Order Ideals on Direct Sum of Three Ordered Abelian Groups under Lexicographic order

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ABSTRACT

Order ideal is one of important concepts in abstract algebra, especially in the study of ordered groups. In the current paper, we investigate order ideals in lexicographic direct sums of totally ordered Abelian groups. We begin by examining the order ideals in $\mathbb{Z}$ and $\mathbb{R}$. It is shown that there are no non-trivial order ideals in both groups. Next, we revisit the order ideals in the lexicographic direct sum of $\mathbb{Z}$ which denoted by $\mathbb{Z} \oplus_{lx} \mathbb{Z}$. Furthermore, our study extends to the direct sum of three totally ordered Abelian groups: $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. We investigate the non-trivial order ideals in these structures. It will be shown that the non-trivial order ideals of $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ are only $0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z})$. Furthermore, the non-trivial order ideals of $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ are only $(0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ and $(0 \oplus_{lx} 0) \oplus_{lx} \mathbb{R}$.

Keywords: lexicographic order; order ideal; direct sum

INTRODUCTION

For about 70 years, researchers have investigated the structure of ordered sets, especially ordered semigroups and ordered groups. Many aspects that have been studied recently regarding ordered semigroups are positive cone, soft ideal, in-soft ideal, C-ideals [1]-[4]. As for ordered groups, several researchers studied the properties of strong ordered groups [5] and cylically ordered groups [6]-[9]. Meanwhile, some of them investigate the connection between lattice ordered Abelian groups and Riesz spaces [10] and the connection between theory of ordered group and its topological aspects [11], [12].

In general, ordered sets are divided into two categories, namely total and non-total. In this case, the characteristic of the order plays an important role for total property. For example, in the context of a set whose members are ordered pairs, the lexicographical order often gives the total condition of the set [13].

One of interesting concept in ordered groups is order ideal which firstly introduced by Frink in his article [14] and has become one of significant concepts in abstract algebra. In [15] it was discussed the order ideal of ordered groups. Subsequent studies related to ordered groups, such as [16]-[20], and most recently [21], [22], the order ideal becomes one of the significant concepts.

Suppose that the Abelian group $(\Gamma, +, \leq)$ be totally ordered with the positive cone Γ^+. A subgroup I of Γ is called order ideal if I preserves the order, that is if $a \in \Gamma^+, b \in I^+$ with

$a \leq b$, then $a \in I$. The set $\{0_\Gamma\}$ and Γ itself are trivial order ideals of Γ. The collection $\sum(\Gamma)$ consists of all order ideals of Γ is totally ordered via inclusion [18].

Suppose G_1 and G_2 are ordered sets. In [23], the lexicographic order in $G_1 \times G_2$ is defined as follows:

$$(x_1, x_2) \leq_{lx} (y_1, y_2) \text{ if and only if } x_1 < y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \leq y_2).$$

If G_1 and G_2 partially ordered Abelian groups, then we denote $G_1 \oplus G_2$ for the direct sum of G_1 and G_2. By using the lexicographic order, the direct sum becomes a partially ordered Abelian group called the lexicographic direct sum denoted by $G_1 \oplus_{lx} G_2$ [24]. Furthermore, if both of G_1 and G_2 are total, then their lexicographic direct sum, $G_1 \oplus_{lx} G_2$, is total too.

METHODS

Our research focused on studying the order ideals of totally ordered Abelian groups. The study proceeded through several stages, starting with a comprehensive examination of the basic concepts of groups, ordering, ordered groups, and order ideals of ordered groups. This provided a solid theoretical foundation for further stages of the research. Subsequently, the research delved into the study of lexicographic order and lexicographic direct sum.

In the next step, we specifically examined the case of order ideals in the totally ordered Abelian group $\mathbb{Z} \oplus_{lx} \mathbb{Z}$, which was previously studied in [19]. It was discovered that this group possesses only one non-trivial order ideal. Further investigations were carried out on the order ideals of the Abelian groups $\mathbb{Z}$ and $\mathbb{R}$, which were found to lack non-trivial order ideals. Based on these findings, the study extended its focus to the order ideals of $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$.

RESULTS AND DISCUSSION

Order Ideals of $\mathbb{Z}$ and $\mathbb{R}$

Our first observation on totally ordered Abelian group suggests that its order ideal must be order-preserving. Consider the group $(\mathbb{Z}, +, \leq)$. Every subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, where $n \in \mathbb{Z}$. It's clear that for $n = 0$ and $n = \pm 1$, the subgroups $n\mathbb{Z}$ are trivial and also order ideals of $\mathbb{Z}$. The following lemma will show the non-trivial subgroups of $\mathbb{Z}$. Note that $n\mathbb{Z} = -n\mathbb{Z}$, so it will be assumed that $n > 0$. Lemma 1 and Corollary 2 are part of folklores.

Lemma 1.
For $n \in \mathbb{N} - \{1\}, n\mathbb{Z}$ is not an order ideal of $\mathbb{Z}$.
Proof. Consider the fact that $0 < n - 1 < n$, but $n - 1 \notin n\mathbb{Z}$. In other words, $n\mathbb{Z}$ does not preserve the order. Therefore, for $n \in \mathbb{N} - \{1\}, n\mathbb{Z}$ is not an order ideal of $\mathbb{Z}$. ∎
Thus, every non-trivial subgroup of $\mathbb{Z}$ is not an order ideal. Therefore, the following result is obtained.

Corollary 2.
The totally ordered Abelian group $(\mathbb{Z}, +, \leq)$ does not have a non-trivial order ideal.
We observed that similar result is obtained for the real numbers $\mathbb{R}$.

Lemma 3.
The totally ordered Abelian group $(\mathbb{R}, +, \leq)$ does not have a non-trivial order ideal.
Proof. Let $\{0\} \neq H < \mathbb{R}$. Assuming H is an order ideal of it, for every $x \in \mathbb{R}^+$ and $u \in H^+$ with $0 \leq x \leq u$, we have $x \in H$. Since H is a group, we have $0 \in H$ and

$-u \in H$. Consequently, we get $(-u, u) \subseteq H$. Since $(-u, u) \subseteq H$ for every $u \in H^+$, but H is not bounded, so we get $(-\infty, \infty) \subseteq H$. This implies that $\mathbb{R} \subseteq H$, which contradicts the initial assumption of H being a proper subset of $\mathbb{R}$. Therefore, H cannot be an order ideal of $\mathbb{R}$. ∎

Order Ideals of $\mathbb{Z} \oplus_{lx} \mathbb{Z}$

Let Γ be the group $\mathbb{Z} \oplus_{lx} \mathbb{Z}$. It is clear that $0 \oplus_{lx} 0$ and Γ itself are trivial order ideals of Γ. Furthermore, in this section, it will be shown that $0 \oplus_{lx} \mathbb{Z}$ is the only non-trivial order ideal of Γ(Rosjanuardi, 2012). To establish this, let I be defined as $0 \oplus_{lx} \mathbb{Z}$. It is clear that $I \subset \Gamma$. Now, suppose $(0, a), (0, b) \in I$. This implies that $(0, a) + (0, -b) = (0, a - b) \in I$, and making I a subgroup of Γ. Moreover, let $(x, y) \in \Gamma^+$ and $(0, a) \in I^+$, such that $(0,0) \leq_{lx} (x, y) \leq_{lx} (0, a)$, then it must be that $x = 0$. Consequently, we have $(x, y) = (0, y) \in I$. Hence, I is an order ideal. Assuming there exists another non-trivial order ideal apart from I, let's say this order ideal is J. Since $I \neq J$, it must be either $I \subset J$ or $I \supset J$.

Case I: $J \subset I$.

If $J \subset I$, then $J = 0 \oplus_{lx} n\mathbb{Z}$ for some $1 < n \in \mathbb{Z}$. Let $(0, n - 1) \in \Gamma^+$ and $(0, n) \in J^+$. It is clear that $(0,0) \leq_{lx} (0, n - 1) \leq_{lx} (0, n)$, but $(0, n - 1) \notin J$. As a result, J is not an order ideal.

Case II: $J \supset I$.

If $J \supset I$, then $J = n\mathbb{Z} \oplus_{lx} \mathbb{Z}$ for some $1 < n \in \mathbb{Z}$. Let $(n - 1, 1) \in \Gamma^+$ and $(n, 1) \in J^+$. It is clear that $(0,0) \leq_{lx} (n - 1, 1) \leq_{lx} (n, 1)$, but $(n - 1, 1) \notin J$. As a result, J is not an order ideal.

The obseravtion above shows that J is not an order ideal of Γ, contradicting the initial assumption that J is a non-trivial order ideal of Γ. Therefore, it must $J = I$. In other words, there are no other non-trivial order ideals of Γ besides I. Hence, it can be inferred that $0 \oplus_{lx} \mathbb{Z}$ is the only non-trivial order ideal of $\mathbb{Z} \oplus_{lx} \mathbb{Z}$.

Order Ideals of $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$

In the previous section, it has been shown that both groups $\mathbb{Z}$ and $\mathbb{R}$ don't possess any non-trivial order ideals. Furthermore, we will show all the order ideals in the $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$. Let us define Γ as $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$, and it is clear that $0 \oplus_{lx} (0 \oplus_{lx} 0)$ and Γ itself are order ideals of Γ. Next, the following two lemmas will present non-trivial order ideals of Γ.

Lemma 4.

Let Γ be defined as $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$. Then, $0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ is an order ideal of Γ.

Proof. Let $I_1 := 0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z}) = \{(0, m) \mid m \in \mathbb{Z} \oplus_{lx} \mathbb{Z}\}$ and $(0, m_1), (0, m_2) \in I_1$. Consider that

$$(0, m_1) + (0, -m_2) = (0, m_1 - m_2) \in I_1,$$

thus I_1 is a subgroup of Γ. Moreover, suppose $(a, b) \in \Gamma^+$ and $(0, m) \in I_1^+$ such that

$$(0,0) \leq_{lx} (a, b) \leq_{lx} (0, m),$$

then it must be that $a = 0$, and we have $(a, b) = (0, b) \in I_1$. Therefore, it can be concluded that $0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ is an order ideal of $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$. ∎

Lemma 5.

Let Γ be defined as $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$. Then, $0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z})$ is an order ideal of Γ.

Proof. Let $I_2 := 0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z}) = \{(0, (0, c)) \mid c \in \mathbb{Z}\}$ and $(0, (0, c_1)), (0, (0, c_2)) \in I_2$. Consider that

$$(0, (0, c_1)) + (0, (0, -c_2)) = (0, (0, c_1 - c_2)) \in I_2,$$

thus I_2 is a subgroup of Γ. Moreover, suppose $(x, (y, z)) \in \Gamma^+$ and $(0, (0, c)) \in I_2^+$ such that

$$(0, (0,0)) \leq_{lx} (x, (y, z)) \leq_{lx} (0, (0, c)),$$

then it must be that $x = 0 = y$, and we get $(x, (y, z)) = (0, (0, z)) \in I_2$. Therefore, it can be concluded that $0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z})$ is an order ideal of $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$. ∎

Furthermore, it will be shown that there are no other non-trivial order ideals of $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ apart from those obtained in the two lemmas above.

Theorem 6.

Let Γ be defined as $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$. The only proper order ideals of Γ are $0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z})$.

Proof. We have previously shown that $I_1 := 0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $I_2 := 0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z})$ are non-trivial order ideals of Γ. Suppose there exists another non-trivial order ideal J apart from I_1 and I_2. Then $J \neq I_1$ and $J \neq I_2$. Since $I_2 \subset I_1$, it follows that $J \subset I_2 \subset I_1$ or $I_2 \subset J \subset I_1$ or $I_2 \subset I_1 \subset J$.

Case I: $J \subset I_2$.

If $J \subset I_2$, then $J = 0 \oplus_{lx} (0 \oplus_{lx} n\mathbb{Z}) = \{(0, (0, c)) \mid c \in n\mathbb{Z}\}$ for some $1 < n \in \mathbb{Z}$. Consider $(0, (0, n - 1)) \in \Gamma^+$ and $(0, (0, n)) \in J^+$. We have

$$(0, (0,0)) \leq_{lx} (0, (0, n - 1)) \leq_{lx} (0, (0, n)),$$

But $(0, (0, n - 1)) \notin J$, which contradicts the assumption of J.

Case II: $I_2 \subset J \subset I_1$.

If $I_2 \subset J \subset I_1$, then $J = \{(0, (b, c)) : b \in n\mathbb{Z}, c \in \mathbb{Z}, n > 1\}$. Take $(0, (n - 1, 1)) \in \Gamma^+$ and $(0, (n, 1)) \in J^+$. We have

$$(0, (0,0)) \leq_{lx} (0, (n - 1, 1)) \leq_{lx} (0, (n, 1)),$$

But $(0, (n - 1, 1)) \notin J$, which is a contradiction.

Case III: $I_1 \subset J$.

If $I_1 \subset J$, then $J = \{(a, (b, c)) : a \in n\mathbb{Z}, b, c \in \mathbb{Z}, 1 < n\}$. Take $(n - 1, (1, 1)) \in \Gamma^+$ and $(n, (1, 1)) \in J^+$. We have

$$(0, (0,0)) \leq_{lx} (n - 1, (1, 1)) \leq_{lx} (n, (1, 1)).$$

Since $(n - 1, (1, 1)) \notin J^+$, this is a contradiction.

From the cases above, J is not an order ideal, which contradicts the initial assumption that J is an order ideal. Therefore, it must be the case that $J = I_1$ or $J = I_2$. In other words, there are no other non-trivial order ideals of Γ besides I_1 and I_2. Hence, the only non-trivial order ideals of $\Gamma := \mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ are $0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z})$. ∎

Order Ideals of $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$

In this section, we will show all the order ideals in the lexicographic direct sum $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. Let us define Γ as $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$, and it is clear that $(0 \oplus_{lx} 0) \oplus_{lx} 0$ and Γ itself are order ideals of Γ. Next, the following two lemmas will present non-trivial order ideals of Γ.

Lemma 7.

Let Γ be defined as $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. Then, $(0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ is an order ideal of Γ.

Proof. Let $I_1 := (0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R} = \{((0, b), c) \mid b \in \mathbb{Z}, c \in \mathbb{R}\}$ and $((0, b_1), c_1), ((0, b_2), c_2) \in I_1$. Consider that

$$((0, b_1), c_1) - ((0, b_2), c_2) = ((0, b_1 - b_2), c_1 - c_2) \in I_1,$$

thus I_1 is a subgroup of Γ. Moreover, suppose $((x, y), z) \in \Gamma^+$ and $((0, b), c) \in I_1^+$ such that

$$((0,0),0) \leq_{lx} ((x,y),z) \leq_{lx} ((0,b),c).$$

This gives $x = 0$, and $((x,y),z) = ((0,y),z) \in I_1$. Therefore, it can be concluded that $(0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ is an order ideal of $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. ∎

Lemma 8.

Let Γ be defined as $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. Then, $(0 \oplus_{lx} 0) \oplus_{lx} \mathbb{R}$ is an order ideal of Γ.

Proof. Let $I_2 := (0 \oplus_{lx} 0) \oplus_{lx} \mathbb{R} = \{((0,0),c) \mid c \in \mathbb{R}\}$ and $((0,0),c_1), ((0,0),c_2) \in I_2$. Consider that

$$((0,0),c_1) - ((0,0),c_2) = ((0,0),c_1 - c_2) \in I_2,$$

thus I_2 is a subgroup of Γ. Moreover, suppose $((x,y),z) \in \Gamma^+$ and $((0,0),c) \in I_2^+$ such that

$$((0,0),0) \leq_{lx} ((x,y),z) \leq_{lx} ((0,0),c).$$

This implies $x = y = 0$, and $((x,y),z) = ((0,0),z) \in I_2$. Therefore, it can be concluded that $(0 \oplus_{lx} 0) \oplus_{lx} \mathbb{R}$ is an order ideal of $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. ∎

Furthermore, it will be shown that there are no other non-trivial order ideals of $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ apart from those obtained in the two lemmas above.

Theorem 9.

Let Γ be defined as $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. The non-trivial order ideal of Γ are only $(0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ dan $(0 \oplus_{lx} 0) \oplus_{lx} \mathbb{R}$.

Proof. From Lemma 7 and Lemma 8, it has been shown that $I_1 := (0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ and $I_2 := (0 \oplus_{lx} 0) \oplus_{lx} \mathbb{R}$ are non-trivial order ideals of Γ. Suppose there exists another non-trivial order ideal apart from I_1 and I_2, let say the order ideal is J. In general, subgroups of Γ are of the form $H_1 \oplus_{lx} H_2$ where $H_1 \leq \mathbb{Z} \oplus_{lx} \mathbb{Z}$ and $H_2 \leq \mathbb{R}$. Recall that H_1 is of the form $d_1\mathbb{Z} \oplus_{lx} d_2\mathbb{Z}$ where d_1, d_2 are integers. Using this fact, we have $J = (p_1\mathbb{Z} \sqcup p_2\mathbb{Z}) \oplus_{lx} \mathbb{R}$ (with p_1 and p_2 are positive integers not both equal to 1) or $J = (0 \oplus_{lx} 0) \oplus_{lx} H_2$ where $H_2 \neq \{0\}$.

Case I: If $J = (p_1\mathbb{Z} \oplus_{lx} p_2\mathbb{Z}) \oplus_{lx} \mathbb{R}$.

Claim that $J = I_1$ or $J = I_2$. In other words, $p_1 = 0$ and ($p_2 = 1$ or $p_2 = 0$). Assume $p_1 \neq 0$. Then there exists $(a,b) \in p_1\mathbb{Z} \oplus_{lx} p_2\mathbb{Z}$ with $a = p_1 k$ for some non-zero $k \in \mathbb{Z}$. Without loss of generality, let k be a positive real number. Consider that $((1,0),x) \in \Gamma^+$ and $((p_1 k, b), c) \in J^+$ with

$$((0,0),0) <_{lx} ((1,0),x) <_{lx} ((p_1 k, b), c).$$

Since J is an order ideal, then $((1,0),x) \in J$. Consequently, $1 = p_1 k'$ for some $k' \in \mathbb{Z}$, which implies $p_1 = 1$. Furthermore, consider $((0,1),x) \in \Gamma^+$ and

$$((0,0),0) <_{lx} ((0,1),x) <_{lx} ((1,0),x).$$

Since J is an order ideal, then $((0,1),x) \in J$. Consequently, $1 = p_2 k''$ for some $k'' \in \mathbb{Z}$, which implies $p_2 = 1$. Thus, $J = (\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R} = \Gamma$, which contradicts the fact that J is a non-trivial ordered ideal. Therefore, $p_1 = 0$.

Moreover, it is clear that if $p_2 = 0$, then $J = I_2$, which is an ordered ideal of Γ. Now, suppose H_2 non-trivial, and $p_2 \neq 0$. Then there exists $(0,b) \in p_1\mathbb{Z} \oplus_{lx} p_2\mathbb{Z}$ with $b = p_2 l$ for some nonzero $l \in \mathbb{Z}$. Suppose that $l > 0$. Consider, the elements $((0,1),y) \in \Gamma^+$, $((0,p_2 l), c) \in J^+$ satisfying

$$((0,0),0) <_{lx} ((0,1),y) <_{lx} ((0,p_2 l),c).$$

Since J is an order ideal, then $((0,1),y) \in J$. Consequently, $1 = p_2 l'$ for some $l' \in \mathbb{Z}$, which implies $p_2 = 1$. Therefore, $J = (0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R} = I_1$. Hence, it is proven that $J = I_1$ or $J = I_2$.

Case II: If $J = (0 \oplus_{lx} 0) \oplus_{lx} H_2$.

Recall that $0 \neq H_2 \neq \mathbb{R}$. From Lemma 3, H_2 is not an order ideal of $\mathbb{R}$, hence there is $z \in \mathbb{R}$ with $0 \leq z < c$ but $z \notin H_2$. Now consider,

$$\big((0,0),0\big) \leq_{lx} \big((0,0),z\big) \leq_{lx} \big((0,0),c\big),$$

but $\big((0,0),z\big) \notin J$. Thus, J is not an order ideal.
Based on cases above, the theorem is proven.

CONCLUSIONS

The totally ordered Abelian groups $(\mathbb{R}, +, \leq)$ and $(\mathbb{Z}, +, \leq)$ do not have non-trivial order ideals. Using this fact, we can identify the order ideals of $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$. The $\mathbb{R} \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ has two nontrivial order ideals, namely $0 \oplus_{lx} (\mathbb{Z} \oplus_{lx} \mathbb{Z})$ and $0 \oplus_{lx} (0 \oplus_{lx} \mathbb{Z})$. Similarly, the $(\mathbb{Z} \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ has two nontrivial ordered ideals, namely $(0 \oplus_{lx} \mathbb{Z}) \oplus_{lx} \mathbb{R}$ and $(0 \oplus_{lx} 0) \oplus_{lx} \mathbb{R}$.

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