f$. Therefore, the morphisms of the objects $(A, f), (B, g), (C, h), \dots$ are $\text{Mor}((A, f), (B, g)), \text{Mor}((B, g), (C, h)), \dots$ with $\text{Mor}((A, f), (B, g)) = \{\varphi_1: A \rightarrow B\}, \text{Mor}((B, g), (C, h)) = \{\varphi_2: B \rightarrow C\}, \dots$. Here is an illustration of the morphism. The states at height 0 in the discrete dynamical system (A, f) will evolve into states at height 0 in the discrete dynamical system (B, g) , and so on.

Table 1. Illustration of morphisms on discrete dynamical systems

	(A, f)	Morphism	(B, g)	Morphism	(C, h)
	0	\rightarrow	0	\rightarrow	0
Height	1	\rightarrow	1	\rightarrow	1
	2	\rightarrow	2	\rightarrow	2

An illustration of the morphisms in the two dynamical systems can be seen in figure 7.

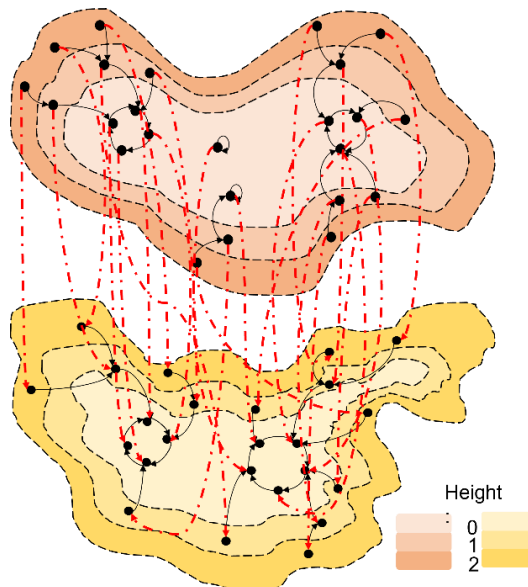


Figure 7. Mapping on discrete dynamical systems (A, f) and (B, g) .

It can be seen in figure 7 that the evolution of the discrete dynamic system (A, f) to the discrete dynamical system (B, g) is a mapping of the state in the discrete

dynamic system (A, f) to the state in the discrete dynamical system (B, g) which has the same height.

- The composition of these morphisms can be seen in the following mapping. For $(\varphi_1, \varphi_2) \in \text{Mor}((A, f), (B, g)) \times \text{Mor}((B, g), (C, h))$, the mapping is

$$(\varphi_1, \varphi_2) \mapsto \varphi_2 \circ \varphi_1$$

with $\varphi_2 \circ \varphi_1 \in \text{Mor}((A, f), (C, h))$, for every object $(A, f), (B, g), (C, h), \dots$. An illustration of function composition in discrete dynamical systems can be seen in Figure 8.

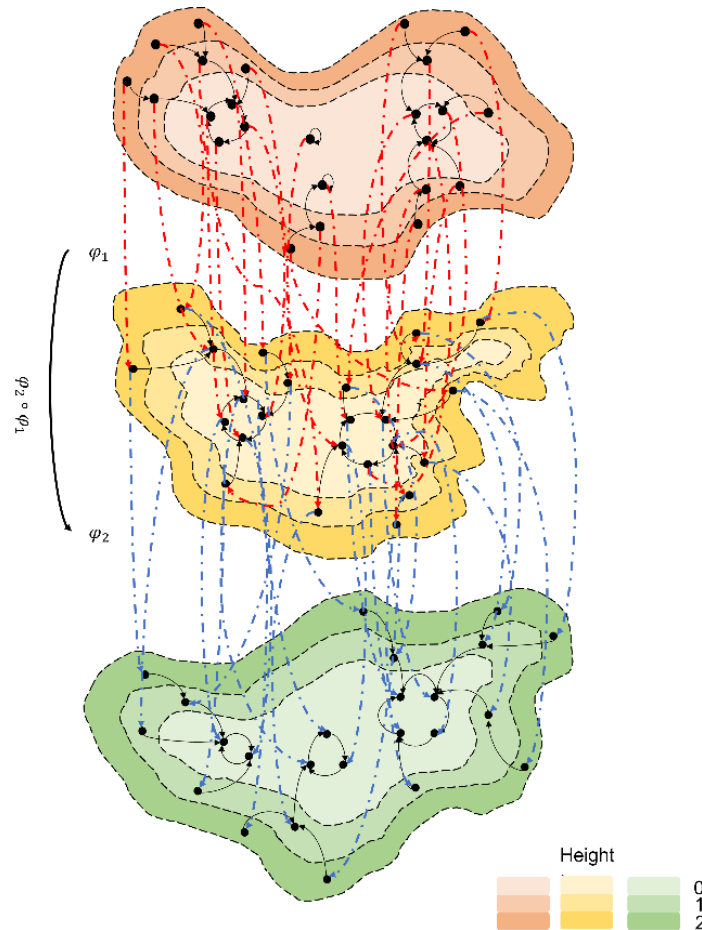


Figure 8. Composition of functions in the category of discrete dynamical systems

In figure 8, we can see the morphism between three discrete dynamical systems, namely discrete dynamical systems (A, f) , (B, g) and (C, h) . The evolution is the same as in figure 7, i.e. the state in discrete dynamic system (A, f) will be mapped to the state in discrete dynamic system (B, g) with the same height and the state in discrete dynamical system (B, g) will be mapped to the state in discrete dynamical system (C, h) with the same height. Therefore, there is an evolution, which is the state of the discrete dynamical system (A, f) is mapped to the state in the discrete dynamical system (C, h) at the same height. Mathematically, if the morphism between the discrete dynamical systems (A, f) and (B, g) is called φ_1 and the morphism between the discrete dynamical systems (B, g) and (C, h) is called φ_2 , then there is a composition function $\varphi_2 \circ \varphi_1$ which indicates the morphism between the discrete dynamical systems (A, f) and (C, h) .

The flow of proving the category of a discrete dynamical system can be seen in Figure 9.

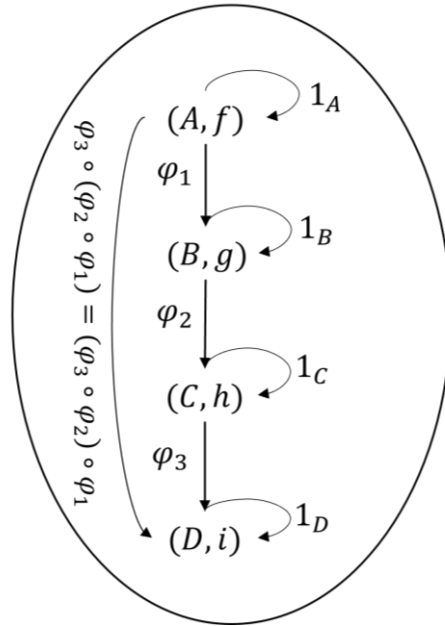


Figure 9. Category of a discrete dynamical system

In figure 9, a category must have objects and morphisms that are clear and can be proven by the applicability of associative rules and identity rules. So for this category of set of discrete dynamical systems, it will be proven that both rules apply.

1. Associative rules

Let $(A, f), (B, g), (C, h), (D, i)$ be objects with morphisms $Mor((A, f), (B, g)), Mor((B, g), (C, h))$ and $Mor((C, h), (D, i))$ of discrete dynamical systems. The morphisms are defined as $Mor((A, f), (B, g)) = \{\varphi_1: A \rightarrow B\}, Mor((B, g), (C, h)) = \{\varphi_2: B \rightarrow C\}, Mor((C, h), (D, i)) = \{\varphi_3: C \rightarrow D\}$.

Therefore, the composition of the morphism is

a. For $(\varphi_2, \varphi_3) \in Mor((B, g), (C, h)) \times Mor((C, h), (D, i))$, then

$$(\varphi_2, \varphi_3) \mapsto \varphi_3 \circ \varphi_2$$

with $\varphi_3 \circ \varphi_2 \in Mor((B, g), (D, i))$.

Thus for $(\varphi_1, (\varphi_2, \varphi_3)) \in Mor((A, f), (B, g)) \times (Mor((B, g), (C, h)) \times Mor((C, h), (D, i)))$, then

$$(\varphi_1, (\varphi_2, \varphi_3)) \mapsto (\varphi_3 \circ \varphi_2) \circ \varphi_1$$

with $(\varphi_3 \circ \varphi_2) \circ \varphi_1 \in Mor((A, f), (D, i))$.

b. For $(\varphi_1, \varphi_2) \in Mor((A, f), (B, g)) \times Mor((B, g), (C, h))$, then

$$(\varphi_1, \varphi_2) \mapsto \varphi_2 \circ \varphi_1$$

with $\varphi_2 \circ \varphi_1 \in Mor((A, f), (C, h))$.

Thus for $((\varphi_1, \varphi_2), \varphi_3) \in (Mor((A, f), (B, g)) \times Mor((B, g), (C, h))) \times Mor((C, h), (D, i))$, then

$$((\varphi_1, \varphi_2), \varphi_3) \mapsto \varphi_3 \circ (\varphi_2 \circ \varphi_1)$$

with $\varphi_3 \circ (\varphi_2 \circ \varphi_1) \in Mor((A, f), (D, i))$.

Based on the two points, it can be concluded that

$$Mor((A, f), (B, g)) \times (Mor((B, g), (C, h)) \times Mor((C, h), (D, i))) = (Mor((A, f), (B, g)) \times Mor((B, g), (C, h))) \times Mor((C, h), (D, i))$$

Therefore, the associative rules are applicable.

2. Identity rules

Let (A, f) dan (B, g) be objects, function $1_A: A \rightarrow A$ is the identity function on the object (A, f) , and $Mor((A, f), (B, g)) = \{\varphi: A \rightarrow B\}$. Hence

- a. For $(1_A, \varphi) \in 1_A \times Mor((A, f), (B, g))$, then $(1_A, \varphi) \mapsto \varphi \circ 1_A = \varphi$ with $\varphi \in Mor((A, f), (B, g))$,
- b. For $(\varphi, 1_A) \in Mor((A, f), (B, g)) \times 1_A$, then $(\varphi, 1_A) \mapsto 1_A \circ \varphi = \varphi$ with $\varphi \in Mor((A, f), (B, g))$.

Therefore, the identity rules are applicable

Based on these two things, it can be concluded that the set of discrete dynamical systems is a category.

These discrete dynamical systems are dynamical systems represented as graphs. Thus, in this category of a set of discrete dynamical systems, operations on graphs apply. The category of a set of discrete dynamical systems has a category product that can be seen in the following equation:

$$(A, f) \times (B, g) = (A \times B, f \times g) \text{ where } (f \times g)(a, b) = (f(a), g(b)) \quad (1)$$

$$(A, f) + (B, g) = (A \uplus B, f + g) \text{ where } (f + g)(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases} \quad (2)$$

with the addition operation is disjoint union and the multiplication operation is tensor product.

A general mathematical theory of structures and systems of structures is known as category theory. Category theory exists because there is a need. In detail, category theory identifies many aspects that are the same in very different areas of mathematics and provides a common unifying language. The purpose of category theory is to provide a way to organize information that applies to a universal concept. Category theory can be a tool to investigate the concepts of a system. Therefore, by using category theory, we can see the concept of discrete dynamical systems. We can find out information about the properties and structure of discrete dynamical systems. Category theory is also a different solution to the same universal problem in discrete dynamical systems. Moreover, we can find out more complex solutions to the problems of discrete dynamical systems that can be further researched as to what the problems of discrete dynamical systems are.

CONCLUSIONS

A discrete dynamical system is represented as a directed graph that has a height that can be seen on the dynamical map. In a discrete dynamical system, there is a profile that is many states at each height. A discrete dynamical system that has the same profile is a category whose morphisms are the evolution of states that have the same height in different dynamical systems. Learning about the category of discrete dynamical systems is very important because we can see the properties and structure of discrete dynamical systems. Furthermore, we can solve problems that exist in discrete dynamical systems.

ACKNOWLEDGMENTS

The authors would like to thank the Rector of Universitas Padjadjaran for financial support to the research results by lecturers and students through the Academic Leadership Grant (ALG) with contract number 1549/UN6.3.1/PT.00/2023.

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