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On $\mathcal{H}$ Irregular Reflexive k-labeling of Graphs

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ABSTRACT

By an irregular reflexive k-labeling, we mean a function $f_e: E(G) \rightarrow \{1, 2, \dots, k_e\}$ and $f_v: V(G) \rightarrow \{0, 2, \dots, 2k_v\}$ such that $f(x) = f_v(x)$ if $x \in V(G)$ and $f(xy) = f_e(xy)$ if $xy \in E(G)$, where $k = \max\{k_e, 2k_v\}$. Let $\mathcal{H} \subset G$, the irregular reflexive k-labeling is called an $\mathcal{H}$-irregular reflexive k-labeling of graph G if every two different subgraphs $\mathcal{H}'$ and $\mathcal{H}"$ isomorphic to $\mathcal{H}$, it holds $\omega(\mathcal{H}') \neq \omega(\mathcal{H})$," where $\omega(\mathcal{H}) = \sum_{x\in V(\mathcal{H})} f_v(x) + \sum_{x\in E(\mathcal{H})} f_e(x)$. The minimum k for graph G which has an $\mathcal{H}$-irregular reflexive k-labeling, is called the reflexive $\mathcal{H}$ strength of graph G and denoted by $r\mathcal{H}s(G)$. In this paper, we initiate to study the lower bound of the reflexive $\mathcal{H}$ strength of graphs and the reflexive $\mathcal{H}$ strength of flower, $Shack(C_t, v, n)$, and book graph, where $\mathcal{H}$ isomorphic to C_3, C_t, and C_4, respectively.

Keywords: $\mathcal{H}$-irregular reflexive k-labeling; Reflexive $\mathcal{H}$ strength; Flower graph; Shackle graph; Book graph.

INTRODUCTION

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph labeling is an assignment of integers to elements of the graph, such as vertices, edges, faces, or sub-graphs, subject to certain conditions [20]. In the paper, certain conditions are constraints put on the weights of the graph element. The weight of a vertex is the sum of the labels of edges incident to the vertex together with the vertex label if it exists. The weight of an edge is the sum of the end vertices together with the edge label if it exists. For vertices u, v, edge uv, and labeling f the vertex weight, $\omega t(v)$ is defined as $\omega t(v) = f(v) + \sum_{\omega\epsilon N(v)} f(vw)$, while the edge weight is denoted $\omega t(uv) = f(u) + f(v) + f(uv)$. Labeling is considered *vertex irregular* if all vertices' weights are pairwise distinct and *edge irregular* if the edge weights are pairwise distinct [4].

Total labeling is a labeling of vertices and edges; a *total k-labeling* is a total labeling in which the largest label is k. The smallest k for which a total (vertex, edge) k-labelling exists is known as the total (vertex, edge) irregular strength of the graph [1]. Many

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results concerning vertex irregular and edge irregular total k-labelings can be found in the survey by Gallian [4].

For $\mathcal{H} \subseteq G$, the weight of $\mathcal{H}$is is given by $\omega t(\mathcal{H}) = \sum_{v\in V(\mathcal{H})} f(v) + \sum_{e\in E(\mathcal{H})} f(e)$. If there exists a partition of G such that every subgraph in the partition, $\mathcal{H}_i$ isomorphic, then the graph G admits an $\mathcal{H}$-covering. Note that each edge of $E(G)$ belongs to at least one of the sub-graphs $\mathcal{H}_i, i = 1, 2, \dots s$. The $\mathcal{H}$-irregular total k-labeling (φ) in which the sub-graph weights are pairwise distinct. For more on subgraph irregular total k-labelings, see [2, 3].

Irregular reflexive k-labelings arose from considering the labels as a function of the graph structure rather than arbitrary assignments. Irregular reflexive k-labelings were introduced by Tanna et al. [8]. Edge labels indicate parallel edges in a multi-graph, while vertex labels represent loops at a vertex. The principal differences to total irregular k-labelings are in the vertex labels. Since in irregular reflexive k-labeling, the vertex labels represent loops; then they must be even numbers since each loop contributes 2 to the degree of the vertex. In addition, the vertex label 0 is permitted as representing a loopless vertex. Edge labels do not permit the 0 labels, as this would just indicate the absence of an edge. Irregularity and weights of vertices of edges, subgraphs, etc, are the same as for total irregular labeling [6]. The smallest k for which a reflexive (vertex or edge) k-labelling exists is known as the irregular reflexive strength of the graph [15].

The formal definition of vertex irregular reflexive k-labeling can be seen in [7, 8]; they also provided some results on $rvs(G)$, the smallest value of k for which such labeling exists is called the reflexive vertex strength of the graph G, where G were prisms, wheels, fans, baskets, any graph with pendant vertex, sunlet graph, helm graph, subdivided star graph, and broom graph. While for a formal definition of edge irregular reflexive k-labeling can be seen in [8, 9, 10, 11, 12, 13, 14, 15]. They determined the lower bound lemma and some previous results of $res(G)$, where G were a star, double star $S_{n,n}$, caterpillar graphs, generalized subdivided star, broom, double star graph $S_{n,m}$, cycle, a cartesian product of cycles, join of cycle graphs and K_1, generalized friendship graphs, wheels, prisms, basket, and fan graphs, the disjoint union of Generalized Petersen graphs. Agustin *et al.* Also determined the reflexive edge strength of some almost regular graphs, namely ladder, triangular ladder, $P_n \times C_3, P_n \odot P_2$, and $P_n \odot C_3$.

Furthermore, we develop a new concept of irregular reflexive k-labeling, namely $\mathcal{H}$-irregular reflexive k-labeling for $\mathcal{H} \subset G$. We evaluate the sub-graph weights under irregular reflexive k-labeling. We initiate to study the lower bound of the reflexive $\mathcal{H}$ strength of graphs and the reflexive $\mathcal{H}$ strength of some graphs, namely flower graph, $Shack(C_t, v, n)$, and book graph. We studied the subgraph $\mathcal{H} \simeq C_3$on determining $r\mathcal{H}s(Fl_n)$, the subgraph $\mathcal{H} \simeq C_t$ on determining $r\mathcal{H}s\big(Shack(C_t, v, n)\big)$, and the subgraph $\mathcal{H} \simeq C_4$ on determining $r\mathcal{H}s(B_n)$. We can look at [5] for more details about shackle operation. The book graph is obtained by edge amalgamation product of C_4 with linked edge [19].

NEW CONCEPTS OF $\mathcal{H}$ IRREGULAR REFLEXIVE k-LABELING

We present a new definition for $\mathcal{H}$ irregular reflexive k-labeling as follows.

Definition 1. Let G be a graph under the function $f_e : E(G) \to \{1, 2, \dots, k_e\}$ and $f_v : V(G) \to \{0, 2, \dots, 2k_v\}$ such that $f(x) = f_v(x)$ if $x \in V(G)$ and $f(xy) = f_e(xy)$ if $xy \in E(G)$, where $k = \max\{k_e, 2k_v\}$. The irregular reflexive k-labeling is called an $\mathcal{H}$-irregular reflexive k-labeling of the graph G if every two different sub-graphs $\mathcal{H}'$ and $\mathcal{H}"$

isomorphic to $\mathcal{H}$, it holds $\omega(\mathcal{H}') \neq \omega(\mathcal{H})$," where $\omega(\mathcal{H}) = \sum_{x\in V(\mathcal{H})} f_v(x) + \sum_{x\in E(\mathcal{H})} f_e(x)$ for the subgraph $\mathcal{H} \subset G$. The minimum k for graph G, which has an $\mathcal{H}$-irregular reflexive k-labelling, is called the reflexive $\mathcal{H}$ strength of graph G and is denoted by $r\mathcal{H}s(G)$.

From the definition of $\mathcal{H}$-irregular reflexive k-labeling, we give the results of our research on $\mathcal{H}$-irregular reflexive k-labeling of graphs. Before prove our obtained theorems, we will show the lower bound of the reflexive $\mathcal{H}$ strength of any graph $r\mathcal{H}s$ in the following Lemma.

Lemma 1 Given that a graph $\mathcal{H} \subset G$. Let $p_{\mathcal{H}}, q_{\mathcal{H}}$ be, respectively number of vertices and edges of $\mathcal{H}$. Let $|\mathcal{H}|$ be the number of sub-graphs. The reflexive $\mathcal{H}$ strength satisfies

$$r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil.$$

Proof. Let G be a graph and $\mathcal{H} \subset G$, with $p_{\mathcal{H}}, q_{\mathcal{H}}$ be, respectively number of vertices and edges of $\mathcal{H}$. Let $|\mathcal{H}|$ be the number of subgraphs. Based on the definition of $\mathcal{H}$-irregular reflexive k-labeling, since we require k-minimum for graph G, which has an $\mathcal{H}$-irregular reflexive k-labeling, the set of an $\mathcal{H}$-weight should be consecutive, otherwise it will not give a minimum $r\mathcal{H}s$. Thus, the set of an $\mathcal{H}$-weight is $\omega(\mathcal{H}) = \{q_{\mathcal{H}}, q_{\mathcal{H}}+1, q_{\mathcal{H}}+2, \dots, q_{\mathcal{H}}+|\mathcal{H}|-1\}$. Since the minimum $k = max\{k_e, 2k_v\}$ is the reflexive $\mathcal{H}$ strength, then the maximum possible $\mathcal{H}$-weight of graph G is $p_{\mathcal{H}}(2k_v) + q_{\mathcal{H}}(k_e) \leq k(p_{\mathcal{H}} + q_{\mathcal{H}})$. It implies

$$\begin{aligned} k(p_{\mathcal{H}}+q_{\mathcal{H}}) &\geq q_{\mathcal{H}}+|\mathcal{H}|-1 \\ k &\geq \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \\ r\mathcal{H}s(G) &\geq \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \end{aligned}$$

Since $r\mathcal{H}s(G)$ should be an integer, and we need a sharpest lower bound, it implies

$$r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil.$$

It completes the proof. □

By the above lemma, we derive the following corollary of the lower bound.

Corollary 1 *Given that a graph* $\mathcal{H} \subset G$. Let $p_{\mathcal{H}}, q_{\mathcal{H}}$ be, respectively number of vertices and edges of $\mathcal{H}$ and $|\mathcal{H}|$ be the number of sub-graphs. If $q_{\mathcal{H}}+|\mathcal{H}|-1 = t(p_{\mathcal{H}}+q_{\mathcal{H}})$, then

$$r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil + 1.$$

where t is an odd natural number.

Proof. Given a graph $\mathcal{H} \subset G$. Let $p_{\mathcal{H}}, q_{\mathcal{H}}$ be the number of vertices and edges of $\mathcal{H}$, respectively, and $|\mathcal{H}|$ be the number of sub-graphs. Let $q_{\mathcal{H}}+|\mathcal{H}|-1 = t(p_{\mathcal{H}}+q_{\mathcal{H}})$, based on Lemma 1, we obtain.

$$r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}} + |\mathcal{H}| - 1}{p_{\mathcal{H}} + q_{\mathcal{H}}} \right\rceil = \left\lceil \frac{t(p_{\mathcal{H}} + q_{\mathcal{H}})}{p_{\mathcal{H}} + q_{\mathcal{H}}} \right\rceil = t$$

Furthermore, suppose t is $r\mathcal{H}s(G)$, then t is the largest label under any $\mathcal{H}$-irregular reflexive k-labeling on graph G. Let $\mathcal{H}_j$ be a sub-graph, such that the $\mathcal{H}$ weight of $\mathcal{H}_j$ is

$$\omega(\mathcal{H}_j) = \sum_{x \in V(\mathcal{H})} f_v(x) + \sum_{x \in E(\mathcal{H})} f_e(x) \leq (t-1)p_{\mathcal{H}} + t(q_{\mathcal{H}})$$

Based on Lemma 1, the largest $\mathcal{H}$ weight under any $\mathcal{H}$-irregular reflexive k-labeling is at least $q_{\mathcal{H}} + |\mathcal{H}| - 1$, such that $q_{\mathcal{H}} + |\mathcal{H}| - 1 \leq (t-1)p_{\mathcal{H}} + t(q_{\mathcal{H}})$. It is a contradiction with $q_{\mathcal{H}} + |\mathcal{H}| - 1 = t(p_{\mathcal{H}} + q_{\mathcal{H}})$. Hence, $t \neq r\mathcal{H}s(G)$, such that $r\mathcal{H}s(G) > \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil = \left\lceil \frac{t(p_{\mathcal{H}}+q_{\mathcal{H}})}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil = t$. Thus, it concludes that $r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil + 1$. □

METHODS

The following method is used to calculate a graph's distance reflexive strength:

1. Graphs are defined as data or research objects and their sub-graphs.
2. Determine vertex and edge sets of graphs.
3. Use the following lemma and corollary to find the lower bound on the reflexive $\mathcal{H}$ strength of a graph.
   Lemma: $r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil$ and Corollary: $r\mathcal{H}s(G) \geq \left\lceil \frac{q_{\mathcal{H}}+|\mathcal{H}|-1}{p_{\mathcal{H}}+q_{\mathcal{H}}} \right\rceil + 1,\ if\ q_{\mathcal{H}} + |\mathcal{H}| - 1 = t(p_{\mathcal{H}} + q_{\mathcal{H}})$.
4. Develop vertex and edge labels using the $\mathcal{H}$ irregular reflexive k-labeling definition.
5. Using the function obtained in step 4 to determine the upper bound of $r\mathcal{H}s(G)$.
6. The exact value of $r\mathcal{H}s(G)$ can be determined if its upper and lower bounds are both equal.
7. If the upper and lower bounds of $r\mathcal{H}s(G)$ are not equal, then step 4 can be repeated until the upper bound and lower bounds of $r\mathcal{H}s(G)$ are equal.

RESULTS AND DISCUSSION

We present the following theorems of several graphs with their respective sub-graphs.

Theorem 1 *Let* $\mathcal{H} \cong C_3$ is a sub graph of Fl_n and for every natural number $n \geq 4$, then

$$r\mathcal{H}s(Fl_n) = \begin{cases} \left\lceil \dfrac{n+2}{6} \right\rceil + 1, & \text{if } n \equiv 3, 4 (mod\ 12) \\ \left\lceil \dfrac{n+2}{6} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. Let $Fl_n, n \geq 4$, be a flower graph with the vertex set $V(Fl_n) = \left\{x_i^1, y_i^1; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil\right\} \cup \left\{x_i^2, y_i^2 : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\right\}$ and the edge set $E(Fl_n) = \left\{x_i^1 x_{i+1}^1, x^1_{\lceil \frac{n}{2} \rceil} x^2_{\lfloor \frac{n}{2} \rfloor} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - \right.$

$1\Big\} \cup \Big\{x_{i+1}^1 y_i^1, y_{\lceil\frac{n}{2}\rceil}^1 x_{\lceil\frac{n}{2}\rceil}^2 : 1 \le i \le \left\lceil\frac{n}{2}\right\rceil - 1\Big\} \cup \Big\{x_i^1 y_i^1 : 1 \le i \le \left\lceil\frac{n}{2}\right\rceil\Big\} \cup \Big\{x_i^2 y_i^2 : 1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor\Big\} \cup$ $\Big\{x_i^2 x_{i+1}^2, x_1^1 x_1^2 : 1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor - 1\Big\} \cup \Big\{x_i^2 y_{i+1}^2, x_1^1 y_1^2 : 1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor - 1\Big\}$.

Based on Corollary 1, we have $r\mathcal{H}s(Fl_n) \ge \left\lceil\frac{n+2}{6}\right\rceil + 1$, if $n \equiv 3, 4 \pmod{12}$ and $r\mathcal{H}s(Fl_n) \ge \left\lceil\frac{n+2}{6}\right\rceil$, otherwise. To show that k is an upper bound for the $\mathcal{H}$-irregular reflexive k-labeling of Fl_n, we define:

Case 1: For every even $n \ge 4$, the graph Fl_n has the vertex labeling as follows:

$$f_v(x_i^1) = \begin{cases} 0, & \text{if } 1 \le i \le 2 \\ 2\left\lceil\dfrac{i-2}{6}\right\rceil, & \text{if } 3 \le i \le \dfrac{n}{2} \end{cases}$$

$$f_v(x_i^2) = \begin{cases} 0, & \text{if } i = 1 \\ 2\left\lceil\dfrac{i-1}{6}\right\rceil, & \text{if } 2 \le i \le \dfrac{n}{2} \end{cases}$$

$$f_v(y_i^1) = f_v(y_i^2) = \begin{cases} 0, & \text{if } 1 \le i \le 3 \\ 2\left\lceil\dfrac{i-3}{6}\right\rceil, & \text{if } 4 \le i \le \dfrac{n}{2} \end{cases}$$

For the edge labeling of Fl_n, suppose $\Big\{x_i^1 x_{i+1}^1 = e_i^1, x_{\frac{n}{2}}^1 x_{\frac{n}{2}}^2 = e_{\frac{n}{2}}^1, x_{i+1}^1 y_i^1 = s_i^1, y_{\frac{n}{2}}^1 x_{\frac{n}{2}}^2 = s_{\frac{n}{2}}^1 : 1 \le i \le \frac{n}{2} - 1\Big\}$ and $\Big\{x_i^2 x_{i+1}^2 = e_{i+1}^2, x_1^1 x_1^2 = e_1^2, x_i^2 y_{i+1}^2 = s_{i+1}^2, x_1^1 y_1^2 = s_1^2 : 1 \le i \le \frac{n}{2} - 1\Big\}$, then we have

$$f_e(e_i^1) = f_e(e_i^2) = \begin{cases} 2t-1, & \text{if } 6t-5 \le i \le 6t-2 \\ 2t, & \text{if } 6t-1 \le i \le 6t \end{cases}$$

$$f_e(s_i^1) = \begin{cases} 2t-1, & \text{if } 6t-5 \le i \le 6t-2 \\ 2t, & \text{if } i = 6t-1 \\ 2t+1, & \text{if } i = 6t \end{cases}$$

$$f_e(x_i^1 y_i^1) = \begin{cases} 2t-1, & \text{if } 6t-5 \le i \le 6t-1 \\ 2t, & \text{if } i = 6t \end{cases}$$

$$f_e(s_i^2) = \begin{cases} 1, & \text{if } 1 \le i \le 4 \\ 2, & \text{if } i = 5 \\ 3, & \text{if } i = 6 \\ 2t+2, & \text{if } 6t+1 \le i \le 6t+4 \\ 2t+3, & \text{if } i = 6t+6 \end{cases}$$

$$f_e(x_i^2 y_i^2) = \begin{cases} 2, & \text{if } 1 \le i \le 5 \\ 3, & \text{if } i = 6 \\ 2t+1, & \text{if } 6t+1 \le i \le 6t+4 \\ 2t+2, & \text{if } i = 6t+5 \\ 2t+3, & \text{if } i = 6t+6 \end{cases}$$

Case 2: For every odd $n \ge 4$, the graph Fl_n has the vertex labeling as follows:

$$f_v(x_i^1) = \begin{cases} 0, & \text{if } 1 \le i \le 2 \\ 2\left\lceil\dfrac{i-2}{6}\right\rceil, & \text{if } 3 \le i \le \dfrac{n+1}{2} \end{cases}$$

$$f_v(x_i^2) = \begin{cases} 0, & \text{if } i = 1 \\ 2, & \text{if } 2 \leq i \leq 6 \\ 2\left\lceil \frac{i-6}{6} \right\rceil + 2, & \text{if } 7 \leq i \leq \frac{n-1}{2} \end{cases}$$

$$f_v(y_i^1) = \begin{cases} 0, & \text{if } 1 \leq i \leq 3 \\ 2\left\lceil \frac{i-3}{6} \right\rceil, & \text{if } 4 \leq i \leq \frac{n+1}{2} \end{cases}$$

$$f_v(y_i^2) = \begin{cases} 0, & \text{if } 1 \leq i \leq 3 \\ 2, & \text{if } 4 \leq i \leq 8 \\ 2\left\lceil \frac{i-8}{6} \right\rceil + 2, & \text{if } 9 \leq i \leq \frac{n-1}{2} \end{cases}$$

For the edge labeling of Fl_n, suppose $\left\{x_i^1 x_{i+1}^1 = e_i^1, x_{\left\lceil \frac{n}{2} \right\rceil}^1 x_{\left\lfloor \frac{n}{2} \right\rfloor}^2 = e_{\left\lceil \frac{n}{2} \right\rceil}^1, x_{i+1}^1 y_i^1 = s_i^1, y_{\left\lceil \frac{n}{2} \right\rceil}^1 x_{\left\lfloor \frac{n}{2} \right\rfloor}^2 = s_{\left\lceil \frac{n}{2} \right\rceil}^1 : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1\right\}$ and $\left\{x_i^2 x_{i+1}^2 = e_{i+1}^2, x_1^1 x_1^2 = e_1^2, x_i^2 y_{i+1}^2 = s_{i+1}^2, x_1^1 y_1^2 = s_1^2 : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\right\}$, then we have

For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$,

$$f_e(e_i^1) = \begin{cases} 2t-1, & \text{if } 6t-5 \leq i \leq 6t-2 \\ 2t, & \text{if } 6t-1 \leq i \leq 6t \end{cases}$$

$$f_e(s_i^1) = \begin{cases} 2t-1, & \text{if } 6t-5 \leq i \leq 6t-2 \\ 2t, & \text{if } i = 6t-1 \\ 2t+1, & \text{if } i = 6t \end{cases}$$

$$f_e(x_i^1 y_i^1) = \begin{cases} 2t-1, & \text{if } 6t-5 \leq i \leq 6t-1 \\ 2t, & \text{if } i = 6t \end{cases}$$

For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,

$$f_e(e_i^2) = \begin{cases} 1, & \text{if } 1 \leq i \leq 4 \\ 2, & \text{if } i = 5 \\ 2t, & \text{if } 6t \leq i \leq 6t+3 \\ 2t+1, & \text{if } i = 6t+4 \\ 2t+2, & \text{if } i = 6t+5 \end{cases}$$

$$f_e(s_i^2) = \begin{cases} 1, & \text{if } 1 \leq i \leq 4 \\ 2, & \text{if } i = 5 \\ 2t+1, & \text{if } 6t \leq i \leq 6t+3 \\ 2t+2, & \text{if } 6t+4 \leq i \leq 6t+5 \end{cases}$$

$$f_e(x_i^2 y_i^2) = \begin{cases} 2, & \text{if } 1 \leq i \leq 5 \\ 2t+1, & \text{if } 6t \leq i \leq 6t+4 \\ 2t+2, & \text{if } i = 6t+5 \end{cases}$$

The function f_e is a periodic function with variable t. At each value of positive integer t, $t = 1, 2, \ldots$, the function f_e is used to give labels on edges of Fl_n.

Then, we get the $\mathcal{H}$-weights:

$$\omega_1 = f_v(x_i^1) + f_v(x_{i+1}^1) + f_v(y_i^1) + f_e(e_i^1) + f_e(x_i^1 y_i^1) + f_e(s_i^1)$$

$$\omega_2 = f_v(x_i^2) + f_v(x_{i+1}^2) + f_v(y_i^2) + f_e(e_i^2) + f_e(x_i^2 y_i^2) + f_e(s_i^2)$$
$$\omega = \omega_1 \cup \omega_2$$

Furthermore, for each value of $t(t = 1, 2, \dots)$, we will obtain the $\mathcal{H}$-weights of Fl_n as follows.

If $Fl_n, n \equiv 0(\text{mod } 12), n \geq 12$, we have
$$\omega = \{3, 4, 5, 6 \dots, 12t + 2\}$$
If $Fl_n, n \equiv 11(\text{mod } 12), n \geq 12$, we have
$$\omega = \{3, 4, 5, 6 \dots, 12t + 1\}$$
The $\mathcal{H}$ weights from the W set are clearly different numbers. Thus, it gives an $\mathcal{H}$-irregular reflexive k-labeling of $Fl_n, n \geq 4$. □

Theorem 2 *Let $\mathcal{H} \cong C_t$ is a sub-graph of G = Shack(C_t, v, n) and for every natural number n ≥ 2, then*
$$rHs(G) = \begin{cases} \left\lceil \dfrac{t+n-1}{2t} \right\rceil + 1, & \text{if } n \equiv 2,3,\dots, t+1 (mod\ 4t) \\ \left\lceil \dfrac{t+n-1}{2t} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. Let G = *Shack*(C_t, v, n), $n \geq 2$, be a graph with the vertex set $V(G) = \{x_i; 0 \leq i \leq n\} \cup \{y_{i,j}: 1 \leq i \leq n, 1 \leq j \leq t-2\}$ and the edge set $E(G) = \{e_{i,j}: 1 \leq i \leq n, 1 \leq j \leq t\}$. Based on Corollary 1, we have $r\mathcal{H}s(G) \geq \left\lceil \frac{t+n-1}{2t} \right\rceil + 1$, if $n = 2,3,4,\cdots, t+1\ (mod\ 4t)$ and $r\mathcal{H}s(G) \geq \left\lceil \frac{t+n-1}{2t} \right\rceil$, otherwise. To show that k is an upper bound for the $\mathcal{H}$-irregular reflexive k-labeling of G, we define:

For $1 \leq i \leq n$,
$$f_v(x_0) = 0$$
$$f_v(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq t+1 \\ 2, & \text{if } t+2 \leq i \leq 4t+1 \\ 2\left\lceil \dfrac{i-(4t+1)}{4t} \right\rceil + 2, & \text{if } 4t+2 \leq i \leq n \end{cases}$$
We define the function in the following for $y_{i,j}$, and $e_{i,j}$.
$$h_1(a_{i,j}) = \begin{cases} 0, & \text{if } i > j \\ 2, & \text{otherwise} \end{cases}$$
$$h_2(a_{i,j}) = \begin{cases} 1, & \text{if } i > j \\ 2, & \text{otherwise} \end{cases}$$
$$h_3(a_{i,j}) = \begin{cases} 2, & \text{if } i > j \\ 1, & \text{otherwise} \end{cases}$$
For $1 \leq j \leq t-2$, then we have,
$$f_v(y_{i,j}) = \begin{cases} 0, & \text{if } 1 \leq i \leq t+1 \\ 2, & \text{if } t+2 \leq i \leq t+3 \\ h_1(a_{i,j}), & \text{if } t+4 \leq i \leq 2t+1 \\ 2, & \text{if } 2t+2 \leq i \leq 4t+1 \\ 2l, & \text{if } (4t)l+2 \leq i \leq (4t)l+3 \\ h_1(a_{i,j}) \oplus 2l, & \text{if } (4t)l+4 \leq i \leq (4t)l+t+1 \\ 2l+2, & \text{if } (4t)l+t+2 \leq i \leq (4t)l+4t+1 \end{cases}$$
For $1 \leq j \leq t$, then we have,

$$f_e(e_{i,j}) = \begin{cases} 1, & \text{if } i = 1 \\ h_2(a_{i,j}), & \text{if } 2 \le i \le t+1 \\ h_3(a_{i,j}), & \text{if } t+2 \le i \le 2t+1 \\ h_2(a_{i,j}), & \text{if } 2t+2 \le i \le 3t+1 \\ h_2(a_{i,j}) \oplus 1, & \text{if } 3t+2 \le i \le 4t+1 \\ h_3(a_{i,j}) \oplus (2l-1), & \text{if } (4t)l+2 \le i \le (4t)l+t+1 \\ h_2(a_{i,j}) \oplus (2l-1), & \text{if } (4t)l+t+2 \le i \le (4t)l+2t+1 \\ h_2(a_{i,j}) \oplus (2l), & \text{if } (4t)l+2t+2 \le i \le (4t)l+3t+1 \\ h_2(a_{i,j}) \oplus (2l+1), & \text{if } (4t)l+3t+2 \le i \le (4t)l+4t+1 \end{cases}$$

The function f_v and f_e is a periodic function with variable *l.* At each value of positive integer *l, l* = 1,2,..., the function f_v and f_e is used to give labels on vertices and edges of *G= Shack*(C_t, v, n)*.* The function f_e with *l* = 1 is used to give labels on vertices and edges of *G= Shack*(C_t, v, n) for $n \equiv 1(mod\ 4t)$ where n is a positive integer $n > 8t + 1$.
Then, we get the $\mathcal{H}$-weights:

$$W = f_v(x_i) + f_v(x_{i+1}) + f_v(v_{i,j}) + f_e(e_{i,j})$$

For each value of l ($l = 1,2,\cdots$), we will obtain the $\mathcal{H}$-weghts of *G= Shack*(C_t, v, n) for $n \equiv 1(mod\ 4t), n \ge 8t + 1$ such that

$$W = \{t, t+1, t+2, \cdots, (4l+5)t\}$$

The $\mathcal{H}$ weights from the W set are clearly different numbers. This, it gives an $\mathcal{H}$-irregular reflexive k-labeling of *G = Shack*(C_t, v, n)*, n* ≥ 2. □

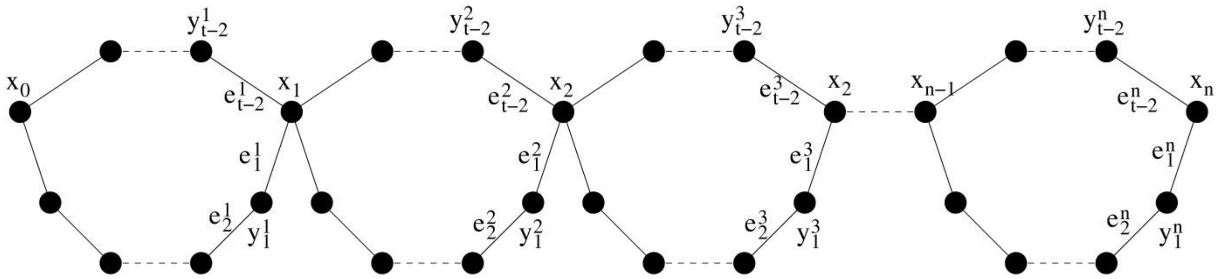

Figure 1. The illustration of $Shack(C_t, v, n)$.

Figure 1 is the illustration of $Shack(C_t, v, n)$, there are n cycles of graph C_t with the linkage vertex is x_i with $1 \le i \le n-1$.

Theorem 3 *Let* $\mathcal{H} \cong C_4$ *be a sub-graph of book graph* B_n *and for every natural number n ≥ 2,*

$$r\mathcal{H}s(B_n) = \begin{cases} \left\lceil \frac{n+2}{5} \right\rceil + 1, & \text{if } n \equiv 2,3(mod\ 10) \\ \left\lceil \frac{n+2}{5} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. The graph $B_n, n \ge 2$, is a book graph with the vertex set $V(B_n) = \{A, B, u_i, v_i: 1 \le i \le n\}$ and the edge set $E(F_n) = \{AB, Au_i, Bv_i, u_iv_i: 1 \le i \le n\}$ with sub graph $\mathcal{H} \simeq C_4$. Edge AB is the edge that is always be part of each sub graph. Since we can give label 0 on vertex A and B*,* label 1 on its edge, we can exclude this edge from determining the lower bound of $r\mathcal{H}s(B_n)$. If n ≡ 2,3(mod 10), then we refer to Corollary 1 to determine the lower bound of $r\mathcal{H}s(B_n)$, such that $r\mathcal{H}s(B_n) \geq \left\lceil\frac{q\mathcal{H}+|\mathcal{H}|-1}{P\mathcal{H}}\right\rceil + 1 = \left\lceil\frac{3+n-1}{5}\right\rceil + 1 = \left\lceil\frac{n+2}{5}\right\rceil +$ 1. For n otherwise, we have $r\mathcal{H}s(F_n) \geq \left\lceil\frac{q\mathcal{H}+|\mathcal{H}|-1}{P\mathcal{H}}\right\rceil = \left\lceil\frac{3+n-1}{5}\right\rceil = \left\lceil\frac{n+2}{5}\right\rceil$. Otherwise, to show upper bound for the $\mathcal{H}$-irregular reflexive k-labeling of B_n,we define:

$$f_v(A) = f_v(B) = 0$$

For $n \in \{2,3,4,5,6,7,8,9,10,11,12\}$, we refer to Table 1 to give labels on vertices and edges of F_n.

Table 1. LABELING OF VERTICES AND EDGES OF $B_n : n \in \{2,3,4,5,6,7,8,9,10,11,12\}$.

i	1	2	3	4	5	6	7	8	9	10	11	12
u_i	0	0	0	0	0	2	2	2	2	2	2	4
v_i	0	0	0	0	0	0	2	2	2	2	2	2
Au_i	1	2	2	2	2	2	2	2	3	3	3	3
Bv_i	1	1	1	2	2	2	2	2	2	2	3	3
u_iv_i	1	1	2	2	1	2	1	2	2	3	3	2

For $n \geq 13$ and $1 \leq i \leq n$, we have

$$f_v(u_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 4 \\ 2, & \text{if } 5 \leq i \leq 11 \\ 2\left\lceil\frac{i-11}{10}\right\rceil + 2, & \text{if } 12 \leq i \leq n \end{cases}$$

$$f_v(v_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 6 \\ 2, & \text{if } 7 \leq i \leq 13 \\ 2\left\lceil\frac{i-13}{10}\right\rceil + 2, & \text{if } 14 \leq i \leq n \end{cases}$$

$$f_e(Au_i) = \begin{cases} 1, & \text{if } i = 1 \\ 2, & \text{if } i = 2,3,4,5 \\ 2\left\lceil\frac{i-5}{10}\right\rceil, & \text{if } i \equiv 6,7,8 \ (mod\ 10) \\ 2\left\lceil\frac{i-9}{10}\right\rceil + 1, & \text{if } i \equiv 0,1,2,3,4,5,9 \ (mod\ 10) \end{cases}$$

$$f_e(u_iv_i) = \begin{cases} 1, & \text{if } i = 1,2,5,7 \\ 2, & \text{if } i = 3,4,6,8,9 \\ 2\left\lceil\frac{i-9}{10}\right\rceil, & \text{if } i \equiv 2,4 \ (mod\ 10) \\ 2\left\lceil\frac{i-9}{10}\right\rceil + 1, & \text{if } i \equiv 0,1,3,5,6 \ (mod\ 10) \\ 2\left\lceil\frac{i-9}{10}\right\rceil + 2, & \text{if } i \equiv 7,8,9 \ (mod\ 10) \end{cases}$$

$$f_e(Bv_i) = \begin{cases} 1, & \text{if } i = 1,2,3 \\ 2, & \text{if } i = 4,5,6,7 \\ 2\left\lceil\frac{i-7}{10}\right\rceil, & \text{if } i \equiv 0,8,9 \ (mod\ 10) \\ 2\left\lceil\frac{i-7}{10}\right\rceil + 1, & \text{if } i \equiv 1,2,3,4,5,6,7 \ (mod\ 10) \end{cases}$$

Then, we get the $\mathcal{H}$-weights.

$$\omega(\mathcal{H}_t) = t + 3 : 1 \leq t \leq n$$

The $\mathcal{H}$-weights from the set $\omega(\mathcal{H}_t)$ are clearly distinct numbers. This, it gives an $\mathcal{H}$-irregular reflexive k-labeling of $B_n, n \geq 2$. □

CONCLUSIONS

We have obtained the lower bound of the reflexive $\mathcal{H}$strength of graphs and the reflexive $\mathcal{H}$strength of some graphs, namely flower graph, $Shack(C_t, v, n)$, and book graph. We studied the sub graph $\mathcal{H} \simeq C_3$ on determining $r\mathcal{H}s(Fl_n)$, the sub graph $\mathcal{H} \simeq C_t$ on determining $r\mathcal{H}s\big(Shack(C_t, v, v)\big)$, and the sub graph $\mathcal{H} \simeq C_4$ on determining $r\mathcal{H}s(B_n)$. We consider our paper has contributed a new novel to the study of the reflexive strength of graphs. However, characterising the exact value of reflexive $\mathcal{H}$-strength for any graph G remains an open problem. Since obtaining the reflexive $\mathcal{H}$-strength of a graph is considered to be an NP-complete problem. As a result, we propose the following open problems.

1. Determine the exact value of reflexive $\mathcal{H}$strenght of any graphs with different subgraphs and characterize the existence of graph for specific r$\mathcal{H}$s(G).
2. Determine the sharper upper bound reflexive $\mathcal{H}$strenght of any graphs with any sub-graph.

REFERENCES

[1] Ali Ahmad and M. Bača. 2013. On Vertex Irregular Total Labelings. *Ars Combinatoria,* **112**:129-139.

[2] F. Ashraf, M. Bača, M. Lascsáková and A. Semaničova-Feňovčıková. 2017. On H-Irregularity Strenght of Graphs. *Discussiones Mathematicae: Graph Theory,* **37**:1067-1078.

[3] Ika Hesti Agustin1, Dafik, Marsidi and Ermita Rizki Albirri, On the total H-irregularity stenght of graph: A new nation, *J. Phys,: Conf. Ser.*885 012004, 2017

[4] Joseph A. Gallian. 2017. A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics,* **20**:1-432.

[5] T. K. Maryati, E. T. Baskoro, A. N. M. Salman, and J. Ryan. 2010. On H-Supermagic Labelings for Certain Shackles and Amalgamations of a Connected Graph. *Utilutas Mathematica,* 1-10.

[6] D. Tanna, J. Ryan, and A. Semaničova-Feňovčıková. 2017. Reflexive Edge Irregular Labelling of Prisms and Wheels. *Australas. J. Combin,* **69**:394-401.

[7] Agustin I.H, Imam Utoyo, Dafik, M. Venkatachalam, and Surahmat. 2020. On the construction of the reflexive vertex k-labeling of any graph with pendant vertex. *International Journal of Mathematics and Mathematical Sciences.* Volume 2020, Article ID 7812812. https://doi.org/10.1155/2020/7812812.

[8] Tanna, D., J. Ryan, A. Semaničova-Feňovčıková dan M. Bača. 2018. Vertex Irregular Reflexive Lebeling of Prisms and Wheels. *AKCE International Journal of Graphs and Combinatorics.*

[9] Ika Hesti Agustin, Imam Utoyo, Dafik, M. Venkatachalam. 2020. Edge irregular reflexive labeling of some tree graphs. *Journal of Physics: Conference Series.* Volume 1543. https:10.1088/1742-6596/1543/1/012008.

[10] M. Bača, M. Irfan, J. Ryan, A. Semaničova-Feňovčıková, D. Tanna, 2017. On Edge

Irregular Reflexive Labelling for the Generalized Friendship Graphs. *Mathematics.* Volume 567. https:10.3390/math5040067.

[11] D. Tanna, J. Ryan, A. Semaničova-Feňovčıková. 2017. Edge Irregular Reflexive Labeling of Prisms and Wheels. *Australasian Journal of Combinatorics.* Volume 69(3).

[12] J. L. G. Guirao, S. Ahmad, M.K.Siddiqui, M. Ibrahim. 2018. Edge Irregular Reflexive Labeling for Disjoint Union of Generalized Petersen Graph. *Mathematics.* Volume 6(304). https:10.3390/math61203004.

[13] M. Bača, M. Irfan, J. Ryan, A. Semaničova-Feňovčıková, D. Tanna. 2019. Note on edge irregular reflexive labelings of graphs. *AKCE International Journal of Graphs and Combinatorics.* Volume 16. https:doi.org/10.1016/j.akcej.2018.01.013.

[14] M. Ibrahim, S. Majeed, M.K. Siddiqui. 2020. Edge Irregular Reflexive Labeling for Star, Double Star and Caterpillar Graphs. *TWMS J. App. and Eng. Math.* Volume 10(3). https:10.3390./math6120304.

[15] Ika Hesti Agustin, Dafik, Imam Utoyo, Slamin, M. Venkatachalam. 2021. The Reflexive Edge Strenght on Some Almost Regular Graphs. *Heliyon* Volume 7. DOI:https://doi.org/10.1016/j.heliyon.2021.e06991.

[16] I.H. Agustin, L. Susilowati, Dafik, I.N. Cangul, and N. Mohanapriya, On the vertex irregular reflexive labeling of several regular and regular-like graphs, J. Discrete Math. Sci. Crytogr 25 (2022), no. 5, 1457-1473.

[17] I.H. Agustin, M.I. Utoyo, Dafik, and M. Venkatachalam, The vertex irregular reflexive labeling of some almost regular graph, Palest. J. Math. 10 (2021), no. Special Issue II, 83-91.

[18] R. Alfarisi, J. Ryan, M.K. Siddiqui, Dafik, and I.H. Agustin, Vertex irregular reflexive labeling of disjoint union of gear and book graphs, Asian-Eur. J. Math. 14 (2021), no. 5, Article ID:2150078.

[19] D. E. Nurvazly and K A Sugeng, Graceful Labelling of Edge Amalgamation of Cycle Graph, IOP Conf. Series: Journal of Physics: Conf. Series 1108 (2018) 012047

[20] Agustin, I. H. Dafik, Marsidi and Albirri, E. R. 2017. "On the total H-irregularity strength of graphs: A new notion", J. Phys.: Conf. Ser.855 012004.