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Bifurcation Analysis of a Discrete Logistic System with Additive Allee Effect and Feedback Control

Nadia Agus Hadinata*, Agus Suryanto, Wuryansari Muharini Kusumawinahyu

Departement of Mathematics, Faculty of Mathematics and Natural Sciences
Brawijaya University, Indonesia

Email: nadiagush17@gmail.com

ABSTRACT

This paper analyzes a discrete logistic system with additive Allee effect and feedback control. The main objective is to examine how the additive Allee effect and feedback control influence the dynamic behavior of the model. The analysis reveals show that the model has a trivial fixed point E_0 and two interior fixed point E_1 and E_2. The results of our stability analysis show that there are topological differences that depend on the step size. Bifurcation analysis is conducted using center manifold theory and the bifurcation theorem. By making step size as a bifurcation parameter, we demonstrate that the model undergoes period-doubling and Neimark-Sacker bifurcations. Some numerical simulations are performed to confirm the result of the analysis.

Keywords: Allee effect; feedback control; flip bifurcation; logistic; Neimark-Sacker

INTRODUCTION

Population dynamics is one of the interesting topic in mathematics becaue it has a direct impact on various aspects of life. Many mathematical models have been studied by researchers, one of which focuses on population growth. Examples of simple population growth models are exponential and logistic growth models. Thomas Robert Malthus introduced the exponential growth model in 1798 by assuming that resources and environmental capacity are unlimited. Then, Pierre Verhulst modified the exponential in 1830 becomes a logistic growth model, by taking the carrying capacity and intrinsic growth rate into account [1].

In the real world, ecosystems can be disturbed by unexpected things and it can cause changes in other components of the ecosystem. This process is named feedback control [2]. This control enables the ecosystem to achieve and sustain the stable condition. Gopalsamy [3] propose a feedback control variable into the logistic models as follows.

$$\frac{dx}{dt} = x\left[r\left(1-\frac{x}{K}\right) - bu\right],$$
$$\frac{du}{dt} = -\alpha u + cx,$$

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where x is the size of a population, r is the intrinsic growth rate, K is the carrying capacity, and u is a feedback control variable; r, K, b, α, and c are positive constants. Liao [4], Hoang [5], and [6-8] also studied on models with feedback control.

On the other hand, the relationship between species in this vast world are complex and diverse. In 1931, Allee pointed out that if the population is too small, it becomes more difficult for species to find mates or food. This situation can lead to reduced natality rates and increased mortality rates, a phenomenon known as the Allee effect [9, 10]. The Allee effect is caused by various biological factors, such as reduced defense capabilities against predators, intraspesific competition, difficulty in finding mates, genetic distortion, decreased foraging efficiency and social dysfunction, which lead the population density low [11, 12]. Denis [13] introduced the following additive Allee effect in the logistic model

$$\frac{dx}{dt} = rx\left[\left(1 - \frac{x}{K}\right) - \frac{m}{x + a}\right],$$

where $\frac{m}{x+a}$ represents the condition for the Allee effect, m and a are positive variables that indicate the strength of the Allee effect.

It is known that the logistic model using feedback control always has a unique positive equilibrium that is globally attractive [14, 15]. Besides that, the Allee effect is one of the most frequently observed phenomena, particularly as an increasing number of species are going extinct. Lv et al. [16] show that additive Allee effects and feedback control significantly affect species permanence, extinction, and stability.

Many researchers have focused on discrete models because it more suitable than the continuous models if the population size is rarely small or the population has no overlapping generations. Furthermore, it is easier to obtain numerical solutions for discrete models, and many researchers have shown that discrete models yield more wealthy analyses than continuous-time models.

The outline of this paper is as follows: In the second section, we present several steps to solve the system. In the third section, we examine the existence of fixed points, investigate the stability of fixed point and investigate that the system has the period-2 and Neimark-Sacker bifurcations. In the fourth section, we perform numerical simulations to confirm the results of our theoretical analysis. At the end, a succinct conclusion is provided in Section 5.

METHODS

In this study, we allow the logistic model with additive Allee effect and feedback control

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - x - \frac{m}{x + a}\right) - bxu, \\ \frac{du}{dt} &= -\alpha u + cx, \end{aligned} \tag{1}$$

where α, a, b, c, and m are all positive constants.

In order to analyze the discrete model of continuous system (1), we perform the following steps.

1) Discretization of the system (1) by implementing forward Euler's scheme.
2) Determination of the fixed points and its existence.
3) Analysis the stability of fixed points.

4) Investigation the existence of period-2 and Neimark-Sacker bifurcation in the model.
5) Conduct numerical simulations to confirm the analytical result.

RESULTS AND DISCUSSION

Implementation of the forward Euler's scheme to system (1), yields below

$$\begin{aligned} x(n+1) &= x(n) + \delta\left(x(n)\left[1 - x(n) - \frac{m}{x(n)+a}\right] - bx(n)u(n)\right), \\ u(n+1) &= u(n) + \delta[-\alpha u(n) + cx(n)], \end{aligned} \tag{2}$$

where δ is the integration step size.

The Existence of the Fixed Points

System (2) has three fixed points, namely one trivial fixed points $E_0(0,0)$ and two interior fixed points $E_i(x_i, u_i), i = 1,2$, where $x_i = \frac{\alpha u_i}{c}$ and u_i is the solutions of the quadratic equation

$$\hat{A}u^2 + \hat{B}u + \hat{C} = 0, \tag{3}$$

where

$$\hat{A} = \alpha^2 + bc\alpha;\ \hat{B} = abc^2 - \alpha c + \alpha ac;\ \hat{C} = c^2(m - a).$$

Consider the discrimination of equation (3), namely $\Delta(m) = (abc + a\alpha + \alpha)^2 - 4\alpha m(bc + \alpha)$. Let m^* be the unique root of $\Delta(m) = 0$, namely

$$m^* = \frac{(abc + a\alpha + \alpha)^2}{4\alpha(bc + \alpha)}.$$

The conditions for the existence of the positive fixed points of system (2) are summarized in proposition 1.

Proposition 1.

1) If $m < m^*$ and $\hat{B} < 0$, then the system (2) has fixed point $E_0(0,0)$ and
   a) $E_1(x_1, u_1)$ if $m < a$;
   b) $E_{1,2}(x_{1,2}, u_{1,2})$ if $m > a$;
   c) $E_3(x_3, u_3)$ if $m = a$, where
   $$x_3 = \frac{\alpha u_3}{c}, u_3 = c\frac{-abc^2 + \alpha c - \alpha ac}{\alpha^2 + bc\alpha}$$
2) If $m = m^*$, then the system (2) has fixed point $E_0(0,0)$ and $E_4(x_4, u_4)$, where
   $$x_4 = \frac{\alpha u_4}{c}, u_4 = c\frac{(\alpha - abc - \alpha ac)}{\alpha^2 + bc\alpha}$$

Proof:

Based on solution of equation (3), $u \in \mathbb{R}$, if $\Delta(m) \geq 0$, which is if and only if $m \leq m^*$.

1. If $m < a$, then $\hat{C} < 0$. So, (2) only has fixed point $E_1(x_1, u_1)$.
2. If $m > a$ and $\hat{B} < 0$, then (2) has two fixed point, i.e., $E_1(x_1, u_1), E_2(x_2, u_2)$.
3. If $m = a$ and $\hat{B} < 0$, then $\hat{C} = 0$ and $u_1 = u_0$. So (2) has a unique fixed point $E_3(x_3, u_3)$.

If $m = m^*$, then $\Delta(m) = 0$. So (2) has a unique fixed point $E_4(x_4, u_4)$.

Stability Analysis

Next we will find out the stability of the system (2) at each fixed point. The Jacobian matrix of (2) at (x^*, u^*) as follows:

(4)

$$J(x^*,u^*) = \begin{pmatrix} 1+\delta-2x^*\delta-\dfrac{ma\delta}{(x^*+a)^2}-bu^*\delta & -bx^*\delta \\ c\delta & 1-\alpha\delta \end{pmatrix}.$$

The local stability of any point is ensured by the following theorems.

Theorem 1. The fixed point $E_0(0,0)$

1) *sink* if $m > a$ and $\delta < 2\min\left\{\frac{a}{m-a},\frac{1}{\alpha}\right\}$,
2) *saddle* if $0 < m < a$ and $0 < \delta < \frac{2}{\alpha}$ or if $m > a$ and $\frac{2}{\alpha} < \delta < \frac{2a}{m-a}$,
3) *source* if $0 < m < a$ and $\delta > \frac{2}{\alpha}$ or if $m > a$ and $\delta > 2\max\left\{\frac{a}{m-a},\frac{1}{\alpha}\right\}$,
4) *non-hyperbolic* if $m = a$ and $\delta = \frac{2}{\alpha}$ or if $m > a$, $\delta = \frac{2a}{m-a}$ and $\delta = \frac{2}{\alpha}$.

For the analysis of the fixed point E_0, substitute fixed point E_0 at (4) and we get

Proof:

$$J(0,0) = \begin{pmatrix} 1+\delta-\dfrac{m}{a}\delta & 0 \\ c\delta & 1-\alpha\delta \end{pmatrix} = \begin{pmatrix} 1+\delta\left(1-\dfrac{m}{a}\right) & 0 \\ c\delta & 1-\alpha\delta \end{pmatrix}.$$

So, the eigen values of matrix are $\lambda_1 = 1+\delta\left(1-\frac{m}{a}\right)$ and $\lambda_2 = 1-\alpha\delta$. The analysis is divided into 3 conditions, i.e., $0 < m < a, m = a$, and $m > a$.

For λ_1,

1) if $0 < m < a$, then $|\lambda_1| > 1$,
2) if $m = a$, then $|\lambda_1| = 1$,
3) if $m > a$, then $|\lambda_1| < 1$ if and only if $0 < \delta < \frac{2a}{m-a}$.

For λ_2,

1) $|\lambda_2| < 1$ if and only if $0 < \delta < \frac{2}{\alpha}$.
2) $|\lambda_2| = 1$ if and only if $\delta = \frac{2}{\alpha}$.
3) $|\lambda_2| > 1$ if and only if $\delta > \frac{2}{\alpha}$.

To determine the stability using eigen analysis method, see Definition and Lemmas in [17,18]. Now, we discuss the stability of fixed point E_1 and E_2. Let the characteristic equation of Jacobian matrix (4) at the interior fixed point $(x_i,u_i), i = 1,2$ can be written as

$$\lambda^2 - Tr(J(E_i))\lambda + Det(J(E_i)) = 0,$$

where

$$Tr(J) = 2+\delta\left(\frac{m}{x_i+a}-\frac{ma}{(x_i+a)^2}-x_i-\alpha\right) = 2+G\delta$$

and

$$\begin{aligned} Det(J) &= 1+\delta\left(\frac{m}{x_i+a}-\frac{ma}{(x_i+a)^2}-x_i-\alpha\right)+\delta^2\left(\frac{ma\alpha}{(x_i+a)^2}-\frac{m\alpha}{x_i+a}+x_i\alpha+bcx_i\right) \\ &= 1+G\delta+H\delta^2, \end{aligned}$$

where

$$G = \frac{m}{x_i+a}-\frac{ma}{(x_i+a)^2}-x_i-\alpha; H = \frac{ma\alpha}{(x_i+a)^2}-\frac{m\alpha}{x_i+a}+x_i\alpha+bcx_i.$$

Now let

$$F(\lambda) = \lambda^2 - Tr\big(J(E_i)\big)\lambda + Det\big(J(E_i)\big) = 0$$
$$\Leftrightarrow \lambda^2-(2+G\delta)\lambda+(1+G\delta+H\delta^2).$$

Therefore $F(1) = H\delta^2, F(0) = 1+G\delta+H\delta^2, F(-1) = 4+2G\delta+H\delta^2$. If $G^2-4H \geq 0$, then $\delta_1 = \frac{-G-\sqrt{G^2-4H}}{H}$ and $\delta_2 = \frac{-G+\sqrt{G^2-4H}}{H}$ lead to $F(-1) = 0$. Clearly, H, G are functions of fixed point and $H(x_1) > 0, H(x_2) < 0$.

The local stability analysis of trivial fixed points E_1 and E_2 can be seen at Theorem 2 and Theorem 3.

Theorem 2. If the fixed point $E_1(x_1, u_1)$ exists, then $E_1(x_1, u_1)$ is

1) *sink*, if one of the following forms satisfy
   a) $G^2 - 4H > 0$ and $0 < \delta < \frac{-G-\sqrt{G^2-4H}}{H}$.
   b) $G^2 - 4H \leq 0$ and $0 < \delta < -\frac{G}{H}$.
2) *source*, if one of the following forms satisfy
   a) $G^2 - 4H > 0$ and $\delta > \frac{-G+\sqrt{G^2-4H}}{H}$.
   b) $G^2 - 4H \leq 0$ and $\delta > -\frac{G}{H}$.
3) *saddle* if following forms $G^2 - 4H > 0$ and $\frac{-G-\sqrt{G^2-4H}}{H} < \delta < \frac{-G+\sqrt{G^2-4H}}{H}$.
4) *non-hyperbolic* if one of the following forms satisfy
   a) $G^2 - 4H > 0$ and $\delta = \frac{-G\pm\sqrt{G^2-4H}}{H}$
   b) $G^2 - 4H \leq 0$ and $\delta = -\frac{G}{H}$

Proof:

1. If $G^2 - 4H > 0$, λ_1 and λ_2 are unequal roots of $F(\lambda) = 0$ and
   a. If $0 < \delta < \delta_1$, then $F(-1) > 0$ and $F(0) < 1$. So, we have $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Therefore, E_1 is a sink.
   b. If $\delta > \delta_2$, then $F(-1) > 0$ and $F(0) > 1$. So, we have $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Therefore, E_1 is a source.
   c. If $\delta = \delta_1$ or δ_2, then $F(-1) = 0$ and $F(0) \neq 1$. So, we have $\lambda_1 = -1$ and $|\lambda_2| \neq 1$. Therefore, E_1 is non-hyperbolic.
   d. If $\delta_1 < \delta < \delta_2$, then $F(-1) < 0$. So, we have $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). Therefore, E_1 is a saddle.
2. If $G^2 - 4H = 0$, λ_1 and λ_2 are equal roots of $F(\lambda) = 0$ and
   a. If $0 < \delta < -\frac{G}{H}$, then $|\lambda_1| < 1$ and $|\lambda_2| < 1$.. Therefore, E_1 is a sink.
   b. If $\delta > -\frac{G}{H}$, then $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Therefore, E_1 is a source.
   c. If $\delta = -\frac{G}{H}$, then $\lambda_1 = \lambda_2 = -1$. Therefore, E_1 is non-hyperbolic.
3. If $G^2 - 4H < 0$, λ_1 and λ_2 are conjugate complex roots of $F(\lambda) = 0$ and
   a. If $0 < \delta < -\frac{G}{H}$, then $|\lambda_1| < 1$ and $|\lambda_2| < 1$.. Therefore, E_1 is a sink.
   b. If $\delta > -\frac{G}{H}$, then $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Therefore, E_1 is a source.
   c. If $\delta = -\frac{G}{H}$, then $\lambda_1 = \lambda_2 = -1$. Therefore, E_1 is non-hyperbolic.

Theorem 3. If the fixed point $E_2(x_2, u_2)$ exists, then $E_2(x_2, u_2)$ is

1) *source* if $\delta > \frac{-G+\sqrt{G^2-4H}}{H}$.
2) *saddle* if $\delta < \frac{-G+\sqrt{G^2-4H}}{H}$.
3) *non-hyperbolic* if $\delta = \frac{-G+\sqrt{G^2-4H}}{H}$.

Proof:

1. If $\delta > \delta_2$, then $F(-1) < 0$. So, we have $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Therefore, E_2 is a source.
2. If $\delta < \delta_2$, then $F(-1) > 0$. So, we have $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). Therefore, E_2 is a saddle.
3. If $\delta = \delta_2$, then $F(-1) = 0$. So, we have $\lambda_1 = -1$ and $|\lambda_2| \neq 1$. Therefore, E_2 is non-hyperbolic.

From the analysis, we can acquire that for the positive fixed point $E_1(x_1, u_1)$, if $(m, b, c, a, \alpha, \delta) \in M_1$, where

$$M_1 = \left\{(m, b, c, a, \alpha, \delta): G^2 - 4H > 0, \delta = \delta_1 = \frac{-G - \sqrt{G^2 - 4H}}{H} \ , m, b, c, a, \alpha > 0\right\},$$

then among the eigen values of the $E_1(x_1, u_1)$ is -1 and other is neither 1 nor -1. Therefore, (2) goes through period-2 bifurcation of $E_1(x_1, u_1)$. Furthermore, if $(m, b, c, a, \alpha, \delta) \in N$, where

$$N = \{(m, b, c, a, \alpha, \delta): G^2 - 4H < 0, \delta^* = -\frac{G}{H}, m, b, c, a, \alpha > 0\},$$

then among of the eigen values of $E_1(x_1, u_1)$ are a pair of complex conjugate with modulus one. Therefore, the model (2) goes through Neimark-Sacker bifurcation.

Period-doubling Bifurcation

Based on the analysis in the previous analyses, now we'll discuss the period-2 bifurcation of $E_1(x_1, u_1)$. We choose δ as a bifurcation parameter to investigate the period-doubling bifurcation and we using the bifurcation theory [19, 20]. Taking parameters $(m, b, c, a, \alpha, \delta)$ arbitrarily from M_1, we consider (2) at $E_1(x_1, u_1)$. From $(m, b, c, a, \alpha, \delta) \in M_1$, we have $\delta = \delta_1$. System (2) turns into

$$\begin{cases} x \to x + \delta_1 \left(x \left[1 - x - \dfrac{m}{x + a}\right] - bxu\right), \\ u \to u + \delta_1[-\alpha u + cx]. \end{cases} \tag{5}$$

Then the map (2) has a unique positive fixed point $E_1(x_1, u_1)$ with eigen values $\lambda_1 = -1$ and $\lambda_2 = 3 + G\delta_1$ with $|\lambda_2| \neq 1$. Choosing a perturbation $\tilde{\delta}$ to the parameter δ , system (5) changes to

$$\begin{cases} x \to x + (\delta_1 + \tilde{\delta}) \left(x \left[1 - x - \dfrac{m}{x + a}\right] - bxu\right) \\ u \to u + (\delta_1 + \tilde{\delta})[-\alpha u + cx] \end{cases} \tag{6}$$

where $|\tilde{\delta}| \ll 1$ is a small perturbation parameter.

Suppose that $X = x - x_1$ and $U = u - u_1$. Then we modify $E_1(x_1, u_1)$ of (6) into the origin a follows

$$\begin{pmatrix} X \\ U \end{pmatrix} \to \begin{pmatrix} a_{11}X + a_{12}U + a_{13}X^2 + a_{14}XU + a_{16}X^3 + b_1X\tilde{\delta} + b_2U\tilde{\delta} + b_3X^2\tilde{\delta} + b_4XU\tilde{\delta} + O((|X| + |U| + |\tilde{\delta}|)^4) \\ a_{21}X + a_{22}U + +c_1X\tilde{\delta} + c_2U\tilde{\delta} + O((|X| + |U| + |\tilde{\delta}|)^4) \end{pmatrix}$$

or

$$\begin{pmatrix} X \\ U \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X \\ U \end{pmatrix} + \begin{pmatrix} f(X, U, \tilde{\delta}) \\ g(X, U, \tilde{\delta}) \end{pmatrix}, \tag{7}$$

where

$$a_{11} = 1 - \delta_1 x_1 - \frac{m x_1 \delta_1}{(x_1 + a)^2}, \qquad a_{12} = -b x_1 \delta_1 \, , a_{13} = -\delta_1 + \frac{m\delta_1}{(x_1 + a)^2} - \frac{m x_1 \delta_1}{(x_1 + a)^3},$$

$$a_{14} = -b\delta_1, \quad a_{16} = -\frac{m\delta_1}{(x_1+a)^3}, \quad b_1 = -\frac{mx_1}{(x_1+a)^2} - x_1, \quad b_2 = -bx_1, \tag{8}$$

$$b_3 = -1 + \frac{m}{(x_1+a)^2} - \frac{mx_1}{(x_1+a)^3}, b_4 = -b, a_{21} = c\delta_1, a_{22} = 1 - \alpha\,\delta_1, c_1 = c, c_2 = -\alpha$$

$$f(X,U,\tilde{\delta}) = a_{13}X^2 + a_{14}XU + a_{16}X^3 + b_1X\tilde{\delta} + b_2U\tilde{\delta} + b_3X^2\tilde{\delta} + b_4XU\tilde{\delta} + O((|X| + |U| + |\tilde{\delta}|)^4)$$

$$g(X,U,\tilde{\delta}) = c_1X\tilde{\delta} + c_2U\tilde{\delta} + O((|X| + |U| + |\tilde{\delta}|)^4)$$

Next, we defining an invertible matrix $T_1 = \begin{pmatrix} a_{12} & a_{12} \\ -1-a_{11} & \lambda_2 - a_{11} \end{pmatrix}$ and using translation $\begin{pmatrix} X \\ U \end{pmatrix} = T_1 \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}$, then the map (6) becomes

$$\begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} + \begin{pmatrix} f(\tilde{x},\tilde{u},\tilde{\delta}) \\ g(\tilde{x},\tilde{u},\tilde{\delta}) \end{pmatrix},$$

where

$$f(\tilde{x},\tilde{u},\tilde{\delta}) = \frac{1}{a_{12}(\lambda_2+1)}\Big\{[(\lambda_2 - a_{11})a_{13} - a_{12}a_{23}]X^2 + [(\lambda_2 - a_{11})a_{14} - a_{12}a_{24}]XU + [(\lambda_2 - a_{11})a_{16}]X^3 + [(\lambda_2 - a_{11})b_1 - a_{12}c_1]X\tilde{\delta} + [(\lambda_2 - a_{11})b_2 - a_{12}c_2]U\tilde{\delta} + [(\lambda_2 - a_{11})b_3]X^2\tilde{\delta} + [(\lambda_2 - a_{11})b_4]XU\tilde{\delta} + O\left(\left(|X| + |U| + |\tilde{\delta}|\right)^4\right)\Big\}, \tag{9}$$

$$g(\tilde{x},\tilde{u},\tilde{\delta}) = \frac{1}{a_{12}(\lambda_2+1)}\Big\{[(a_{11}+1)a_{13}]X^2 + [(a_{11}+1)a_{14}]XU + [(a_{11}+1)a_{16}]X^3 + [(a_{11}+1)b_1 + a_{12}c_1]X\tilde{\delta} + [(a_{11}+1)b_2 + a_{12}c_2]U\tilde{\delta} + [(a_{11}+1)b_3]X^2\tilde{\delta} + [(a_{11}+1)b_4]XU\tilde{\delta} + O\left(\left(|X| + |U| + |\tilde{\delta}|\right)^4\right)\Big\},$$

and $X = a_{12}(\tilde{x} + \tilde{u}), U = -(1 + a_{11})\tilde{x} + (\lambda_2 - a_{11})\tilde{u}$.

Now, we apply the center manifold theorem [19]. We set center manifold $W^c(0,0)$ of the system (2) in the small neighborhood of $\tilde{\delta} = 0$. There exists a center manifold that has the following expression for it.

$$W^c(0,0) = \{(\tilde{x},\tilde{u},\tilde{\delta}) \in \mathbb{R}^3 : \tilde{u} = h(\tilde{x},\tilde{\delta}), h(0,0) = 0, Dh(0,0) = 0\}.$$

Assume that

$$h(\tilde{x},\tilde{\delta}) = a_1\tilde{x}^2 + a_2\tilde{x}\tilde{\delta} + a_3\tilde{\delta}^2 + O\left(\left(|\tilde{x}| + |\tilde{\delta}|\right)^3\right), \tag{10}$$

Then the center manifold must satisfy

$$\mathcal{F}\left(h(\tilde{x},\tilde{\delta})\right) = h[A\tilde{x} + f(\tilde{x}, h(\tilde{x},\tilde{\delta}),\tilde{\delta}),\tilde{\delta})] - Bh(\tilde{x},\tilde{\delta}) - g(\tilde{x}, h(\tilde{x},\tilde{\delta}),\tilde{\delta}),\tilde{\delta}) = 0 \tag{11}$$

By substituting (9) and (10) into (11) and comparing the coefficients of like powers of

(10), we obtain that

$$a_1 = \frac{(a_{11}+1)}{1-\lambda_2^2}\{a_{12}a_{13} - a_{14}(a_{11}+1)\},$$

$$a_2 = \frac{1}{a_{12}(\lambda_2+1)^2}\{-a_{12}[(a_{11}+1)b_1 + a_{12}c_1] + (a_{11}+1)[(a_{11}+1)b_2 + a_{12}c_2]\},$$

$$a_3 = 0.$$

Thus, we consider (7) restricted to the $W^c(0,0)$ is given by

$$G : \tilde{x} \to -\tilde{x} + h_1\tilde{x}^2 + h_2\tilde{x}\tilde{\delta} + h_3\tilde{x}^2\tilde{\delta} + h_4\tilde{x}\tilde{\delta}^2 + h_5\tilde{x}^2 + O\left(\left(|\tilde{x}| + |\tilde{\delta}|\right)^3\right), \qquad (11)$$

where

$$h_1 = \frac{(\lambda_2 - a_{11})}{a_{12}(\lambda_2+1)}\{a_{12}a_{13} - a_{14}(a_{11}+1)\},$$

$$h_2 = \frac{1}{a_{12}(\lambda_2+1)}\{a_{12}[(\lambda_2 - a_{11})b_1 - a_{12}c_1] - (a_{11}+1)[(\lambda_2 - a_{11})b_2 - a_{12}c_2]\},$$

$$h_3 = \frac{a_2(\lambda_2 - a_{11})}{(\lambda_2+1)}\{2a_{12}a_{13} + (\lambda_2 - 2a_{11} - 1)a_{14}\} + \frac{a_1}{a_{12}(\lambda_2+1)}\{a_{12}[(\lambda_2 - a_{11})b_1 - a_{12}c_1] + (\lambda_2 - a_{11})[(\lambda_2 - a_{11})b_2 - a_{12}c_2]\} + \frac{(\lambda_2 - a_{11})}{(\lambda_2+1)}\{a_{12}b_3 - b_4(a_{11}+1)\},$$

$$h_4 = \frac{a_2}{a_{12}(\lambda_2+1)}\{a_{12}[(\lambda_2 - a_{11})b_1 - a_{12}c_1] + (\lambda_2 - a_{11})[(\lambda_2 - a_{11})b_2 - a_{12}c_2]\},$$

$$h_5 = \frac{a_1}{(\lambda_2+1)}\{2a_{12}[(\lambda_2 - a_{11})a_{13}] + (\lambda_2 - 2a_{11} - 1)[(\lambda_2 - a_{11})a_{14}]\} + \frac{a_{12}^2[(\lambda_2 - a_{11})a_{16}]}{(\lambda_2+1)}.$$

In order to show that the map (11) undergoes a period- 2 bifurcation, we need to exhibit that α_1 and $\alpha_2 \neq 0$ [20], where

$$\alpha_1 = \left(2G_{\tilde{x}\tilde{\delta}} + G_{\tilde{\delta}}G_{\tilde{x}\tilde{x}}\right)|_{(0,0)} = 2h_2,$$

$$\alpha_2 = \frac{1}{3}G_{\tilde{x}\tilde{x}\tilde{x}} + \frac{1}{2}(G_{\tilde{x}\tilde{x}})^2|_{(0,0)} = 2h_5 + 2h_1^2.$$

Neimark-Sacker Bifurcation

Next we will investigate the Neimark-Sacker bifurcation of $E_1(x_1, u_1)$ when the parameters $(m, b, c, a, \alpha, \delta)$ vary in a small neighborhood of N. Considering the parameters arbitrary from N, the system (2) represented by

$$\begin{cases} x \to x + \delta^*\left(x\left[1 - x - \dfrac{m}{x+a}\right] - bxu\right), \\ u \to u + \delta^*[-\alpha u + cx], \end{cases} \qquad (13)$$

where $\delta^* = -\frac{G}{H}$. Choosing $\bar{\delta}$ as a bifurcation parameter, we consider a perturbation of the map (12) as follows:

$$\begin{cases} x \to x + (\delta^* + \bar{\delta})\left(x\left[1 - x - \dfrac{m}{x+a}\right] - bxu\right) \\ u \to u + (\delta^* + \bar{\delta})[-\alpha u + cx] \end{cases} \quad (14)$$

where $|\bar{\delta}| \ll 1$ is a small perturbation parameter.
Let $X = x - x_1$ and $U = u - u_1$, then the fixed point $E_1(x_1, u_1)$ is transformed into the origin represented by

$$\begin{pmatrix} X \\ U \end{pmatrix} \to \begin{pmatrix} a_{11}X + a_{12}U + a_{13}X^2 + a_{14}XU + a_{16}X^3 + O((|X| + |U|)^4) \\ a_{21}X + a_{22}U + O((|X| + |U|)^4) \end{pmatrix} \quad (15)$$

where all coeficients are given in (7) by substituting $\delta_1 = \delta^* + \bar{\delta}$. The characteristic equation associated with the linearization of model (15) at $(X, U) = (0,0)$ is given by

$$\lambda^2 + p(\bar{\delta})\lambda + q(\bar{\delta}) = 0,$$

where

$$p(\bar{\delta}) = -2 - G(\delta^* + \bar{\delta}), q(\bar{\delta}) = 1 + G(\delta^* + \bar{\delta}) + H\left((\delta^* + \bar{\delta})\right)^2.$$

Hence, since parameters belong to N, we have a pair of complex conjugate eigenvalues λ and $\bar{\lambda}$ with modulus 1 at $(X, U) = (0,0)$ which is

$$\lambda, \bar{\lambda} = \frac{-p(\bar{\delta}) \pm \sqrt{p^2(\bar{\delta}) - 4q(\bar{\delta})}}{2} = \frac{-p(\bar{\delta}) \pm i\sqrt{4q(\bar{\delta}) - p^2(\bar{\delta})}}{2}.$$

and $|\lambda, \bar{\lambda}| = \sqrt{q(\bar{\delta})}$. Then we have $\frac{d|\lambda,\bar{\lambda}|}{d\bar{\delta}}|_{\bar{\delta}=0} = l = -\frac{G}{2}$.

Moreover, it is required that when $\bar{\delta} = 0$, then $\lambda^n, \bar{\delta} = 0 \neq 1, n = 1,2,3,4$. This is equivalent to $p(0) \neq -2,0,1,2$. Note if parameters belong to N then $p(0)^2 < 4q(0) = 4$. Therefore we have $p(0) \neq \pm 2$. Thus we just need to show that $p(0) \neq 0,1$, which leads to $G^2 \neq 2H, 3H$.
Next, we study the normal form of (13) when $\bar{\delta} = 0$. Put

$$\mu = 1 + \frac{G\delta^*}{2}, \omega = \frac{\delta^*}{2}\sqrt{4H - G^2}, T_2 = \begin{pmatrix} a_{12} & 0 \\ \mu - a_{11} & -\omega \end{pmatrix},$$

and T_2 is invertible. Using the translation $\begin{pmatrix} X \\ U \end{pmatrix} = T_2 \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}$, the map (14) becomes

$$\begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} \to \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} + \begin{pmatrix} f(\tilde{x}, \tilde{u}) \\ g(\tilde{x}, \tilde{u}) \end{pmatrix}, \quad (16)$$

where

$$f(\tilde{x}, \tilde{u}) = \frac{1}{a_{12}}\{a_{13}X^2 + a_{14}XU + a_{16}X^3 + O((|X| + |U| + |\tilde{\delta}|)^4\},$$

$$g(\tilde{x}, \tilde{u}) = \frac{1}{a_{12}\omega}\{[(\mu - a_{11})a_{13}]X^2 + [(\mu - a_{11})a_{14}]XU + [(\mu - a_{11})a_{16}]X^3 + O((|X| + |U|)^4)\},$$

and

$$X = a_{12}\tilde{x}, U = (\mu - a_{11})\tilde{x} - \omega\tilde{u}.$$

Then we get.

$$\tilde{f}_{\tilde{x}\tilde{x}} = \frac{2}{a_{12}}[a_{12}^2 a_{13} + a_{14}a_{12}(\mu - a_{11})], \tilde{f}_{\tilde{x}\tilde{u}} = -\frac{\omega a_{12}a_{14}}{a_{12}}, \tilde{f}_{\tilde{u}\tilde{u}} = 0, \tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} = 6a_{16}a_{12}^2,$$

$$\tilde{f}_{\tilde{x}\tilde{x}\tilde{u}}, \tilde{f}_{\tilde{x}\tilde{u}\tilde{u}}, \tilde{f}_{\tilde{u}\tilde{u}\tilde{u}} = 0, \tilde{g}_{\tilde{x}\tilde{x}} = \frac{2(\mu - a_{11})}{\omega}\{a_{12}a_{13} + a_{14}(\mu - a_{11})\},$$

$$\tilde{g}_{\tilde{x}\tilde{u}} = -a_{14}(\mu - a_{11}), \tilde{g}_{\tilde{u}\tilde{u}} = 0, \tilde{g}_{\tilde{x}\tilde{x}\tilde{x}} = \frac{6}{\omega}\{a_{12}^2[(\mu - a_{11})a_{16}]\}, \tilde{g}_{\tilde{x}\tilde{x}\tilde{u}}, \tilde{g}_{\tilde{x}\tilde{u}\tilde{u}}, \tilde{g}_{\tilde{u}\tilde{u}\tilde{u}} = 0.$$

The map (14) can undergoes the Neimark-Sacker bifurcation when $\gamma \neq 0$ [8]:

(17)

$$\gamma = -Re\left[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\xi_{11}\xi_{20}\right] - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\bar{\lambda}\xi_{21}) \neq 0,$$

where

$$\xi_{20} = \frac{1}{8}\left[\left(\tilde{f}_{\tilde{x}\tilde{x}} - \tilde{f}_{\tilde{u}\tilde{u}} + 2\tilde{g}_{\tilde{x}\tilde{u}}\right) + i\left(\tilde{g}_{\tilde{x}\tilde{x}} - \tilde{g}_{\tilde{u}\tilde{u}} - 2\tilde{f}_{\tilde{x}\tilde{u}}\right)\right],$$
$$\xi_{11} = \frac{1}{4}\left[\left(\tilde{f}_{\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{u}\tilde{u}}\right) + i(\tilde{g}_{\tilde{x}\tilde{x}} + \tilde{g}_{\tilde{u}\tilde{u}})\right],$$
$$\xi_{02} = \frac{1}{8}\left[\left(\tilde{f}_{\tilde{x}\tilde{x}} - \tilde{f}_{\tilde{u}\tilde{u}} - 2\tilde{g}_{\tilde{x}\tilde{u}}\right) + i\left(\tilde{g}_{\tilde{x}\tilde{x}} - \tilde{g}_{\tilde{u}\tilde{u}} + 2\tilde{f}_{\tilde{x}\tilde{u}}\right)\right],$$
$$\xi_{21} = \frac{1}{16}\left[\left(\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{x}\tilde{u}\tilde{u}} + \tilde{g}_{\tilde{x}\tilde{x}\tilde{u}} + \tilde{g}_{\tilde{u}\tilde{u}\tilde{u}}\right) + i\left(\tilde{g}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{g}_{\tilde{x}\tilde{u}\tilde{u}} - \tilde{f}_{\tilde{x}\tilde{x}\tilde{u}} - \tilde{f}_{\tilde{u}\tilde{u}\tilde{u}}\right)\right].$$

Numerical Simulations

Now we will show the bifurcation diagrams and phase portraits of system (2) to confirm our theoretical analysis and illustrate the complex dynamical behaviors using numerical simulations. The bifurcation parameters will be examined in the following two cases.

1) Varying δ in the range $2 \leq \delta \leq 3$ and fixing $m = 0.1, b = 0.21, c = 0.005, a = 20, \alpha = 0.053$.
2) Varying δ in the range $1.2 \leq \delta \leq 1.6$ and fixing $m = 0.01, b = 1.4, c = 1.2, a = 20, \alpha = 0.9$.

Case (i) On the basis of Proposition 1, we find out that the map (2) has a unique positive fixed point $E_1(x_1, u_1)$. By calculation, the flip bifurcation appear from the fixed point $E_1(0.976,0.092)$ at $\delta = \delta_1 = 2.05$ with $\alpha_1 = -1.95 \neq 0$ and $\alpha_2 = 1.465 > 0$. The period-doubling bifurcation happens and the orbits are stable. From bifurcation diagram at Fig. 1, the fixed point of system (2) is stable when $\delta_1 < 2.05$. The phase portraits which are associated with Fig. 1 are diposed in Fig. 2. If $\delta = 1.8 < \delta_1$, then system (2) stable. Also we observe from Fig. 2, when $\delta = 2.1$ that there are period-2, when $\delta = 2.55$ that there are period-4, when $\delta = 2.8$ the chatic sets are seen.

Case (ii) For case (ii), we choosing the parameter values as $m = 0.01, b = 1.4, c = 1.2, a = 20, \alpha = 0.9$. The initial value $(x_0, u_0) = (0.35,0.5)$ and after a simple calculation for the fixed point $E_1(0.349,0.465)$, we get $\delta^* = 1.388$, $\lambda, \bar{\lambda} = 0.1333750375 \pm 0.9910656375\, i$. For $\delta = \delta^* = 1.388$, we've got $|\lambda, \bar{\lambda}| = 1, l = 0.62432 > 0, \gamma = -0.9915 \neq 0$.

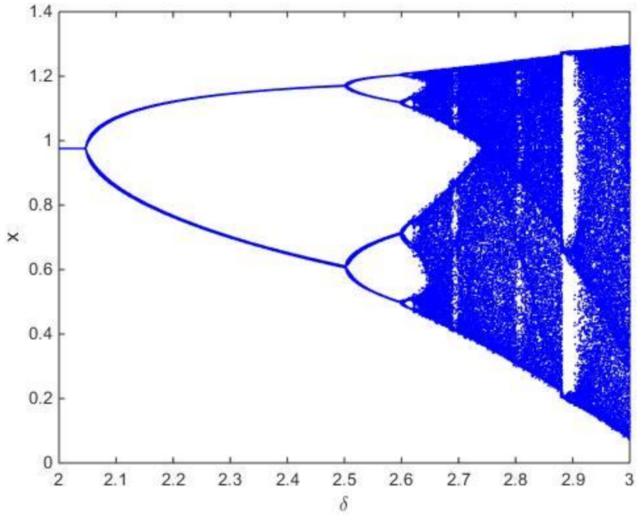
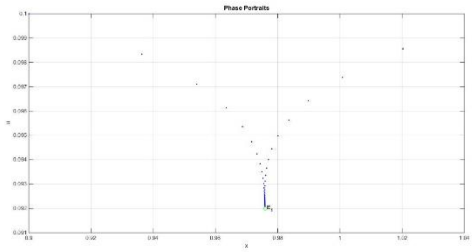
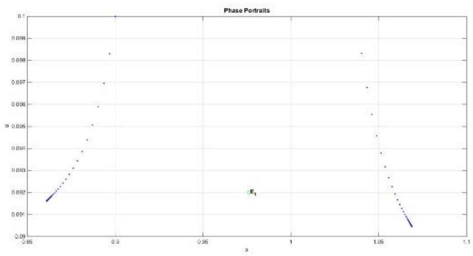


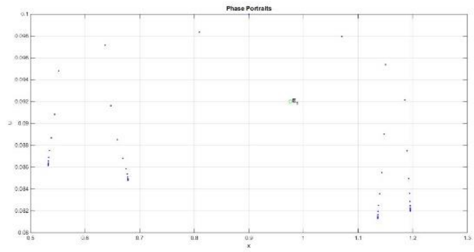
Figure 1. Period doubling bifurcation diagram of system (2) for case (i).



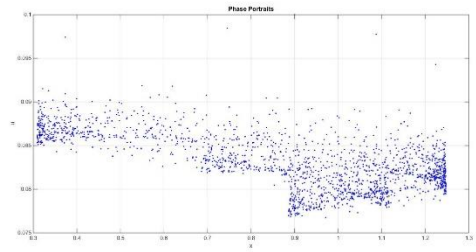
$\delta = 1.8$



$\delta = 2.1$



$\delta = 2.55$



$\delta = 2.8$

Figure 2. Phase portraits for various values of δ.

We conduct the bifurcation diagrams in (δ, x) plane in Fig 3. We can see that the fixed point $E_1(0.349{,}0.465)$ of system (2) is stable for $\delta < 1.388$, that it loses its stability at $\delta = 1.388$, and that an invariant circle emerges if the parameter δ pass 1.388. The phase portraits which are associated with Fig. 3 are diposed in Fig. 4 and the attractive cycle is smooth, as can be seen.

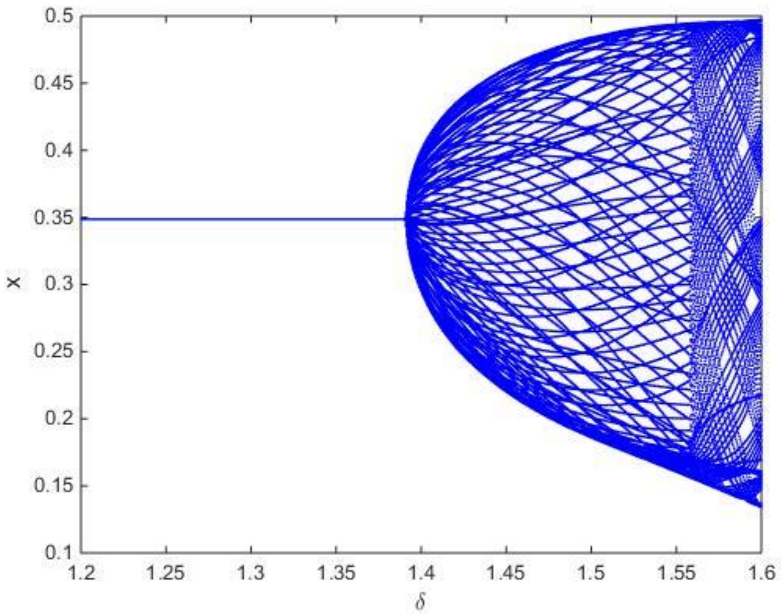
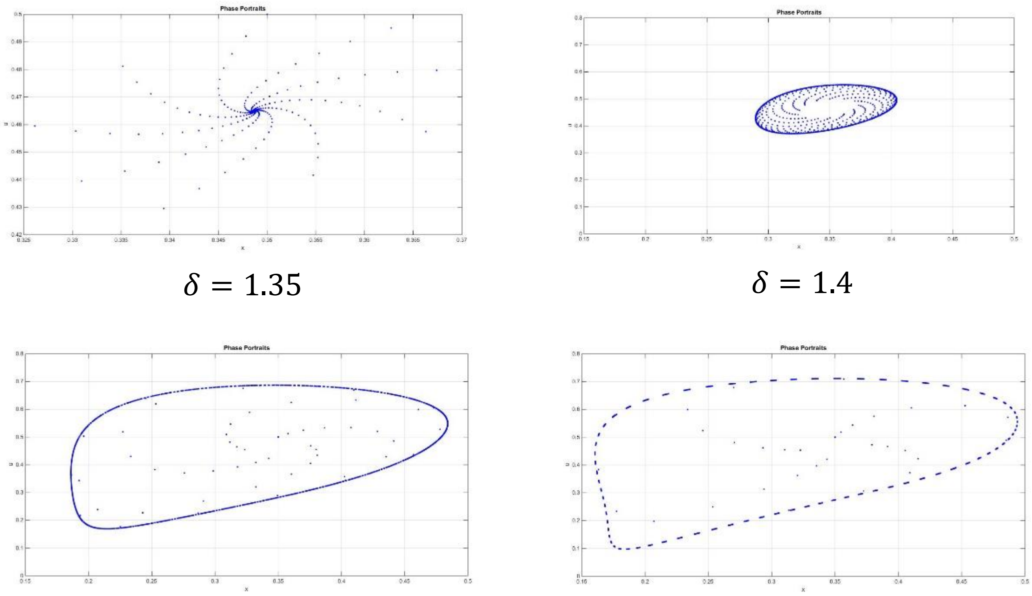


Figure 3. Neimark-Sacker bifurcation diagram of system (2) for case (ii).



$\delta = 1.35$ $\delta = 1.4$

$\delta = 1.5$ $\delta = 1.55$

Figure 4. Phase portraits for various values of δ.

CONCLUSIONS

The primary focus of this study has been the dynamic behaviour of a discrete logistic model with additive Allee effect and feedback control. Through the use of the bifurcation theory, we exhibit that the positive fixed point E_1 can undergoes flip and Neimark-Sacker bifurcation. Moreover, when δ is choosen as a bifurcation parameter, numerical simulations show that system (2) much interesting dynamical behaviors, including period-doubling orbits, the chaotic sets, and attracting invariant circles.

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