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On Group-Vertex-Magic Labeling of Simple Graphs

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ABSTRACT

Let G be a graph and Abelian group $(\mathcal{A},+)$ with identity 0. The $\mathcal{A}$-vertex-magic labeling of G is a mapping from its set of vertices to $\mathcal{A}-\{0\}$ such that for every vertex u in G, the total of the labels of every adjacent vertex with u is equal. In this article, we discuss group-vertex-magic labeling of simple graphs by using the Abelian group $\mathbb{Z}_k$, with natural numbers $k>1$. Some simple graphs that we observed are path graphs, complete graphs, cycle graphs, and star graphs. This research methodology as follows: determining the $\mathbb{Z}_k$ −vertex-magic labeling of path graphs, complete graphs, and cycle graphs, investigating the special properties of $\mathbb{Z}_k$ −vertex-magic of complete graphs and cycle graphs, and determining the $\mathbb{Z}_k$ −vertex-magic labeling of star graphs. We obtain the complete graphs, cycle graphs, and star graphs have $\mathbb{Z}_k$ −vertex-magic labeling, while path graphs have $\mathbb{Z}_k$ −vertex-magic labeling only for $n=2{,}3$.

Keywords: Abelian group; group-vertex-magic labeling; simple graph

INTRODUCTION

A graph G is a system consisting of a finite non-empty set of vertices and a finite set of edges such that the edge is identified with a pair of vertices [1]. Graph theory can be applied in various types of fields, such as networks [2], transportation [3], coding theory [4], chemistry [5], crystallography [6], cryptography [7], and information systems [8]. Graph labeling is one of the important topics in graph theory. Graph labeling can be applied to circuit design [9], communication network design [10], computer sciences [11], and cryptography [12].

Graph labeling is a mapping from the set of vertices, the set of edges, or both into the set of non-negative integers such that it satisfies a certain condition [13]. Rosa first introduced graph labeling in 1966 [14]. Several kinds of graph labeling have been explored up to this point, including magic labeling, which Kotzig and Rosa initially presented in 1970 as magic-valuation [15]. Following that, there are various types of magic labeling of G, such as vertex-magic labeling [16], anti-magic labeling [17], graceful labeling [18], edge-magic labeling [19], and so on. The concept of graph labeling can be

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related to groups as Lee et al. first introduced the concept of graph labeling using Abelian groups in 2001, which is called group-magic labeling [20]. Later, in 2006, Low et al. developed the idea of group-magic labeling of the cartesian product of graphs [21]. In 2009, Shiu and Low investigated about group-magic labeling of fan graphs and wheel graphs [22].

In this article, we specifically talk about group-vertex-magic labeling of G which Kamatchi et al. first described in 2020 [23]. Group-vertex-magic labeling of G is a mapping from the set of its vertices to an Abelian group that assigns each vertex in the graph to a non-identity element of the Abelian group, such that the total of the labels of all adjacent vertices of each vertex in the graph is equal [23]. We called that graph a group-vertex-magic graph [23]. Kamatchi et al. gave results on group-vertex-magic graphs for complete bipartite graphs, complete $m-$partite graphs, and trees with diameters no greater than 4 [23]. Besides providing research results, Kamatchi et al. also provided some observations which are still open until now [23]. Navas et al. in 2020 studied vertex-magic labeling for a class of caterpillar graphs using the Abelian group $\mathbb{Z}_k$ [24].

The research topic about group-vertex-magic labeling G using Abelian groups is relatively new and has the possibility to grow rapidly. Motivated by observations given by Kamatchi et al. [23], this article explores group-vertex-magic labeling of simple graphs using Abelian group $\mathbb{Z}_k$. The simple graph classes include path graphs, complete graphs, cycle graphs, and star graphs. There are four sections in this article. In the second section, we give a literature review and describe the research's methodology. In the third section, we obtain some results about group-vertex-magic labeling of some simple graphs. The last section concludes the contents of this article.

METHODS

In this work, we study some literature related to group vertex-magic graphs. The research methodology is given as follows:

i. Determining the $\mathbb{Z}_k-$vertex-magic labeling of path graphs.
ii. Determining the $\mathbb{Z}_k-$vertex-magic labeling of complete graphs and cycle graphs.
iii. Investigating the special properties of $\mathbb{Z}_k-$vertex-magic labeling of complete graphs and cycle graphs.
iv. Determining the $\mathbb{Z}_k-$vertex-magic labeling of star graphs.

This study of graph theory is restricted to simple graphs, that is, undirected graphs that have no loops and multiple edges. Most of the material in this graph theory is cited from [1], while the study of algebraic structure is cited from [25]. Graph $G = (V(G), E(G))$ is a system containing a finite non-empty set of vertices $V(G)$ and a finite set of edges $E(G)$, which is the subset of $V(G) \times V(G)$. The edge $(u, v) \in E(G)$ is generally simply written as uv or vu. If $uv \in E(G)$, then the vertices u and v are called adjacent. The size and order of G are the number of edges and vertices in G, respectively. The degree of $u \in V(G)$, denoted by $\deg(u)$, is the number of vertices that are adjacent to u. The open neighborhood of $u \in V(G)$, denoted by $N(u)$, is the set of vertices that are adjacent with u.

In graph theory, some kinds of simple graphs play an important purpose. A complete graph of order n, denoted by K_n, is a graph in which every two vertices are adjacent. A path graph of order n, denoted by P_n, is a graph of size $n-1$, with the vertices labeled $u_1, u_2, \dots, u_n$, and its edges u_iu_{i+1} for $i = 1, \dots, n-1$. A cycle graph of order n, denoted by

C_n, is a graph of size n whose vertices are labeled $u_1, u_2, \dots, u_n$, and whose edges are $u_i u_{i+1}$ for $i = 1, \dots, n-1$ and $u_n u_1$. A r −regular graph is a graph in which every vertex has degree r. The cycle graph C_n is an example of a 2 −regular graph. A star graph of order $n+1$, denoted by $K_{1,n}$, is a graph of size n whose vertices are labeled $u_0, u_1, u_2, \dots, u_n$, and its edges $u_0 u_i$ for $i = 1,2, \dots, n$. The vertex u_0 is then called the center of $K_{1,n}$.

The concept of graph labeling using Abelian groups was first introduced by Lee et al. and is referred to as group-magic labeling with Abelian groups [20]. This group-magic labeling of G is an edge labeling such that the label of each vertex is sum of the labeling of incident edge and a constant value [20]. Later, Kamatchi et al. introduced the new concept of graph labeling using Abelian groups as follows:

Definition 1 [23] Let $(\mathcal{A}, +)$ be an Abelian group with identity 0 and $G = (V, G)$ be a graph. The $\mathcal{A}$ −vertex-magic labeling of G is a mapping $\ell: V(G) \to \mathcal{A} - \{0\}$ such that the mapping $w: V(G) \to \mathcal{A}$ given by

$$w(u) = \sum_{v \in N(h)} \ell(v),$$

is constant map, that is there exists $a \in \mathcal{A}$ such that $w(u) = a$ for every $u \in V(G)$. The graph G that have $\mathcal{A}$-vertex-magic labeling is called $\mathcal{A}$-vertex-magic graph.

Next, we introduce the definition of a vertex-magic integer set. This definition is special case for graphs labeled with the group $\mathbb{Z}_k$.

Definition 2 [23] Let G be a graph. The set of vertex integer magic of G, denoted by $VIM(G)$, is the set of all positive integer k such that G is $\mathbb{Z}_k$ −vertex-magic graph, or can be written as

$$VIM(G) = \{k \in \mathbb{N} | G \text{ is } \mathbb{Z}_k - \text{vertex} - \text{magic graph}\}.$$

We notice some observations and theorems by Kamatchi et al. [23]. This observation and theorem motivate results and discussions.

Observation 1 [23] If G is a graph that contains vertices $u_1, u_2, u_3,$ and u_4 with $\deg(u_1) = 1$ and $\deg(u_3) = 2$ such that $u_1u_2, u_2u_3,$ and u_3u_4 are edges, then G is not $\mathcal{A}$ −vertex-magic graph for every Abelian group $\mathcal{A}$.

Observation 2 [23] For every Abelian group $\mathcal{A}$, the r −regular graph G is group-vertex-magic graph.

Theorem 1 [23] Let T be a tree of order n and diameter 2.

i. The tree T is V_4 −vertex-magic graph, where V_4 is Klein four-group.
ii. If $n = 2k$ for natural numbers k, then T is group-vertex-magic graph.

RESULTS AND DISCUSSION

In this section, we give some results about group-vertex-magic labeling of simple graphs using the Abelian group $\mathbb{Z}_k$. These results are motivated by the observation given by Kamatchi et al. [23]. Based on Observation 1, we give some properties of $\mathbb{Z}_k$ −vertex-magic labeling of P_n.

Proposition 1

i. For every integer $k > 1$, P_2 is $\mathbb{Z}_k$ −vertex-magic graph.
ii. For every integer $k > 2$, P_3 is $\mathbb{Z}_k$ −vertex-magic graph.

Proof.

i. Let $k > 1$ be an integer. Define a mapping $\ell: V(P_2) \rightarrow \mathbb{Z}_k - \{\overline{0}\}$ as $\ell(u_1) = \ell(u_2) = \overline{a}$, with $\overline{a} \in \mathbb{Z}_k - \{\overline{0}\}$, and $u_1, u_2 \in V(P_2)$. Clearly, $w(u_1) = \overline{a} = w(u_2)$. Thus, P_2 is $\mathbb{Z}_k$vertex-magic graph.

ii. Let $k > 2$ be an integer. Note that for every integer $k > 2$, we can always find $\overline{a} \in \mathbb{Z}_k$ with $2 \cdot \overline{a} \neq \overline{0}$. Now define a mapping $\ell: V(P_3) \rightarrow \mathbb{Z}_k - \{\overline{0}\}$ as $\ell(u_1) = \ell(u_3) = \overline{a}$, and $\ell(u_2) = 2 \cdot \overline{a}$ with $u_1, u_2, u_3 \in V(P_3), \overline{a} \in \mathbb{Z}_k - \{\overline{0}\}$ and $2 \cdot \overline{a} \neq \overline{0}$. It follows that

$$w(u_3) = w(u_2) = w(u_1) = 2 \cdot \overline{a}.$$

Thus, P_3 is $\mathbb{Z}_k$ −vertex-magic graph.

For $k = 2$, the only possible label for P_3 is $\ell(u_3) = \ell(u_2) = \ell(u_1) = \overline{1}$. But this result implies

$$\overline{0} = \ell(u_1) + \ell(u_3) = w(u_2) = w(u_1) = \ell(u_2) = \overline{0},$$

which is impossible.

■

Theorem 2 For every integer $k > 1$ and $n > 3$, P_n is not $\mathbb{Z}_k$ −vertex-magic graph.

Proof.

Consider a path graph P_n with $n > 3$. Suppose P_n is $\mathbb{Z}_k$ −vertex-magic graphs. It follows that for $u_1, u_2, u_3, u_4 \in V(P_n)$ holds

$$\ell(u_2) = w(u_1) = w(u_3) = \ell(u_2) + \ell(u_4),$$

which implies $\ell(u_4) = \overline{0}$, which is impossible. So, P_n is not a $\mathbb{Z}_k$ −vertex-magic graph. ■

Based on Proposition 1 and Theorem 2, we can determine the vertex-integer-magic set of P_n as desired in the following:

Corollary 2 For every integer $n > 1$, holds

$$VIM(P_n) = \begin{cases} \mathbb{N} \setminus \{1\}, & \text{if } n = 2, \\ \mathbb{N} \setminus \{1,2\}, & \text{if } n = 3, \\ \emptyset, & \text{if } n > 3. \end{cases}$$

In Observation 2, Kamatchi et al. stated that every r −regular graph is group-vertex-magic graph [23]. In the following, we investigate a $\mathbb{Z}_k$ −vertex-magic labeling of the r −regular graphs for K_n and C_n graphs.

Theorem 3 Let $k > 1$ and $n > 2$ be integers, the complete graph K_n, and map $\ell: V(K_n) \rightarrow \mathbb{Z}_k - \{\overline{0}\}$. K_n is $\mathbb{Z}_k$ −vertex-magic graphs if and only if ℓ is non zero constant map.

Proof.

Let K_n be a complete graph and $u_1, u_2, \dots, u_n \in V(K_n)$, with $n > 2$.

(⇒) Suppose K_n is $\mathbb{Z}_k$ −vertex-magic graph. This means that

$$\ell(u_1) + \ell(u_3) + \cdots + \ell(u_n) = w(u_2) = w(u_1) = \ell(u_2) + \ell(u_3) + \cdots + \ell(u_n).$$

By cancellation law, we have $\ell(u_1) = \ell(u_2)$. In the same way, we obtain

$$\ell(u_1) = \ell(u_2) = \cdots = \ell(u_n).$$

Thus, $\ell(u_i) = \overline{a}$ for some $\overline{a} \in \mathbb{Z}_k - \{\overline{0}\}$.

(⇐) Suppose that ℓ is non zero constant map, this means $\ell(u_i) = \overline{a}$, for all $u_i \in V(K_n)$, and some $\overline{a} \in \mathbb{Z}_k - \{\overline{0}\}$. It's clear that for all $u_i \in V(K_n)$ holds

$$w(u_i) = \sum_{v \in N(u_i)} \ell(v) = (n-1) \cdot \overline{a}.$$

Thus, K_n is $\mathbb{Z}_k$ −vertex-magic graph.

■

Based on the equivalence condition in Theorem 3, we can compute the number of mappings ℓ and determine all integers k such that the graph K_n is a $\mathbb{Z}_k$ −vertex-magic-graph as desired in the following:

Corollary 4 For every integer $k > 1$ and $n > 2$, there exists $k - 1$ distinct labeling such that K_n is a $\mathbb{Z}_k$ −vertex-magic graph.

Corollary 5 For every integer $n > 2$, holds

$$VIM(K_n) = \mathbb{N} \setminus \{1\}.$$

Theorem 4 For every integer $k > 1$ and $n > 2$, C_n is $\mathbb{Z}_k$ −vertex-magic graphs.

Proof.

Consider a cycle graph C_n, with $n > 2$. Define a mapping $\ell: V(C_n) \to \mathbb{Z}_k - \{\overline{0}\}$ as $\ell(u_i) = \overline{a}$, for all $u_i \in V(C_n)$ and $\overline{a} \in \mathbb{Z}_k - \{\overline{0}\}$. It's clear that for all $i = 2, \dots, n-1$ holds

$$w(u_i) = \ell(u_{i-1}) + \ell(u_{i+1}) = 2 \cdot \overline{a},$$

and

$$w(u_1) = \ell(u_n) + \ell(u_2) = w(u_n) = \ell(u_{n-1}) + \ell(u_1) = 2 \cdot \overline{a}.$$

Thus, C_n is $\mathbb{Z}_k$ −vertex-magic graph. ■

Next, we discuss the properties of $\mathbb{Z}_k$ −vertex-magic labeling of cycle graphs.

Theorem 5 Let $n > 2$ and $k > 1$ be integers. If $4 \nmid n$, then there exists $k - 1$ distinct labeling such that C_n is $\mathbb{Z}_k$ −vertex-magic graph.

Proof.

By Theorem 4, C_n is a $\mathbb{Z}_k$ −vertex-magic graph. We consider three cases.

Case 1: $n = 4m + 1$, for some integer m.

Note that for $u_1, u_2, u_3, u_4, u_5 \in V(C_n)$ holds

$$\ell(u_3) + \ell(u_5) = w(u_4) = w(u_2) = \ell(u_3) + \ell(u_1),$$

which implies $\ell(u_1) = \ell(u_5)$. In a similar way, for all $u_i \in V(C_n)$ we obtain

$$\ell(u_1) = \dots = \ell(u_{4m+1}) = \ell(u_4) = \dots = \ell(u_{4m}) = \dots = \ell(u_{4m-1}) = \dots = \ell(u_{4m-2}).$$

Case 2: $n = 4m + 2$, for some integer m.

Note that for $u_1, u_2, u_3, u_4, u_5 \in V(C_n)$ holds

$$\ell(u_3) + \ell(u_5) = w(u_4) = w(u_2) = \ell(u_3) + \ell(u_1),$$

which implies $\ell(u_1) = \ell(u_5)$. In a similar way, for all $u_i \in V(C_n)$ we obtain

$$\ell(u_1) = \ell(u_5) = \dots = \ell(u_{4m+1}) = \ell(u_3) = \ell(u_7) = \dots = \ell(u_{4m-1}),$$

and

$$\ell(u_2) = \ell(u_6) = \dots = \ell(u_{4m+2}) = \ell(u_4) = \ell(u_8) = \dots = \ell(u_{4m}).$$

Then, since

$$\ell(u_2) + \ell(u_4) = w(u_3) = w(u_2) = \ell(u_3) + \ell(u_1),$$

holds $\ell(u_1) = \ell(u_2)$.

Case 3: $n = 4m + 3$, for some integer m.

Note that for $u_1, u_2, u_3, u_4, u_5 \in V(C_n)$ holds

$$\ell(u_3) + \ell(u_5) = w(u_4) = w(u_2) = \ell(u_3) + \ell(u_1),$$

which implies $\ell(u_1) = \ell(u_5)$. In a similar way, for all $u_i \in V(C_n)$ we obtain

$$\ell(u_1) = \dots = \ell(u_{4m+1}) = \ell(u_2) = \dots = \ell(u_{4m+2}) = \dots = \ell(u_{4m+3}) = \dots = \ell(u_{4m}).$$

It can be concluded that ℓ must be a constant map. Since there are $k - 1$ non zero elements in $\mathbb{Z}_k$, then there are $k - 1$ different possible labeling such that C_n is $\mathbb{Z}_k$ −vertex-magic graph. ■

Based on Theorem 4, we can determine the vertex-integer-magic set of cycle graphs as desired in the following:

Corollary 6 For every integer $n > 2$, holds

$$VIM(C_n) = \mathbb{N} \setminus \{1\}.$$

In the next discussion, we observed the $\mathbb{Z}_k$ −vertex-magic labeling of $K_{1,n}$. This study is motivated by Theorem 1 [23].

Proposition 2 Let $K_{1,n}$ be a star graph, with $n > 2$.

i. If n even, then $K_{1,n}$ is not a $\mathbb{Z}_2$ −vertex-magic graph.
ii. If n odd, then $K_{1,n}$ is $\mathbb{Z}_2$ −vertex-magic graph.

Proof.

The only possible label for $K_{1,n}$ is $\ell(u_i) = \overline{1}$, for all $u_i \in V(K_{1,n})$, where $i = 0,1, \dots, n$.

i. Suppose n even. Note that

$$\overline{1} = \ell(u_0) = w(u_1) = w(u_0) = \ell(u_1) + \cdots + \ell(u_n) = n \cdot \overline{1} = \overline{0}.$$

This is impossible, so $K_{1,n}$ is not a $\mathbb{Z}_2$ −vertex-magic graph.

ii. Suppose n odd. Note that

$$w(u_1) = \cdots = w(u_n) = \ell(u_0) = w(u_0) = \ell(u_1) + \cdots + \ell(u_n) = n \cdot \overline{1} = \overline{1}.$$

Thus, $K_{1,n}$ is $\mathbb{Z}_2$ −vertex-magic graph.

■

Theorem 6 For every integer $k > 2$ and $n > 2$, $K_{1,n}$ is $\mathbb{Z}_k$ −vertex-magic graph.

Proof.

Consider a star graph $K_{1,n}$, with $n > 2$. Define a mapping $\ell: V(K_{1,n}) \to \mathbb{Z}_k - \{\overline{0}\}$ as $\ell(u_i) = \overline{a_i}$, for all $u_i \in V(K_{1,n})$ with $i = 1,2, \dots, n$ such that $\ell(u_0) = \sum_{i=1}^{n} \overline{a_i} \neq \overline{0}$. It is clear that

$$w(u_i) = \ell(u_0) = w(u_0) = \ell(u_1) + \ell(u_2) + \cdots + \ell(u_n) = \sum_{i=1}^{n} \overline{a_i},$$

for all $i = 1,2, \dots, n$. Thus, $K_{1,n}$ is $\mathbb{Z}_k$ −vertex-magic graph. ■

Note that for $n = 1$, then $K_{1,1} = P_2$ and for $n = 2$, then $K_{1,2} = P_3$. Based on Proposition 1, Proposition 2, and Theorem 6, we can determine the vertex-integer-magic set of star graphs as desired in the following:

Corollary 7 For every positive integer n, holds

$$VIM(K_{1,n}) = \begin{cases} \mathbb{N} \setminus \{1\}, & n \text{ odd}, \\ \mathbb{N} \setminus \{1,2\}, & n \text{ even}. \end{cases}$$

CONCLUSIONS

Based on the results, we obtain some conclusion. First, the path graphs P_n is not a $\mathbb{Z}_k$ −vertex-magic graph in general. Second, the complete graphs K_n is a $\mathbb{Z}_k$ −vertex-magic graph with constant labeling. Third, the cycle graphs C_n is a $\mathbb{Z}_k$ −vertex-magic graph and has only constant labeling for $4 \nmid n$. Last, the star graphs $K_{1,n}$ is a $\mathbb{Z}_k$ −vertex-magic graph, except for n even and $k = 2$. For further research, it can be investigated the group-vertex-magic labeling of graphs constructed from algebraic structures.

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