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The Reflexive $\mathcal{H}$ - Strength on Some Graphs

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ABSTRACT

Let G be a connected, simple, and undirected graph with a vertex set $V(G)$ and an edge set $E(G)$. The function f_e defines the irregular reflexive k-labeling $f_e: E(G) \to \{1, 2, 3, \dots, k_e\}$ and $f_v: V(G) \to \{0, 2, 4, \dots, 2k_v\}$ such that $f(x) = f_v(x)$ if $x \in V(G)$ and $f(xy) = f_e(xy)$ if $xy \in E(G)$, where $k = \max\{k_e, 2k_v\}$. The irregular reflexive k labeling is called an $\mathcal{H}$ irregular reflexive k-labeling of the graph G if every two different subgraphs $\mathcal{H}'$ and $\mathcal{H}"$ isomorphic to $\mathcal{H}$, it holds $w(\mathcal{H}') \neq w(\mathcal{H}")$, where $w(\mathcal{H}) = \sum_{x \in V(\mathcal{H})} f_v(x) + \sum_{x \in E(\mathcal{H})} f_e(x)$ for the subgraph $\mathcal{H} \subset G$. The minimum k for graph G, which has an $\mathcal{H}$ irregular reflexive k-labeling, is called the reflexive $\mathcal{H}$-strength of graph G and is denoted by $r\mathcal{H}s(G)$. In this paper, we determine the lower bound of the reflexive $\mathcal{H}$-strength of some subgraphs, $\mathcal{H} \simeq P_3$ on $r\mathcal{H}s(P_n)$, the subgraph $\mathcal{H} \simeq C_3$ on $r\mathcal{H}s\big((W_n) \text{ and } (DF_n)\big)$, the subgraph $\mathcal{H} \simeq C_4$ on $r\mathcal{H}s(Tl_n)$ and the subgraph $\mathcal{H} \simeq C_6$ on $r\mathcal{H}s(L_n)$.

Keywords: $\mathcal{H}$ irregular reflexive k-labeling; reflexive $\mathcal{H}$-strength; path; wheel; double fan; triangular ladder; ladder graph

INTRODUCTION

A graph G is a finite non-empty set $V(G)$ whose members are called vertices and a finite set $E(G)$ whose members are unordered (not necessarily different) pairs of $V(G)$ members called edges [1]. For vertex u, v, sides uv, and a labeling vertex weight f, $wt(v)$ is defined as $wt(v) = f(v) + \sum_{w \in N(v)} f(vw)$, while side weights are denoted by $wt\ (uv) = f(u) + f(v) + f(uv)$. A labeling is considered vertex irregular if the weights of all vertices are different in pairs and edge irregular if the weights of the sides are different in pairs [2].

Labeling is one aspect of graph theory. Labeling a graph is a mapping that brings the elements in a graph as a domain to positive or non-negative integers as a codomain. Graph labeling consists of point labeling, a set of vertices as a domain; edge labeling, and total labeling, a set of vertices and edges as a domain [3]. However, types of graph labeling have developed, one of which is totally disordered labeling. Total irregular

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labeling consists of total irregular labeling of edges and vertices [4]. Vertex irregular reflexive labeling and edge irregular reflexive labeling on a graph were recently introduced by Ryan *et al.* [5].

Vertex $\mathcal{H}$-irregularity strength and Edge $\mathcal{H}$-irregularity strength developed from irregular labeling. According to [6], Vertex $\mathcal{H}$-irregularity strength is an irregular labeling that labels vertex on a graph with positive integers sequentially and repeatedly so that each $\mathcal{H}$ subgraph has a different weight. Furthermore, [7] explained the total k $\mathcal{H}$ -irregular labeling as an irregular labeling that labels the vertices and edges of a graph with positive integers sequentially and repeatedly so that each $\mathcal{H}$ subgraph has a different weight. The weight of the subgraph $\mathcal{H}$ is the result of the sum of the dot and edge labels in one subgraph $\mathcal{H}$, and the minimum k value so that graph G has a total labeling of k $\mathcal{H}$ irregulars called total $\mathcal{H}$-irregularity strength, which can be denoted as $ths(G, H)$. The study of H-irregularity strength labeling has produced many findings from many researchers. It can be found in [4], [7], [8], [9], [10], [11].

For $\mathcal{H} \subseteq G$, the weight of $\mathcal{H}$is given by $wt(\mathcal{H}) = \sum_{v\in V(\mathcal{H})} f(v) + \sum_{e\in E(\mathcal{H})} f(e)$. If there exists a partition of G such that every subgraph in the partition, $\mathcal{H}_i$ is isomorphic, then the graph G admits an $\mathcal{H}$-covering. Note that each edge of $E(G)$ belongs to at least one of the subgraphs $\mathcal{H}_i, i = 1, 2, \dots s$. The $\mathcal{H}$ irregular total k-labeling (φ) in which the subgraph weights are pairwise distinct. For more on subgraph irregular total labelings [6], [8].

Martin Bača *et al.* have studied irregular k-labeling on some graphs [12]. If the weights are different for every two vertex u and v, $wt(u) \neq wt(v)$, the total k-labeling is called vertex irregular k-labeling. The total vertex irregularity value in graph G, $tvs(G)$, is the smallest k value, so graph G has an irregular $vertex$ k-labeling [13]. In 2017, Joseph Ryan *et al.* began labeling vertex as loops, known as irregular reflexive labeling, which combines the discussion of $s(G)$ and $tvs(G)$. This impacts the vertex label. First, because each loop adds 2 to the vertex degrees, the vertex labels are even positive numbers. Second, the vertex label 0 can represent a vertex with no loops [14]. In 2018, Dushyant Tanna *et al.* discuss irregular reflexive k-labeling of the vertex on a graph G. If, for every two vertices u and v, the weights are different, $wt(u) \neq wt(v)$, the total k-labeling is called vertex irregular reflexive k-labeling [15]. The vertex-reflexive disorder value in graph G, $rvs(G)$, is the smallest k-value, so graph G has irregular vertex-reflexive labeling [16]. Irregularity and weights of edges, subgraphs, etc. vertices, are the same as for total irregular labeling. The smallest k for which a reflexive (vertex or edge) k-labeling exists is known as the irregular reflexive strength of the graph [13].

Furthermore, this paper develops a new concept of irregular reflexive k-labeling, namely the $\mathcal{H}$ irregular reflexive k-labeling for $\mathcal{H} \subset G$. Mathematically, the irregular reflexive k-labeling is defined by the function $f_e : E(G) \to \{1, 2, \dots, k_e\}$ and $f_v : V(G) \to \{0, 2, \dots, 2k_v\}$ such that $f(x) = f_v(x)$ if $x \in V(G)$ and $f(xy) = f_e(xy)$ if $xy \in E(G)$, where $k = \max\{k_e, 2k_v\}$ [13], [15]. The irregular reflexive k labeling is called an $\mathcal{H}$ irregular reflexive k-labeling of the graph G if every two different subgraphs $\mathcal{H}'$ and $\mathcal{H}$" isomorphic to $\mathcal{H}$, it holds $w(\mathcal{H}') \neq w(\mathcal{H}")$, where $w(\mathcal{H}) = \sum_{x\in V(\mathcal{H})} f_v(x) + \sum_{x\in E(\mathcal{H})} f_e(x)$ for the subgraph $\mathcal{H} \subset G$ [9], [17]. The minimum k for graph G, which has an $\mathcal{H}$ irregular reflexive k-labeling, is called the reflexive $\mathcal{H}$-strength of graph G and denoted by $r\mathcal{H}s(G)$.

In this paper, we initiate to study the lower bound of the reflexive $\mathcal{H}$-strength of graphs and the reflexive $\mathcal{H}$-strength of some graphs, namely path graph, wheel graph, double fan graph, triangular ladder graph, and ladder graph. We studied subgraph $\mathcal{H} \simeq P_3$ on determining $r\mathcal{H}s(P_n)$, the subgraph $\mathcal{H} \simeq C_3$ on determining $r\mathcal{H}s\big((W_n)$ and $(DF_n)\big)$, subgraph $\mathcal{H} \simeq C_4$ on determining $r\mathcal{H}s(Tl_n)$, and the subgraph $\mathcal{H} \simeq C_6$ on determining $r\mathcal{H}s(L_n)$.

METHODS

Following is the method used to determine the distance reflexive strength of a graph:

1. Defines a graph as a research object.
2. Establish the graph's vertex and edge sets.
3. Identify a graph's lower bound of reflexive $\mathcal{H}$-strength with the following Lemma and Corollary [18].
   Lemma: $r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil$
   Corollary: $r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil + 1$, if $q_{\mathcal{H}} + |\mathcal{H}| - 1 = t(p_{\mathcal{H}} + q_{\mathcal{H}})$.
4. Create edge and vertex labels based on the $\mathcal{H}$ irregular reflexive labeling specification.
5. Identify the upper bound of a graph's reflexive $\mathcal{H}$-strength using the function found at point 4.
6. The value of the reflexive $\mathcal{H}$-strength can be calculated from the graph if the graph's upper and lower bounds for its reflexive $\mathcal{H}$-strength are equal.
7. If the graph's higher bound for reflexive $\mathcal{H}$-strength and lower bound for reflexive $\mathcal{H}$-strength are not equal, point 4 is repeated until the upper bound for reflexive $\mathcal{H}$-strength and lower bound for reflexive $\mathcal{H}$-strength are equal.

RESULTS AND DISCUSSION

This study aims to determine the reflexive $\mathcal{H}$-strength of several graphs, including the path graph, wheel graph, double fan graph, triangular ladder graph, and ladder graph.

Theorem 1

Let $\mathcal{H} \simeq P_3$ be a subgraph of P_n, and for every natural number $n \geq 4$, then

$$r\mathcal{H}s(P_n) = \begin{cases} \left\lceil \frac{n-1}{5} \right\rceil + 1, & \text{if } n \equiv 6 (\text{mod } 10) \\ \left\lceil \frac{n-1}{5} \right\rceil, & \text{if n} \equiv 7,8,9 \text{ (mod 10)} \end{cases}$$

Proof. Let P_n, $n \geq 4$, be a path graph with the vertex set $V(P_n) = \{x_i : 1 \leq i \leq n\}$ and the edge set $E(P_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\}$, with subgraph $\mathcal{H} \simeq P_3$, such that $|\mathcal{H}| = n - 2, p_{\mathcal{H}} = 3, q_{\mathcal{H}} = 2$. Based on the lower bound Lemma, we obtain $r\mathcal{H}s(P_n) \geq \left\lceil \frac{q\mathcal{H}+|\mathcal{H}|-1}{p\mathcal{H}+qH} \right\rceil = \left\lceil \frac{2+n-2-1}{5} \right\rceil = \left\lceil \frac{n-1}{5} \right\rceil$.
For $n \equiv 6 (mod\ 10)$, there is an integer a such that $n = 10a + 6$ such that $r\mathcal{H}s(P_n) \geq \left\lceil \frac{n-1}{5} \right\rceil = \left\lceil \frac{10a+6-1}{5} \right\rceil = \left\lceil \frac{10a+5}{5} \right\rceil = 2a + 1 = t$. Suppose t is $r\mathcal{H}s(P_n)$, then t is the largest label in labeling irregular reflexive subgraphs of the graph P_n. Since t is an odd number, the largest weight in a subgraph $\mathcal{H}_l$ is $wt(\mathcal{H}_l) = 3(t-1) + 2t = 5t - 3 = 5(2a+1) - 3 = 10a + 2$. On the other hand, to obtain the minimum $r\mathcal{H}s$, the subgraph weights must form an arithmetic sequence with a difference 1, that is, $wt(\mathcal{H}_i) = q_{\mathcal{H}} + i - 1 : 1 \leq i \leq (n-2)$. Since the subgraph has the largest weight, $q_{\mathcal{H}} + n - 2 - 1 \leq wt(\mathcal{H}_l)$, a

contradiction occurs with $q_{\mathcal{H}} + n - 2 - 1 = n - 1 = 10a + 5 \leq 10a + 2$. Therefore, $t \neq r\mathcal{H}s(P_n)$ is such that $r\mathcal{H}s(P_n) > \left\lceil \frac{n-1}{5} \right\rceil$. It gives $r\mathcal{H}s(P_n) \geq \left\lceil \frac{n-1}{5} \right\rceil + 1$. Furthermore, determine the upper bound $r\mathcal{H}s(P_n)$, labeling vertices and edges on the graph P_n is constructed.

For $n \equiv 10 \pmod{20}$, labeling of vertices and edges in graph, P_n is constructed as follows.

$$f_v(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 5 \\ 2, & \text{if } 6 \leq i \leq 12 \\ 2\left\lceil \frac{i-12}{9} \right\rceil + 2, & \text{if } i \in \{0,1,2,3,4,5,6,7,8\}, \text{i(mod 9) for } i \geq 12 \end{cases}$$

$$f_e(x_i x_{i+1}) = \begin{cases} 1, & \text{if } i \in \{1,2,6,7\} \\ 2, & \text{if } i \in \{3,4,5\} \\ 2\left\lceil \frac{i-7}{9} \right\rceil, & \text{if } i \in \{0,4,5,8\}, i \pmod 9 \text{ for } i \geq 7 \\ 2\left\lceil \frac{i-7}{9} \right\rceil + 1, & \text{if } i \in \{1,2,3,6,7\}, i \pmod 9 \text{ for } i \geq 7 \end{cases}$$

For n otherwise, labeling vertices and edges in graph P_n is constructed as follows.

$$f_v(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 5 \\ 4\left\lceil \frac{i-5}{20} \right\rceil - 2, & \text{if } i \in \{6,7,9,10,11,13,14\}, i \pmod{20} \\ 4\left\lceil \frac{i-5}{20} \right\rceil - 4, & \text{if } i \equiv 8 \pmod{20} \\ 4\left\lceil \frac{i-5}{20} \right\rceil, & \text{if } i \in \{0,1,2,3,4,5,12,15,16,17,18,19\}, i \pmod{20} \text{ for } i \geq 5 \end{cases}$$

$$f_e(x_i x_{i+1}) = \begin{cases} 1, & \text{if } i \in \{1,2,5,6\} \\ 2, & \text{if } i \in \{3,4,7,8\} \\ 4\left\lceil \frac{i-8}{20} \right\rceil - 1, & \text{if } i \in \{9,13,14,17,18\}, i \pmod{20} \\ 4\left\lceil \frac{i-8}{20} \right\rceil - 3, & \text{if } i \equiv 10 \pmod{20} \\ 4\left\lceil \frac{i-5}{20} \right\rceil - 2, & \text{if } i \in \{11,12,15,16\}, i \pmod{20} \\ 4\left\lceil \frac{i-5}{20} \right\rceil, & \text{if } i \in \{0,19\}, i \pmod{20} \text{ for } i \geq 8 \\ 4\left\lceil \frac{i-5}{20} \right\rceil + 1, & \text{if } i \in \{1,2,5,6\}, i \pmod{20} \text{ for } i \geq 8 \\ 4\left\lceil \frac{i-5}{20} \right\rceil + 2, & \text{if } i \in \{3,4,7,8\}, i \pmod{20} \text{ for } i \geq 8 \end{cases}$$

Based on the vertex and edge labeling above, we obtained $r\mathcal{H}s(P_n) \leq \left\lceil \frac{n-1}{5} \right\rceil$ and $r\mathcal{H}s(P_n) \leq \left\lceil \frac{n-1}{5} \right\rceil + 1$ for $n \equiv 6 \pmod{10}$. The set of subgraph weights on a graph P_n is $wt(\mathcal{H}_i) = 1 + i : 1 \leq i \leq n-2$. Since all subgraph weights P_3 on the graph P_n are different, it is an irregular reflexive subgraph labeling. Since $r\mathcal{H}s(P_n) \geq \left\lceil \frac{n-1}{5} \right\rceil$, $r\mathcal{H}s(P_n) \geq \left\lceil \frac{n-1}{5} \right\rceil + 1$ for $n \equiv 6 \pmod{10}$ and $r\mathcal{H}s(P_n) \leq \left\lceil \frac{n-1}{5} \right\rceil$, $r\mathcal{H}s(P_n) \leq \left\lceil \frac{n-1}{5} \right\rceil + 1$ for $n \equiv 6 \pmod{10}$, $r\mathcal{H}s(P_n) = \left\lceil \frac{n-1}{5} \right\rceil$ and $r\mathcal{H}s(P_n) = \left\lceil \frac{n-1}{5} \right\rceil + 1$ for $n \equiv 6 \pmod{10}$. ■

Figure 1 illustrates the $\mathcal{H}$ irregular reflexive labeling on path graph P_8 with subgraph $\mathcal{H} \simeq P_3$. The number of vertices and edges of P_8 are 8 and 7, respectively. Since the number of subgraphs is 6, based on Theorem 1, the reflexive strength on P_8 is 2. The vertex set on this labeling is $V(P_8) = \{0, 2\}$, and the edge set is $E(P_8) = \{1, 2\}$. Therefore, the $\mathcal{H}$ irregular reflexive labeling on path graph P_8 satisfies Theorem 1.

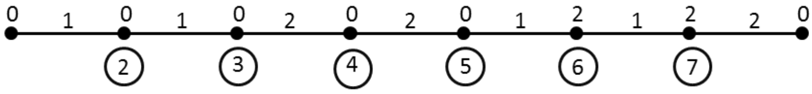

Figure 1. The illustration of $\mathcal{H}$ Irregular Reflexive Labeling on P_8

Theorem 2

Let $\mathcal{H} \simeq \mathrm{C}_3$ be a subgraph of W_n, and for every natural number $\mathrm{n} \geq 3$, then

$$r\mathcal{H}s(W_n) = \begin{cases} \left\lceil \frac{n+1}{5} \right\rceil + 1, & \text{if } n \in \{3,4\}, i \text{ (mod 10)} \\ \left\lceil \frac{n+1}{5} \right\rceil, & \text{otherwise} \end{cases}$$

Proof. Let W_n, $n \geq 3$, be a wheel graph with the vertex set $V\,(W_n) = \{x_i, y_i, z\colon 1 \leq i \leq n\}$ and the edge set $E\,(W_n) = \{x_i z, y_i z\colon 1 \leq i \leq n\} \cup \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\}$, with subgraph $\mathcal{H} \simeq C_3$, such that $|\mathcal{H}| = n - 1$. Since one of the vertex (z) in subgraph C_3 is an element that always exists in all subgraphs and is labeled 0 such that it can be said, $p\mathcal{H} = 2, q\mathcal{H} = 3$, Based on the lower bound Lemma, we obtain $r\mathcal{H}s(W_n) \geq \left\lceil \frac{q\mathcal{H}+|\mathcal{H}|-1}{p\mathcal{H}+q\mathcal{H}} \right\rceil = \left\lceil \frac{3+n-1-1}{5} \right\rceil = \left\lceil \frac{n+1}{5} \right\rceil$.
For $n \equiv 4(mod\ 10)$, there is an integer a such that $n = 10a + 4$ such that $r\mathcal{H}s(W_n) \geq \left\lceil \frac{n+1}{5} \right\rceil = \left\lceil \frac{10a+4+1}{5} \right\rceil = \left\lceil \frac{10a+5}{5} \right\rceil = 2a + 1 = t$. Suppose t is $r\mathcal{H}s(W_n)$, then t is the largest label in labeling irregular reflexive subgraphs of the graph W_n. Since t is an odd number, the largest weight in a subgraph $\mathcal{H}_l$ is $wt(\mathcal{H}_l) = 2(t-1) + 3t = 5t - 2 = 5(2a+1) - 2 = 10a + 3$. On the other hand, to obtain the minimum $r\mathcal{H}s$, the subgraph weights must form an arithmetic sequence with a difference 1, that is $wt(\mathcal{H}_i) = q_{\mathcal{H}} + i - 1\ : 1 \leq i \leq (n-1)$. Since the subgraph has the largest weight $q_{\mathcal{H}} + n - 1 - 1 \leq wt(\mathcal{H}_l)$, a contradiction occurs with $q_{\mathcal{H}} + n - 1 - 1 = n = 10a + 5 \leq 10a + 3$. Therefore, $t \neq r\mathcal{H}s(W_n)$ such that $r\mathcal{H}s(W_n) > \left\lceil \frac{n+1}{5} \right\rceil$. It gives $r\mathcal{H}s(W_n) \geq \left\lceil \frac{n+1}{5} \right\rceil + 1$. For $n \equiv 3\ (mod\ 10)$, it is obtained $r\mathcal{H}s(W_n) \geq \left\lceil \frac{n+1}{5} \right\rceil + 1$ for the same reason. Furthermore, determine the upper bound $r\mathcal{H}s(W_n)$, labeling vertices and edges on the graph W_n is constructed.
For $n \geq 3$, the labeling of vertices and edges in graph W_n is constructed as follows.

$$f_v(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 3 \\ 2, & \text{if } 4 \leq i \leq 7 \\ 2\left\lceil \frac{i-7}{5} \right\rceil + 2, & \text{if } i \in \{0,1,2,3,4\}, i \text{ (mod 5) for } i \geq 7 \end{cases}$$

$$f_v(y_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 3 \\ 2, & \text{if } 4 \leq i \leq 7 \\ 2\left\lceil \frac{i-7}{5} \right\rceil + 2, & \text{if } i \in \{0,1,2,3,4\}, i \text{ (mod 5) for } i \geq 7 \end{cases}$$

$$f_e(x_i z) = \begin{cases} 1, & \text{if } i \in \{1,2,4\} \\ 2, & \text{if } i = 3 \\ 2\left\lceil \dfrac{i-4}{5} \right\rceil, & \text{if } i \equiv 0 \ (\text{mod } 5) \text{ for } i \geq 4 \\ 2\left\lceil \dfrac{i-4}{5} \right\rceil + 1, & \text{if } i \in \{1,2,3,4\}, i \ (\text{mod } 5) \text{ for } i \geq 4 \end{cases}$$

$$f_e(y_i z) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } 2 \leq i \leq 5 \\ 2\left\lceil \dfrac{i-5}{5} \right\rceil + 1, & \text{if } i \in \{1,3\}, i \ (\text{mod } 5) \text{ for } i \geq 5 \\ 2\left\lceil \dfrac{i-5}{5} \right\rceil + 2, & \text{if } i \in \{0,2,4\}, i \ (\text{mod } 5) \text{ for } i \geq 5 \end{cases}$$

$$f_e(x_i x_{i+1}) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } i \in \{2,3\} \\ 2\left\lceil \dfrac{i-3}{5} \right\rceil, & \text{if } i \in \{0,4\} \ (\text{mod } 5) \\ 2\left\lceil \dfrac{i-3}{5} \right\rceil + 1, & \text{if } i \in \{1,2,3\}, i \ (\text{mod } 5) \text{ for } i \geq 3 \end{cases}$$

$$f_e(y_i y_{i+1}) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } i \in \{2,3\} \\ 2\left\lceil \dfrac{i-3}{5} \right\rceil, & \text{if } i \equiv 4 \ (\text{mod } 5) \\ 2\left\lceil \dfrac{i-3}{5} \right\rceil + 1, & \text{if } i \in \{0,1,2,3\}, i \ (\text{mod } 5) \text{ for } i \geq 3 \end{cases}$$

Based on the vertex and edge labeling above, we obtained $r\mathcal{H}s(W_n) \leq \left\lceil \frac{n+1}{5} \right\rceil$ and $r\mathcal{H}s(W_n) \leq \left\lceil \frac{n+1}{5} \right\rceil + 1$ for $n \equiv 3, 4 \ (\text{mod } 10)$. The set of subgraph weights on a graph W_n is $wt(\mathcal{H}_i) = 2 + i : 1 \leq i \leq n - 1$. Since all subgraph weights C_3 on the graph W_n are different, it is an irregular reflexive subgraph labeling. Since $r\mathcal{H}s(W_n) \geq \left\lceil \frac{n+1}{5} \right\rceil, r\mathcal{H}s(W_n) \geq \left\lceil \frac{n+1}{5} \right\rceil + 1$ for $n \equiv 3, 4 \ (\text{mod } 10)$, $r\mathcal{H}s(W_n) = \left\lceil \frac{n+1}{5} \right\rceil$ and $r\mathcal{H}s(W_n) = \left\lceil \frac{n+1}{5} \right\rceil + 1$ for $n \equiv 3, 4 \ (\text{mod } 10)$. ∎

Figure 2 illustrates the $\mathcal{H}$ irregular reflexive labeling on path graph W_{16} with subgraph $\mathcal{H} \simeq C_3$. The number of vertices and edges of W_{16} are 17 and 16, respectively. Since the number of subgraphs is 16, based on Theorem 2, the reflexive strength on W_{16} is 4. The vertex set on this labeling is $V(W_{16}) = \{0, 2, 4\}$, and the edge set is $E(W_{16}) = \{1, 2, 3, 4\}$. Therefore, the $\mathcal{H}$ irregular reflexive labeling on wheel graph W_{16} satisfies Theorem 2.

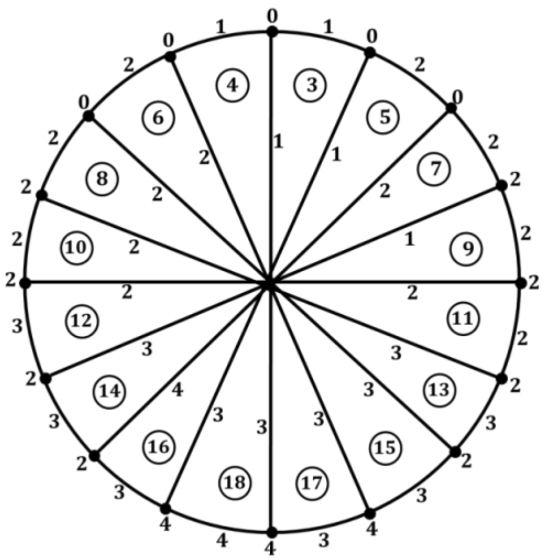

Figure 2. The illustration of $\mathcal{H}$ Irregular Reflexive Labeling on W_{16}

Theorem 3

Let $\mathcal{H} \simeq C_3$ be a subgraph of DF_n, and for every natural number $n \geq 3$, then

$$r\mathcal{H}s(DF_n) = \begin{cases} \left\lceil \frac{2n}{5} \right\rceil + 1, & \text{if } n \equiv 7 \text{ (mod 10)} \\ \left\lceil \frac{2n}{5} \right\rceil, & \text{otherwise} \end{cases}$$

Proof. Let DF_n, $n \geq 3$, be a double fan graph with the vertex set $V\,(DF_n) = \{a, b, x_i : 1 \leq i \leq n\}$ and the edge set $E\,(DF_n) = \{ax_i, bx_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\}$, with subgraph $\mathcal{H} \simeq C_3$, such that $|\mathcal{H}| = 2n - 2$. Since the vertex (a and b) in subgraph C_3 is an element that always exists in all subgraphs and is labeled 0 such that it can be said, $p\mathcal{H} = 2, q\mathcal{H} = 3$, Based on the lower bound Lemma, we obtain $r\mathcal{H}s(W_n) \geq \left\lceil \frac{q\mathcal{H}+|\mathcal{H}|-1}{p\mathcal{H}+qH} \right\rceil = \left\lceil \frac{3+2n-2-1}{5} \right\rceil = \left\lceil \frac{2n}{5} \right\rceil$.

For $n \equiv 7\ (mod\ 10)$, there is an integer a such that $n = 10a + 7$ such that $r\mathcal{H}s(DF_n) \geq \left\lceil \frac{2n}{5} \right\rceil = \left\lceil \frac{2(10a+7)}{5} \right\rceil = \left\lceil \frac{20a+14}{5} \right\rceil = 4a + 3 = t$. Suppose t is $r\mathcal{H}s(DF_n)$, then t is the largest label in labeling irregular reflexive subgraphs of the graph DF_n. Since t is an odd number, the largest weight in a subgraph $\mathcal{H}_l$ is $wt(\mathcal{H}_l) = 2(t-1) + 3t = 5t - 2 = 5(2a+1) - 2 = 10a + 3$. Conversely, to obtain the minimum $r\mathcal{H}s$, the subgraph weights must form an arithmetic sequence with difference 1, $wt(\mathcal{H}_i) = q_{\mathcal{H}} + i - 1 \ : 1 \leq i \leq (2n-2)$. Since the subgraph has the largest weight, $q_{\mathcal{H}} + 2n - 2 - 1 \leq wt(\mathcal{H}_l)$, a contradiction occurs with $q_{\mathcal{H}} + 2n - 2 - 1 = 2n = 20a + 14 \leq 10a + 3$. Therefore, $t \neq r\mathcal{H}s(DF_n)$ such that $r\mathcal{H}s(DF_n) > \left\lceil \frac{2n}{5} \right\rceil$. It gives $r\mathcal{H}s(DF_n) \geq \left\lceil \frac{2n}{5} \right\rceil + 1$. Furthermore, determine the upper bound $r\mathcal{H}s(DF_n)$, labeling vertices and edges on the graph DF_n is constructed.

For $n \geq 3$, the labeling of vertices and edges in graph DF_n is constructed as follows.

$$f_v(a) = 0$$

$$f_v(b) = 0$$

$$f_v(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 3 \\ 2\left\lceil \frac{i-3}{5} \right\rceil, & \text{if } i \in \{0,1,2,3,4\}, \text{i (mod 5) for } i \geq 3 \end{cases}$$

$$f_e(ax_i) = \begin{cases} 4\left\lceil \frac{i}{10} \right\rceil - 3, & \text{if } i \in \{1, 2, 4\}, i \text{ (mod 10)} \\ 4\left\lceil \frac{i}{10} \right\rceil - 2, & \text{if } i \in \{3, 5, 6\}, i \text{ (mod 10)} \\ 4\left\lceil \frac{i}{10} \right\rceil - 1, & \text{if } i \in \{0, 8\}, i \text{ (mod 10)} \\ 4\left\lceil \frac{i}{10} \right\rceil, & \text{if } i \in \{7, 9\}, i \text{ (mod 10)} \end{cases}$$

$$f_e(bx_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } 2 \le i \le 5 \\ 2\left\lceil \frac{i-5}{5} \right\rceil + 1, & \text{if } i \equiv 1 \text{ (mod 5) for } i \ge 5 \\ 2\left\lceil \frac{i-5}{5} \right\rceil + 2, & \text{if } i \in \{0, 2, 3, 4\}, i \text{ (mod 5) for } i \ge 5 \end{cases}$$

$$f_e(x_i x_{i+1}) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } 2 \le i \le 4 \\ 2\left\lceil \frac{i-4}{5} \right\rceil + 1, & \text{if } i \in \{0, 1\}, i \text{ (mod 5) for } i \ge 4 \\ 2\left\lceil \frac{i-4}{5} \right\rceil + 2, & \text{if } i \in \{2, 3, 4\}, i \text{ (mod 5) for } i \ge 4 \end{cases}$$

Based on the vertex and edge labeling above, we obtained $r\mathcal{H}s(DF_n) \le \left\lceil \frac{2n}{5} \right\rceil$ and $r\mathcal{H}s(DF_n) \le \left\lceil \frac{2n}{5} \right\rceil + 1$ for $n \equiv 7 \text{ (mod 10)}$. The set of subgraph weights on a graph DF_n is $wt(\mathcal{H}_i) = 2 + i : 1 \le i \le 2n$. Since all subgraph weights C_3 on the graph DF_n are different, it is an irregular reflexive subgraph labeling. Since $r\mathcal{H}s(DF_n) \ge \left\lceil \frac{2n}{5} \right\rceil$, $r\mathcal{H}s(DF_n) \ge \left\lceil \frac{2n}{5} \right\rceil + 1$ for $n \equiv 7 \text{ (mod 10)}$, $r\mathcal{H}s(DF_n) = \left\lceil \frac{2n}{5} \right\rceil$ and $r\mathcal{H}s(DF_n) = \left\lceil \frac{2n}{5} \right\rceil + 1$ for $n \equiv 7 \text{ (mod 10)}$. ■

Figure 3 illustrates the $\mathcal{H}$ irregular reflexive labeling on double fan graph DF_8 with sub-graph $\mathcal{H} \simeq C_3$. The number of vertices and edges of DF_8 are 10 and 23, respectively. Since the number of subgraphs is 16, based on Theorem 3, the reflexive strength on DF_8 is 4. The vertex set on this labeling is $V(DF_8) = \{0, 2\}$, and the edge set is $E(DF_8) = \{1, 2,3,4\}$. Therefore, the $\mathcal{H}$ irregular reflexive labeling on double fan graph DF_8 satisfies Theorem 3.

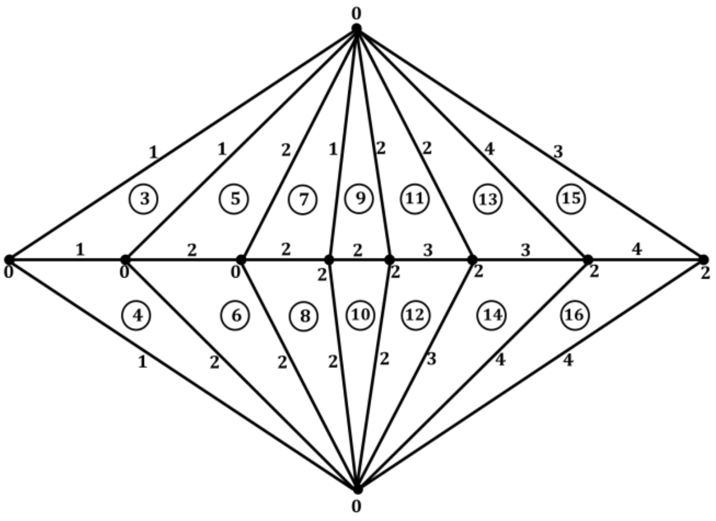

Figure 3. The illustration of $\mathcal{H}$ Irregular Reflexive Labeling on DF_8

Theorem 4
Let $\mathcal{H} \simeq C_4$ be a subgraph of TL_n, and for every natural number $n \geq 4$, then

$$r\mathcal{H}s(TL_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil + 1, & \text{if } n \in \{3,4\}, i \text{ (mod 8)} \\ \left\lceil \frac{n}{4} \right\rceil, & \text{otherwise} \end{cases}$$

Proof. Let TL_n, $n \geq 4$, be a triangular ladder graph with the vertex set $V(TL_n) = \{x_i, y_i : 1 \leq i \leq n\}$ and the edge set $E(TL_n) = \{x_i y_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1} : 1 \leq i \leq n-1\}$, with subgraph $\mathcal{H} \simeq C_4$, such that $|\mathcal{H}| = 2n - 3, p\mathcal{H} = 4, q\mathcal{H} = 4$. Based on the lower bound Lemma, we obtain $r\mathcal{H}s(TL_n) \geq \left\lceil \frac{q\mathcal{H}+|\mathcal{H}|-1}{p\mathcal{H}+qH} \right\rceil = \left\lceil \frac{4+2n-3-1}{8} \right\rceil = \left\lceil \frac{n}{4} \right\rceil$.
For $n \equiv 4 \ (mod\ 8)$, there is an integer that $n = 8a + 4$ such that $r\mathcal{H}s(TL_n) \geq \left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{8a+4}{4} \right\rceil = 2a + 1 = t$. Suppose t is $r\mathcal{H}s(TL_n)$, then t is the largest label in labeling irregular reflexive sub-graphs of the graph TL_n. Since t is an odd number, the largest weight in a subgraph $\mathcal{H}_l$ is $wt(\mathcal{H}_l) = 4(t-1) + 4t = 8t - 4 = 8(2a+1) - 4 = 16a + 4$. Conversely, to obtain the minimum $r\mathcal{H}s$, the subgraph weights must form an arithmetic sequence with difference 1, $wt(\mathcal{H}_i) = q_{\mathcal{H}} + i - 1 : 1 \leq i \leq (2n-3)$. Since the sub-graph has the largest weight, $q_{\mathcal{H}} + 2n - 3 - 1 \leq wt(\mathcal{H}_l)$, a contradiction occurs with $q_{\mathcal{H}} + 2n - 3 - 1 = 2n = 16a + 8 \leq 16a + 4$. Therefore, $t \neq r\mathcal{H}s(TL_n)$ such that $r\mathcal{H}s(TL_n) > \left\lceil \frac{n}{4} \right\rceil$. It gives $r\mathcal{H}s(TL_n) \geq \left\lceil \frac{n}{4} \right\rceil + 1$. For $n \equiv 3 \ (mod\ 8)$, it is obtained $r\mathcal{H}s(TL_n) \geq \left\lceil \frac{n}{4} \right\rceil + 1$ for the same reason. Furthermore, determine the upper bound $r\mathcal{H}s(TL_n)$, labeling vertices and edges on the graph TL_n is constructed.
For $n \geq 4$, the labeling of vertices and edges in graph TL_n is constructed as follows.

$$f_v(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 6 \\ 2, & \text{if } 7 \leq i \leq 13 \\ 2\left\lceil \frac{i-13}{8} \right\rceil + 2, & \text{if } i \in \{0,1,2,3,4,5,6,7\}, i \text{ (mod 8) for } i \geq 13 \end{cases}$$

$$f_v(y_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 4 \\ 2, & \text{if } 5 \leq i \leq 11 \\ 2\left\lceil \frac{i-11}{8} \right\rceil + 2, & \text{if } i \in \{0,1,2,3,4,5,6,7\}, i \text{ (mod 8) for } i \geq 11 \end{cases}$$

$$f_e(x_i y_i) = \begin{cases} 1, & \text{if } i = 1,2,7 \\ 2, & \text{if } i = 3,4,5,6 \\ 3, & \text{if } i = 8 \\ 2\left\lceil \frac{i-8}{8} \right\rceil + 1, & \text{if } i \in \{1,2,4\}, i \text{ (mod 8) for } i \geq 8 \\ 2\left\lceil \frac{i-8}{8} \right\rceil + 2, & \text{if } i \in \{0,3,5,7\}, i \text{ (mod 8) for } i \geq 8 \\ 2\left\lceil \frac{i-8}{8} \right\rceil, & \text{if } i \equiv 6 \text{ (mod 8) for } i \geq 8 \end{cases}$$

$$f_e(x_i x_{i+1}) = \begin{cases} 2\left\lceil \frac{i}{8} \right\rceil - 1, & \text{if } i \equiv 1 \text{ (mod 8)} \\ 2\left\lceil \frac{i}{8} \right\rceil, & \text{if } i \in \{0,2,3,4,5,6,7\}, i \text{ (mod 8)} \end{cases}$$

$$f_e(y_iy_{i+1}) = \begin{cases} 1, & \text{if } i = 1,2 \\ 2, & \text{if } i = 3,4,5,7,8 \\ 3, & \text{if } i = 6 \\ 2\left\lceil \frac{i-8}{8} \right\rceil + 1, & \text{if } i \in \{1,2,3,4\}, i \ (\text{mod } 8) \text{ for } i \geq 8 \\ 2\left\lceil \frac{i-8}{8} \right\rceil + 2, & \text{if } i \in \{5,6,7\}, i \ (\text{mod } 8) \text{ for } i \geq 8 \\ 2\left\lceil \frac{i-8}{8} \right\rceil + 3, & \text{if } i \equiv 0 \ (\text{mod } 8) \text{ for } i \geq 8 \end{cases}$$

$$f_e(x_iy_{i+1}) = \begin{cases} 1, & \text{if } i = 1,3,5 \\ 2, & \text{if } i = 2,4,7 \\ 3, & \text{if } i = 6 \\ 2\left\lceil \frac{i-7}{8} \right\rceil + 1, & \text{if } i \in \{0,1,5,7\}, i \ (\text{mod } 8) \text{ for } i \geq 7 \\ 2\left\lceil \frac{i-7}{8} \right\rceil + 2, & \text{if } i \in \{2,4,6\}, i \ (\text{mod } 8) \text{ for } i \geq 7 \\ 2\left\lceil \frac{i-7}{8} \right\rceil, & \text{if } i \equiv 3 \ (\text{mod } 8) \text{ for } i \geq 7 \end{cases}$$

Based on the vertex and edge labeling above, we obtained $r\mathcal{H}s(TL_n) \leq \left\lceil \frac{n}{4} \right\rceil$ and $r\mathcal{H}s(TL_n) \leq \left\lceil \frac{n}{4} \right\rceil + 1$ for $n \equiv 3,4 \ (\text{mod } 8)$. The set of sub-graph weights on a graph TL_n is $\omega t(\mathcal{H}_i) = 3 + i : 1 \leq i \leq 2n - 3$. Since all sub-graph weights C_4 on the graph TL_n are different, it is an irregular reflexive sub-graph labeling. Since $r\mathcal{H}s(TL_n) \geq \left\lceil \frac{n}{4} \right\rceil$, $r\mathcal{H}s(TL_n) \geq \left\lceil \frac{n}{4} \right\rceil + 1$ for $n \equiv 3,4 \ (\text{mod } 8)$, then $r\mathcal{H}s(TL_n) = \left\lceil \frac{n}{4} \right\rceil$ and $r\mathcal{H}s(TL_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ for $n \equiv 3,4 \ (\text{mod } 8)$. ■

Figure 4 illustrates the $\mathcal{H}$ irregular reflexive labeling on triangular ladder graph TL_8 with sub-graph $\mathcal{H} \simeq C_4$. The number of vertices and edges of TL_8 are 16 and 29, respectively. Since the number of subgraphs is 13, based on Theorem 4, the reflexive strength on TL_8 is 3. The vertex set on this labeling is $V(TL_8) = \{0,2\}$ and the edge set is $E(TL_8) = \{1,2,3\}$. Therefore, the $\mathcal{H}$ irregular reflexive labeling on triangular ladder graph TL_8 satisfies Theorem 4.

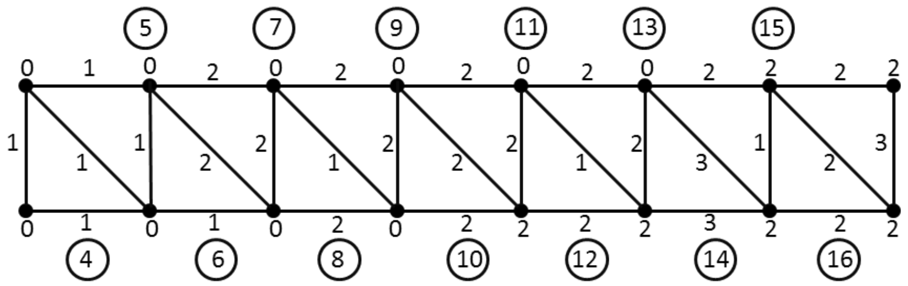

Figure 4. The illustration of $\mathcal{H}$ Irregular Reflexive Labeling on TL_8

Theorem 5

Let $\mathcal{H} \simeq C_6$ be a subgraph of L_n, and for every natural number $n \geq 4$, then

$$r\mathcal{H}s(L_n) = \begin{cases} \left\lceil \frac{n+3}{12} \right\rceil + 1, & \text{if } n \in \{1,3,4,5,6,7,8,9\}, i \ (\text{mod } 24) \\ \left\lceil \frac{n+3}{12} \right\rceil, & \text{otherwise} \end{cases}$$

Proof. Let L_n , $n \geq 4$, be a ladder graph with the vertex set $V(L_n) = \{x_i, y_i : 1 \leq i \leq n\}$ and the edge set $E(L_n) = \{x_i y_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\}$, with subgraph $\mathcal{H} \simeq C_6$, such that $|\mathcal{H}| = n-2, p\mathcal{H} = 6, q\mathcal{H} = 6$. Based on the lower bound Lemma, we obtain $r\mathcal{H}s(L_n) \geq \left\lceil \frac{q\mathcal{H}+|\mathcal{H}|-1}{p\mathcal{H}+qH} \right\rceil = \left\lceil \frac{6+n-2-1}{12} \right\rceil = \left\lceil \frac{n+3}{12} \right\rceil$.

For $n \equiv 9 \ (mod\ 24)$, there is an integer a such that $n = 24a + 9$ such that $r\mathcal{H}s(L_n) \geq \left\lceil \frac{n+3}{12} \right\rceil = \left\lceil \frac{24a+9+3}{12} \right\rceil = \left\lceil \frac{24a+12}{12} \right\rceil = 2a + 1 = t$. Suppose t is $r\mathcal{H}s(L_n)$, then t is the largest label in labeling irregular reflexive subgraphs of the graph L_n. Since t is an odd number, the largest weight in a subgraph $\mathcal{H}_l$ is $wt(\mathcal{H}_l) = 6(t-1) + 6t = 12t - 6 = 12(2a + 1) - 6 = 24a + 6$. On the other hand, to obtain the minimum $r\mathcal{H}s$, the subgraph weights must form an arithmetic sequence with a difference 1, that is, $wt(\mathcal{H}_i) = q_{\mathcal{H}} + i - 1 \ : 1 \leq i \leq (n-2)$. Since the subgraph has the largest weight, $q_{\mathcal{H}} + n - 2 - 1 \leq wt(\mathcal{H}_l)$, a contradiction occurs with $q_{\mathcal{H}} + n - 2 - 1 = n + 3 = 24a + 12 \leq 24a + 6$. Therefore, $t \neq r\mathcal{H}s(L_n)$ such that $r\mathcal{H}s(L_n) > \left\lceil \frac{n+3}{12} \right\rceil$. It gives $r\mathcal{H}s(L_n) \geq \left\lceil \frac{n+3}{12} \right\rceil + 1$. For $n \equiv 1,3,4,5,6,7,8 \ (mod\ 24)$, it is obtained $r\mathcal{H}s(L_n) \geq \left\lceil \frac{n+3}{12} \right\rceil + 1$ for the same reason.

Furthermore, determine the upper bound $r\mathcal{H}s(L_n)$, labeling vertices and edges on the graph L_n is constructed.

For $n \geq 4$, the labeling of vertices and edges in graph L_n is constructed as follows.

$$f_v(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 9 \\ 2\left\lceil \dfrac{i-9}{24} \right\rceil, & \text{if } i \in \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15, \\ & 16,17,18,19,20,21,22,23\}, i \text{ (mod 24) for } i \geq 9 \end{cases}$$

$$f_v(y_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 10 \\ 2\left\lceil \dfrac{i-10}{24} \right\rceil, & \text{if } i \in \{0,1,2,3,4,5,6,7,8,9,10,11,13,14,15,16,17 \\ & 18,19,20,21,22,23\}, i \text{ (mod 24) for } i \geq 10 \\ 2\left\lceil \dfrac{i-10}{24} \right\rceil - 2, & \text{if } i \equiv 12 \text{ (mod 24)} \end{cases}$$

$$f_e(x_i y_i) = \begin{cases} 1, & \text{if } i \in \{1,3,4,11,12,13,15,16\} \\ 2, & \text{if } i \in \{2,5,6,7,8,9,10,14,17,18,19,20,21,22\} \\ 2\left\lceil \dfrac{i-22}{24} \right\rceil + 1, & \text{if } i \in \{0,1,3,4,11,12,13,15,16,23\}, i \text{ (mod 24) for } i \geq 22 \\ 2\left\lceil \dfrac{i-22}{24} \right\rceil + 2, & \text{if } i \in \{2,5,6,7,8,9,10,14,17,18, \\ & 19,20,21,22\}, i \text{ (mod 24) for } i \geq 22 \end{cases}$$

$$f_e(x_i x_{i+1}) = \begin{cases} 1, & \text{if } 1 \leq i \leq 3 \\ 2, & \text{if } 4 \leq i \leq 9 \\ 2\left\lceil \dfrac{i-9}{24} \right\rceil - 1, & \text{if } i \in \{10,11,12,13,14,15\}, i \text{ (mod 24)} \\ 2\left\lceil \dfrac{i-9}{24} \right\rceil, & \text{if } i \in \{0,16,17,18,19,20,22\}, i \text{ (mod 24) for } i \geq 9 \\ 2\left\lceil \dfrac{i-9}{24} \right\rceil + 1, & \text{if } i \in \{1,2,3,21,23\}, i \text{ (mod 24) for } i \geq 9 \\ 2\left\lceil \dfrac{i-9}{24} \right\rceil + 2, & \text{if } i \in \{4,5,6,7,8,9\}, i \text{ (mod 24) for } i \geq 9 \end{cases}$$

$$f_e(y_i y_{i+1}) = \begin{cases} 1, & \text{if } 1 \le i \le 6 \\ 2\left\lceil \dfrac{i-6}{24} \right\rceil, & \text{if } i \in \{0,7,8,19,20,21,22\}, i \pmod{24} \text{ for } i \ge 6 \\ 2\left\lceil \dfrac{i-6}{24} \right\rceil - 1, & \text{if } i \in \{9,10,11,12,13,14,15,16,17,18\}, i \pmod{24} \\ 2\left\lceil \dfrac{i-3}{5} \right\rceil + 1, & \text{if } i \in \{1,2,3,4,5,6,23\}, i \pmod{24} \text{ for } i \ge 6 \end{cases}$$

Based on the vertex and edge labeling above, we obtained $r\mathcal{H}s(L_n) \le \left\lceil \frac{n+3}{12} \right\rceil$ and $r\mathcal{H}s(L_n) \le \left\lceil \frac{n+3}{12} \right\rceil + 1$ for $n \in \{1,3,4,5,6,7,8,9\}, n \pmod{24}$. The set of subgraph weights on a graph L_n is $\omega t(\mathcal{H}_i) = 5 + i : 1 \le i \le n-2$. Since all subgraph weights C_6 on graph L_n are different, it is an irregular reflexive subgraph labeling. Since $r\mathcal{H}s(L_n) \ge \left\lceil \frac{n+3}{12} \right\rceil, r\mathcal{H}s(L_n) \ge \left\lceil \frac{n+3}{12} \right\rceil + 1$ for $n \in \{1,3,4,5,6,7,8,9\}, n \pmod{24}$, $r\mathcal{H}s(L_n) = \left\lceil \frac{n+3}{12} \right\rceil$ and $r\mathcal{H}s(L_n) = \left\lceil \frac{n+3}{12} \right\rceil + 1$ for $n \in \{1,3,4,5,6,7,8,9\}, n \pmod{24}$. ■

Figure 5 illustrates the $\mathcal{H}$ irregular reflexive labeling on ladder graph L_8 with sub-graph $\mathcal{H} \simeq C_6$. The number of vertices and edges of L_8 are 16 and 22, respectively. Since the number of subgraphs is 6, based on Theorem 5, the reflexive strength on L_8 is 2. The vertex set on this labeling is $V(L_8) = \{0\}$, and the edge set is $E(L_8) = \{1,2\}$. Therefore, the $\mathcal{H}$ irregular reflexive labeling on ladder graph L_8 satisfies Theorem 5.

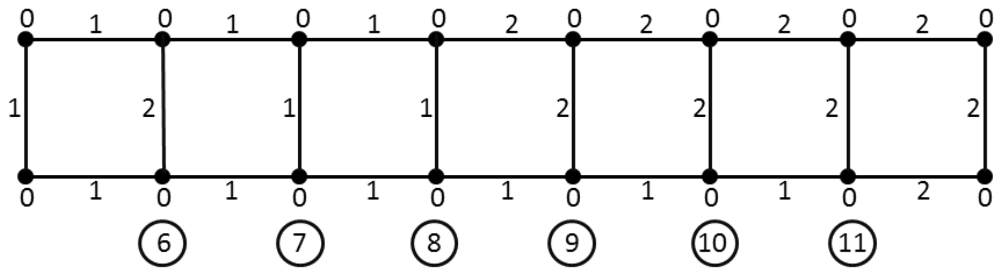

Figure 5. The illustration of $\mathcal{H}$ Irregular Reflexive Labeling on L_8

CONCLUSIONS

We have obtained the lower bound of the reflexive $\mathcal{H}$-strength of graphs and the reflexive $\mathcal{H}$-strength of some graphs, namely the path graph, wheel graph, double fan graph, triangular ladder graph, and ladder graph. We studied subgraph $\mathcal{H} \simeq P_3$ on determining $r\mathcal{H}s(P_n)$, the subgraph $\mathcal{H} \simeq C_3$ on determining $r\mathcal{H}s\big((W_n) \text{ and } (DF_n)\big)$, subgraph $\mathcal{H} \simeq C_4$ on determining $r\mathcal{H}s(Tl_n)$, and the subgraph $\mathcal{H} \simeq C_6$ on determining $r\mathcal{H}s(L_n)$. We believe our article has contributed to the field of irregular reflexive graph research in a novel way. However, the exact value of $r\mathcal{H}s(G)$ for any graph G is still open for research because determining a graph's reflexive $\mathcal{H}$-strength is considered an NP-complete problem. As a result, we propose the following open problems.

1. Find the $r\mathcal{H}s(P_n)$ with subgraph $\mathcal{H} \simeq P_3$, where $n \equiv 0,1,2,3,4,5 \pmod{10}$.
2. With any subgraph, find the sharper upper bound of every graph's reflexive $\mathcal{H}$-strength.
3. Consider any graph with a given $r\mathcal{H}s(G)$ and determine the precise value of its reflexive $\mathcal{H}$-strength.

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