



# The Boundedness of Generalized Fractional Integral Operators on Small Morrey Spaces

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## ABSTRACT

Morrey space was first introduced by C.B. Morrey in 1938 which is the solution space of an elliptic partial differential equation. Morrey space can be said to be a generalization of Lebesgue space. Morrey spaces are generalized into generalized Morrey spaces, small Morrey spaces, weighted Morrey spaces, and Bourgain Morrey spaces. One of the studies in Morrey space is the boundedness of operators in Morrey space. One such operator is the fractional integral operator. One of the generalizations of fractional integral operators is the generalized fractional integral operator. The generalized fractional integral operator is finite on the generalized Morrey space. In this study, it is extended on small Morrey spaces. The small Morrey space is the set of locally Lebesgue integrable functions with norm defined supremum over radius of ball  $r \in (0,1)$ . This paper aims to prove the boundedness properties of the generality of fractional integral operators on small Morrey spaces using Hedberg-type inequality. The first, in this paper will be discuss to prove Hedberg-type inequality on small Morrey spaces using dyadic decomposition, Hölder inequality, and doubling condition. The proof method used is using Hedberg's inequality. The results imply that the generalized fractional integral operator is finite on small Morrey spaces.

**Keywords:** Fractional integral operators; Hedberg-type inequality; small Morrey spaces

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## INTRODUCTION

Morrey space was first introduced by C. B. Morrey in 1938. Morrey found a solution space of elliptic partial differential equations called Morrey space. Morrey space can be said to be a generalization of Lebesgue space. In the development of research, Morrey space develops into several kinds of spaces such as small Morrey space [1]. Morrey spaces are generalized into generalized Morrey spaces, small Morrey spaces, weighted Morrey spaces, and Bourgain Morrey spaces. One of the studies in Morrey space is the boundedness of operators in Morrey space. One such operator is the fractional integral operator. One of the generalizations of fractional integral operators is the generalized fractional integral operator.

The fractional integral operator is one type of singular integral operator. Research on fractional integral operators, first started by Hardy and Littlewood. Hardy and Littlewood proved the boundedness of fractional integral operators on Lebesgue spaces

using dyadic decomposition [2]. Furthermore, the research was continued by Adams [3] and Chiarenza and Frasca [4] who proved the boundedness of fractional integral operators on Morrey space using dyadic decomposition. Nakai generalized the space of Adams and Chiarenza and Frasca. Nakai proved the boundedness property of fractional integral operators on the generalized Morrey spaces using dyadic decomposition. The result of Nakai research is also called Spanne inequality [5]. Gunawan and Eridani extended the results of Adams and Chiarenza and Frasca [6]. Karim and Apriliani continued their research on the Spanne inequality of the  $p = q$  case [7].

Sawano extended Morrey space into a small Morrey space for  $r \in (0,1)$ . In Sawano's research, it also gives the properties of inclusions among small Morrey spaces. In Sawano's research, it also gives the properties of inclusions among small Morrey spaces [8]. Based on the inclusion properties, Karim et. al. research further discussed the inclusion of Lebesgue spaces, Morrey spaces, and small Morrey spaces. Karim, et.al. has proved the boundedness of fractional integral operator in small Morrey spaces using Hedberg-type Inequality [9]. The purpose of this study is to extend Karim's results, namely to investigate the boundedness of the fractional integral operator on small Morrey spaces using Hedberg-type Inequality, in particular from small Morrey spaces to other small Morrey spaces. This result then implies the boundedness of the generalized fractional integral operator on small Morrey spaces.

## METHODS

The method used in writing this research article is a literature study of several related articles and books. In some of the literature, the definition and the boundedness properties of fractional integral operators have been discussed and the boundedness properties and definition of Morrey space have also been discussed. In this work, the research methodology is given as follows:

- i. Constructing the Hedberg-type inequality on small Morrey spaces for generalized fractional integral operators.
- ii. Proving Hedberg-type inequality on small Morrey spaces for generalized fractional integral operators using dyadic decomposition, Hölder inequality, and doubling condition.
- iii. Constructing the boundedness theorem of fractional integral operator in small Morrey space.
- iv. Proving the boundedness theorem of fractional integral operator in small Morrey space using Hedberg-type inequality on small Morrey spaces for generalized fractional integral operators.

Therefore, by using the definitions and properties that have been discussed in the previous literature study, it can be used to prove the boundedness properties in this article. The first, we will discuss about Lebesgue spaces and Morrey spaces below.

### **Definition 1** (Lebesgue Spaces, [10])

Let  $(X, \mathcal{A}, \mu)$  is measurable spaces and  $1 \leq p < \infty$ . The Lebesgue spaces  $L^p(X)$  are consists all equivalence class of function  $f : X \rightarrow \mathbb{R}$  such that

$$\|f\|_{L^p} := \left( \int_X |f|^p dx \right)^{\frac{1}{p}} < \infty.$$

**Definition 2** (Local Lebesgue Spaces, [11])

Let  $(X, \mathcal{A}, \mu)$  is measurable spaces. The Lebesgue spaces  $L^p_{loc}(X)$  are consists all equivalence class of function  $f : X \rightarrow \mathbb{R}$  such that for every compact subset  $S \subset X$ ,

$$\int_S |f|^p dx < \infty.$$

**Theorem 1** (Hölder Inequality, [11])

If  $f \in L^p(\mathbb{R}^n)$  dan  $g \in L^q(\mathbb{R}^n)$  with  $p > 1$ , then  $\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$ .

*Proof.* The proof of this theorem can be seen in [11].

**Definition 2** (Fractional Integral Operator, [12])

Let  $0 < \beta < n$ . The integral fractional operator  $I_\beta f(x)$  is defined by

$$I_\beta f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy$$

for every  $f \in L^p_{loc}(\mathbb{R}^n)$  with  $1 \leq p < \infty$  and for every  $x \in \mathbb{R}^n$ .

**Theorem 2** (Hedberg-type inequality, [13])

Let  $1 \leq q < p < \infty$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ , then for every  $f \in L^p(\mathbb{R}^n)$ ,

$$|I_\beta f(x)| \leq C_{p,q} (Mf(x))^{1-\frac{\beta p}{n}} \|f(x)\|_{L^p}^{\frac{\beta p}{n}}.$$

*Proof.* Suppose  $1 \leq q < p < \infty$ , take any  $f \in L^p(\mathbb{R}^n)$  and open ball  $O(x, r)$ . From definition integral fractional operator, write

$$I_\beta f(x) := I_1 f(x) + I_2 f(x),$$

with

$$I_1 f(x) := \int_{O(x,r)} \frac{f(y)}{|x-y|^{n-\beta}} dy,$$

and

$$I_2 f(x) := \int_{O(x,r)^c} \frac{f(y)}{|x-y|^{n-\beta}} dy.$$

Using dyadic decomposition, partition of the open ball  $O(x, r)$  is  $2^k r \leq |x-y| < 2^{k+1} r$  with  $-\infty < k \leq -1$  such that

$$|I_1 f(x)| \leq C \sum_{k=-\infty}^{-1} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{f(y)}{|x-y|^{n-\beta}} dy \leq Cr^\beta (Mf(x)).$$

Partition of the complement of open ball  $O(x, r)^c$ , we obtain  $2^k r \leq |x-y| < 2^{k+1} r$  with  $0 \leq k < \infty$  such that

$$|I_2 f(x)| \leq C \sum_{k=0}^{\infty} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{f(y)}{|x-y|^{n-\beta}} dy \leq C r^{\beta-\frac{n}{p}} \|f(x)\|_{L^p}$$

Choose  $r := \left(\frac{\|f(x)\|_{L^p}}{Mf(x)}\right)^{\frac{p}{n}}$ , such that we obtain

$$\begin{aligned} |I_{\beta} f(x)| &\leq |I_1 f(x)| + |I_2 f(x)| \\ &\leq C \left( r^{\beta} (Mf(x)) + r^{\beta-\frac{n}{p}} \|f(x)\|_{L^p} \right) \\ &\leq C (Mf(x))^{1-\frac{\beta p}{n}} \|f(x)\|_{L^p}^{\frac{\beta p}{n}}. \end{aligned}$$

So, Hedberg-type Inequality is holds.

**Definition 3** (Doubling Condition, [14])

In defining the operators  $I_{\psi}$ , the function  $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$  satisfy the doubling condition if holds

1. If  $\frac{1}{2} \leq \frac{r}{s} \leq 2$  then  $\frac{1}{C_1} \leq \frac{\psi(r)}{\psi(s)} \leq C_1$  for every  $r, s > 0$ .
2. For  $r > 0$ ,  $\int_r^{\infty} \frac{\psi(t)}{t} dt$ .

**Definition 4** (Morrey Spaces, [15])

Let  $0 \leq q \leq p < \infty$  and measurable spaces  $X = \mathbb{R}^n$ . The Morrey spaces  $\mathcal{M}_q^p(\mathbb{R}^n)$  consists of function  $f \in L_{loc}^p(\mathbb{R}^n)$  with

$$\|f\|_{\mathcal{M}_q^p} := \sup_{x \in \mathbb{R}^n, r > 0} |O(x, r)|^{\frac{1}{q}-\frac{1}{p}} \left( \int_{O(x, r)} |f(x)|^p dx \right)^{\frac{1}{p}},$$

is finite.

**Definition 5** (Small Morrey Spaces, [15])

Let  $0 \leq q \leq p < \infty$  and measurable spaces  $X = \mathbb{R}^n$ . The small Morrey spaces  $m_q^p(\mathbb{R}^n)$  consists of function  $f \in L_{loc}^p(\mathbb{R}^n)$  with

$$\|f\|_{m_q^p} := \sup_{x \in \mathbb{R}^n, r \in (0,1)} |O(x, r)|^{\frac{1}{q}-\frac{1}{p}} \left( \int_{O(x, r)} |f(x)|^p dx \right)^{\frac{1}{p}},$$

is finite.

**RESULTS AND DISCUSSION**

In this section, we will discuss the boundedness properties of the fractional integral operator. However, we will first prove Hedberg-type inequality in small Morrey spaces. Eridani [12] generalized the fractional integral operator. Given function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  that satisfies the doubling condition. The generalized fractional integral operator is defined by

$$I_\psi f(x) := \int_{\mathbb{R}^n} \frac{\psi(|x-y|)}{|x-y|^n} f(y) dy,$$

for every  $f \in L^p_{loc}(\mathbb{R}^n)$  with  $1 \leq p < \infty$ . If  $\psi(|x-y|) := |x-y|^\beta$  with  $0 < \alpha < n$  then  $I_\psi f(x) = I_\beta f(x)$  [14].

Since  $\rho$  function satisfies the doubling condition, such that for every  $k \in \mathbb{Z}$  and for every  $r > 0$  with  $2^k r \leq |x-y| < 2^{k+1} r$  holds

1. If  $\frac{1}{2} \leq \frac{|x-y|}{2^{k+1} r} \leq 2$  then

$$\frac{1}{C} \leq \frac{\psi(|x-y|)}{\psi(2^{k+1} r)} \leq C.$$

As a result, we obtain

$$\frac{\psi(2^{k+1} r)}{C} \leq v(|x-y|) \leq C\psi(2^{k+1} r). \tag{1}$$

2. Let  $t := |x-y|$ . The inequality (1) can be expressed as

$$\frac{\psi(2^{k+1} r)}{C} \leq \psi(t) \leq C\psi(2^{k+1} r). \tag{2}$$

The next, from inequality (2), we have

$$\begin{aligned} \int_{2^{k_r}}^{2^{k+1} r} \frac{\psi(2^{k+1} r)}{C2^{k+1} r} dt &\leq \int_{2^{k_r}}^{2^{k+1} r} \frac{\psi(t)}{t} dt \leq \int_{2^{k_r}}^{2^{k+1} r} \frac{C\psi(2^{k+1} r)}{2^{k_r}} dt \\ \frac{\psi(2^{k+1} r)}{2C} &\leq \int_{2^{k_r}}^{2^{k+1} r} \frac{\psi(t)}{t} dt \leq C\psi(2^{k+1} r). \end{aligned} \tag{3}$$

**Theorem 3 (Hedberg-type Inequality)**

Let function  $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$  satisfies doubling condition. If  $1 \leq p < q < \infty$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ , then

$$|I_\psi f(x)| \leq C_{p,q} |Mf(x)|^{1-\frac{\beta q}{n}} \|f(x)\|_{m_q^p}^{\frac{\beta q}{n}}$$

for every  $f \in m_q^p(\mathbb{R}^n)$ , with  $Mf(x)$  is maximal Hardy-Littlewood operator.

*Proof.* Take any  $f \neq 0 \in m_q^p(\mathbb{R}^n)$  and let  $1 \leq p < q < \infty$ . Defined a function  $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$ , is  $\psi(t) := t^\beta$  with  $0 < \beta < n$  satisfies doubling condition such that  $\psi(t)$  satisfies inequality (1) and (3). Write

$$I_\psi f(x) := I_1 f(x) + I_2 f(x),$$

With

$$I_1 f(x) := \int_{O(x,r)} \frac{\psi(|x-y|)}{|x-y|^n} f(y) dy,$$

and

$$I_2 f(x) := \int_{O(x,r)^c} \frac{\psi(|x-y|)}{|x-y|^n} f(y) dy.$$

For open ball  $O(x,r)$ , can be partitioned  $2^k r \leq |x-y| < 2^{k+1} r$  for every  $-\infty < k \leq -1$  such that we obtain

$$|I_1 f(x)| \leq \sum_{k=-\infty}^{-1} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{\psi(|x-y|)}{|x-y|^n} |f(y)| dy$$

$$\begin{aligned}
 &\leq C \sum_{k=-\infty}^{-1} \psi(2^{k+1}r) \frac{1}{(2^{k+1}r)^n} \left( \int_{|x-y| < 2^{k+1}r} |f(y)| dy \right) \\
 &= C \sum_{k=-\infty}^{-1} (2^{k+1}r)^\beta \left( \frac{1}{(2^{k+1}r)^n} \int_{|x-y| < 2^{k+1}r} |f(y)| dy \right) \\
 |I_1 f(x)| &\leq C \sum_{k=-\infty}^{-1} (2^{k+1}r)^\beta \left( \sup_{O(x, 2^{k+1}r)} \frac{1}{(2^{k+1}r)^n} \int_{|x-y| < 2^{k+1}r} |f(y)| dy \right) \\
 &= C \sum_{k=-\infty}^{-1} (2^{k+1}r)^\beta Mf(x).
 \end{aligned}$$

Because the series  $\sum_{k=-\infty}^{-1} 2^{\beta(k+1)}$  converges to one for  $k < 0$ , then inequality can be expressed by

$$|I_1 f(x)| \leq Cr^\beta Mf(x).$$

For  $O(x, r)^c$ , the ball  $O(x, r)^c$  can be partitioned  $2^k r \leq |x - y| < 2^{k+1} r$  with  $0 \leq k < \infty$ . Observe that

$$\begin{aligned}
 |I_2 f(x)| &\leq \sum_{k=0}^{\infty} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{\psi(|x-y|)}{|x-y|^n} |f(y)| dy \\
 &\leq C \sum_{k=0}^{\infty} \frac{\psi(2^k r)}{(2^k r)^n} \int_{|x-y| < 2^{k+1} r} \frac{\psi(|x-y|)}{|x-y|^n} |f(y)| dy,
 \end{aligned}$$

with constant  $C > 0$  and using Hölder inequality we have

$$\begin{aligned}
 \frac{1}{(2^k r)^n} \int_{|x-y| < 2^{k+1} r} f(y) dy &\leq \left( \frac{1}{(2^k r)^n} \int_{|x-y| < 2^{k+1} r} |f(y)|^p dy \right)^{\frac{1}{p}} \\
 &\leq (2^k r)^{-\frac{n}{q}} \|f\|_{m_q^p}.
 \end{aligned}$$

Because  $\psi(t) := t^\beta$ , consequently can be expressed as

$$\begin{aligned}
 |I_2 f(x)| &\leq C \sum_{k=0}^{\infty} \frac{\psi(2^k r)}{(2^k r)^n} \int_{|x-y| < 2^{k+1} r} \frac{\psi(|x-y|)}{|x-y|^n} |f(y)| dy \\
 &\leq C \sum_{k=0}^{\infty} (2^k r)^\beta (2^k r)^{-\frac{n}{q}} \|f\|_{m_q^p} \\
 &\leq Cr^{\beta-\frac{n}{q}} \sum_{k=0}^{\infty} (2^k)^{\beta-\frac{n}{q}} \|f\|_{m_q^p},
 \end{aligned}$$

and because  $0 < \beta < n$  then  $\beta - n < \beta - \frac{n}{q} < 0$  for  $1 < p < q < \infty$  such that the series  $\sum_{k=0}^{\infty} (2^k)^{\beta-\frac{n}{q}}$  converges to one. The result is

$$|I_2 f(x)| \leq Cr^{\beta-\frac{n}{q}} \|f\|_{m_q^p}.$$

Next, using triangle inequality we obtain

$$\begin{aligned}
 |I_\psi f(x)| &\leq |I_1 f(x)| + |I_2 f(x)| \\
 &\leq Cr^\beta Mf(x) + Cr^{\beta-\frac{n}{q}} \|f\|_{m_q^p}
 \end{aligned}$$

$$\leq C \left( r^\beta Mf(x) + r^{\beta - \frac{n}{q}} \|f\|_{m_q^p} \right).$$

Choose  $r := \left( \frac{\|f\|_{m_q^p}}{Mf(x)} \right)^{\frac{q}{n}}$ , such that

$$|I_\psi f(x)| \leq C \left( r^\beta Mf(x) + r^{\beta - \frac{n}{q}} \|f\|_{m_q^p} \right) = C(Mf(x))^{1 - \frac{\beta q}{n}} \|f\|_{m_q^p}^{\frac{\beta q}{n}}.$$

So, Hedberg-type inequality on small Morrey spaces for fractional integral operator is proved.

The following presents the limitation theorem of the generalized fractional integral operator in small Morrey spaces.

**Theorem 4**

Let positive function  $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$  satisfies doubling condition and  $1 < p_1 < q_1 < \frac{n}{\beta}$  with  $0 < \beta < n$ . If  $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$  and  $\frac{1}{p_2} = \frac{1}{p_1} \left( 1 - \frac{\beta q_1}{n} \right)$ , then exist constant  $C_{p_1, q_1} > 0$  such that for every  $f \in m_q^p(\mathbb{R}^n)$  holds

$$\|I_\psi f\|_{m_{q_2}^{p_2}} \leq C_{p_1, q_1} \|f\|_{m_{q_1}^{p_1}}.$$

*Proof.* Let  $1 < p_1 < q_1 < \frac{n}{\beta}$  with  $0 < \beta < n$ . Defined function  $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$ , is  $\psi(t) := t^\beta$ . Take any function  $f \neq 0 \in m_{q_1}^{p_1}(\mathbb{R}^n)$ . According to Hedberg-type Inequality from Theorem 3, holds

$$|I_\psi f(x)| \leq C_{p, q} |Mf(x)|^{1 - \frac{\beta q}{n}} \|f(x)\|_{m_q^p}^{\frac{\beta q}{n}}.$$

Defined

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n},$$

and

$$\frac{1}{p_2} = \frac{1}{p_1} \left( 1 - \frac{\beta q_1}{n} \right).$$

Therefore, it is obtained

$$\begin{aligned} |I_\psi f(x)| &\leq C(Mf(x))^{1 - \frac{\beta n}{q_1}} \|f\|_{m_{q_1}^{p_1}}^{\frac{\beta n}{q_1}} \\ &= C(Mf(x))^{\frac{p_1}{p_2}} \|f\|_{m_{q_1}^{p_1}}^{1 - \frac{p_1}{p_2}}, \end{aligned}$$

Next, both segments are raised by  $p_2$  and integrated over  $O(x, r)$  such that

$$\begin{aligned} \int_{O(x, r)} |I_\psi f(x)|^{p_2} dx &\leq C^{p_2} (Mf(x))^{p_1} \|f\|_{m_{q_1}^{p_1}}^{p_2 - p_1} \\ &= C^{p_2} \|f\|_{m_{q_1}^{p_1}}^{p_2 - p_1} \int_{B(x, r)} (Mf(x))^{p_1} dx \\ &\leq C^{p_2} \|f\|_{m_{q_1}^{p_1}}^{p_2 - p_1} \sup_{x \in \mathbb{R}^n, r \in (0, 1)} |O(x, r)|^{\frac{1}{q_1} - \frac{1}{p_1}} \int_{O(x, r)} (Mf(x))^{p_1} dx \end{aligned}$$

$$\begin{aligned}
 &= C^{p_2} \|f\|_{m_{q_1}^{p_1}}^{p_2-p_1} \|Mf\|_{m_{q_1}^{p_1}}^{p_1} \\
 &\leq C^{p_2} \|f\|_{m_{q_1}^{p_1}}^{p_2-p_1} \|f\|_{m_{q_1}^{p_1}}^{p_1} \\
 &\leq C^{p_2} \|f\|_{m_{q_1}^{p_1}}^{p_2}.
 \end{aligned}$$

Rank both segments by  $\frac{1}{p_2}$  then multiply by  $|O(x,r)|^{\frac{1}{q_2} - \frac{1}{p_2}}$  and take the supremum over radius  $r \in (0,1)$  of both segments such that we get

$$\|I_\psi f\|_{m_{q_2}^{p_2}} \leq C_{p_1, q_1} \|f\|_{m_{q_1}^{p_1}}.$$

Thus, it is proven that  $I_\psi f(x)$  is finite in the small Morrey spaces.

## CONCLUSIONS

The generalization of the fractional integral operators by defining the positive function  $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$  that satisfies the doubling condition. The proof of the boundedness properties of the generalized fractional integral operator in small Morrey space uses the Hedberg-type inequality. Therefore, in proving it, it is necessary to prove first that the Hedberg-type inequality applies in small Morrey space and applies to the generalized fractional integral operator.

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