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Jensen m-Convexity on Set-Valued Function

Eko Dedi Pramana*, Ratno Bagus Edy Wibowo, Mohamad Muslikh

[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya, Indonesia.

Email: eko.d.pramana@gmail.com

ABSTRACT

Jensen's inequality is a concept in convex analysis, providing essential insights into the behavior of convex functions. This paper discusses the class of Jensen m-convex functions and their properties, leading to enhancements of these functions and Jensen's inequalities for set-valued functions. One of the research areas in m-convexity focuses on the value of t depending on m, which belongs to the interval (0,1] which is Jensen m-convexity. By combining the properties of m-convexity in set-valued functions with Jensen m-convexity in real-valued functions, we provide several characterizations and explores various algebraic properties. Additionally, it introduces a discrete Jensen-type inclusion.

Keywords: Jensen m-convex; Jensen-type inclusion; Jensen m-convex set; Set-valued function

INTRODUCTION

Convexity theory is a fundamental aspect of mathematical analysis with wide-ranging applications across various disciplines[1]. One significant development within this theory is the notion of m-convexity, a generalization of convexity introduced to extend the scope of traditional convex functions [2]. The refinements of the well-known Jensen inequality for convex mappings were introduced by Dragomir and Toader and leading development concept of m-convexity[3]. Then, some inequalities for m-convex functions are obtained by Dragomir[4]. Furthermore, the research was continued by Bakula, Ozdemir, and Pečarić who established the product of two m-convex function which called (α, m)-convex[5].

Lara has provided the class of convexity in various fields such as operator theory, linear spaces, and functional analysis. By setting some algebraic and topological properties for m-convex functions, Lara, Rosales, and Sanchez obtained some properties for m- convex functions, algebraic, inequalities of Fejer-type[6]. Lara, et. al. extended this research of m-convexity on Hilbert space by defining m-convex function of self-adjoint operator[7]. The concept of m-convexity, initially introduced by Toader in 1984, has been further developed in real linear spaces by Lara et al., beyond its applications in Hilbert space[8]. Furthermore, the properties and characterization of m-concave function in real linear space also introduced[9].

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In recent years, interest has grown significantly in studying m-convex functions and their properties, particularly in the context of set-valued functions. Lara, et. al. introduced a m-convex set-valued function which defined in non-empty m-convex subset of a real linear spaces. The properties and algebraic operation for m-convex function also introduced as well as with Jensen inclusion of m-convex functions[10]. They, extended the result and obtained strong version of m-convex set-valued function[11].

Jensen's inequality is a fundamental concept in convex analysis, offering crucial perspectives on the behavior of convex functions[12]. The extension of this inequality to set-valued functions is a dynamic research area, motivated by its potential applications in optimization, economics, statistics, and various other disciplines. Comprehending the attributes and features of set-valued functions within the framework of m-convexity is pivotal for tackling intricate optimization challenges and representing the uncertainties that are intrinsic to real-world scenarios[13].

Lara et. al. continued the research about m-convexity in a specific class called Jensen m-convex which defining the value of m belongs to $(0,1]$. This function generate new kind of functional convexity[14]. In functional analysis a functional equation of Jensen type which called Jensen m-convex inequality had certain characterization properties in its solution[15]. On the other hand, Lara delivered a strong version of Jensen m-convexity[14], [16], which remains unexplored in certain areas, particularly in the concept of Set-Valued functions.

The primary goal of this research is to present the concept of a Jensen m-convex set-valued function and to elucidate specific properties of these functions, thereby generalizing the established concept of convexity for set-valued functions. These functions are defined on a nonempty Jensen m-convex subset of a real linear space, taking values in the set of nonempty subsets of another real linear space.

METHODS

The methodology employed in composing this research article involves a literature review of various related articles and books. Within this literature, both the definition and properties of the m-convex function have been explored, as well as the characteristics of m-convexity in set-valued functions. The research methodology for this study is outlined as follows:

i. Construct a Jensen m-convex set-valued function with a domain that is a Jensen m-convex subset of linear spaces.
ii. Investigate the algebraic characteristics of the Jensen m-convex function, including operations within its domain by incorporating additional properties, such as starshaped characteristics.
iii. Explore the relationship between a graph and the Jensen m-convex set-valued function. Subsequently, formulate the theorem pertaining to the Jensen m-convex set-valued function.
iv. Proving the closedness of a collection of Jensen m-convex set-valued functions under operations such as addition, union, intersection, Cartesian product, and function composition.
v. Construct theorem of Jensen inclusion discrete-type of Jensen m-convex set-valued function

Therefore, by utilizing the definitions and properties outlined in prior studies, this research investigates Jensen m-convexity in set-valued functions. Initially, the discussion focuses on Jensen m-convexity and m-convexity.

Let Ω be a non-empty subset of $\mathbb{R}^n$. Ω is defined as a convex set if, for any x, y in Ω and any t in [0,1], the combination $tx + (1-t)y$ also belongs to Ω. Geometrically, this implies that the line segment connecting any two points in Ω lies entirely within Ω. Interval is denoted I.

Definition 1 (m-Convex set, [10])
Let X be a real linear space and $m \in [0,1]$. A non-empty set $D \subseteq X$ is called m-convex set if for any $x, y \in D$ and $t \in [0,1]$ the point $tx + m(1-t)y \in D$.

Definition 2 (m-Convex function,[3], [4], [6], [8])
Let $m \in [0,1]$. A function $f: [0, b] \to \mathbb{R} (b > 0)$ is said to be m-convex in the interval $[0, b]$, if for any $x, y \in [0, b]$ and $t \in [0,1]$ we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1}$$

The concept of an m-convex function can be interpreted geometrically. a function $f: [0, b] \to \mathbb{R}$ is considered m-convex if, for any $x, y \in [0, b]$ with $x \leq y$, the line segment connecting the points $(x, f(x))$ and $(my, mf(y))$ lies above the graph of f on the interval $[x, my]$.

The foundations of convex function theory are attributed to Jensen. It is also noted that the original definition of a real-valued convex function, as established by Jensen, is what is now commonly referred to as a mid-convex function or Jensen convex. A Jensen convex function is a function $f: I \to \mathbb{R}$ that satisfies the following inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \tag{2}$$

for all $x, y \in I$.

Definition 3 (Jensen m-convex function,[16])
Let $m \in (0,1]$. A function $f: [0, +\infty) \to \mathbb{R}$ is called Jensen m-convex function in the interval $[0, +\infty)$ if satisfies the inequality

$$f\left(\frac{x+y}{c_m}\right) \leq \frac{f(x)+f(y)}{c_m}, \tag{3}$$

for all $x, y \in [0, +\infty)$.

The concept of a Jensen m-convex function pertains to each m within the interval (0,1], for a specific value t in the open interval (0,1), which is denoted with respect to m as t_m. Given any $(x, y) \in [0, b] \times [0, b]$, $t_m x + m(1-t_m)y$ can be represented as

$$t_m x + m(1-t_m)y = \frac{x+y}{c_m}, \tag{4}$$

where c_m is a constant that depends only on m. By rewriting Equation (4) as

$$\left(t_m - \frac{1}{c_m}\right)x + \left(m(1-t_m) - \frac{1}{c_m}\right)y = 0,$$

and solving for c_m and t_m we obtain

$$c_m = 1 + \frac{1}{m}, \quad \text{and} \quad t_m = \frac{m}{m+1}. \tag{5}$$

Definition 4 (*starshaped* function,[16])
A function $f: I \to \mathbb{R}$ that satisfies the inequality

$$f(tx) \leq tf(x), \tag{6}$$

for all $x \in I$ and $t \in [0,1]$ will be called *starshaped* function in the interval I.Since Jensen m-convex functions exhibit properties as outlined in (5), the concept of *starshaped*

functions are utilized to produce algebraic properties within Jensen m-convex functions.
Definition 5 (m-Convex set-valued function,[10])
Let $m \in [0,1]$. Let D be any non-empty m-convex subset of X. A set-valued function $F: D \to n(Y)$ is called m-convex if satisfies the inclusion

$$tF(x) + m(1-t)F(y) \subseteq F(tx + m(1-t)y), \tag{7}$$

for all $x, y \in D$ and $t \in [0,1]$.

RESULTS AND DISCUSSION

In this section, we present some results on Jensen m-convexity for set-valued functions. Our focus extends the concept of Jensen m-convexity to set-valued functions. Let X and Y denote real linear spaces, $n(Y)$ represent the family of all non-empty subsets of Y, and let m and t_m be fixed real numbers in the interval (0,1]. The Jensen m-convex set is defined by substituting the variables t and m with a positive constant, denoted as c_m.

Definition 6 Let X be a real linear space and $m \in (0,1]$. A non-empty set $D_m \subseteq X$ is called Jensen m-convex if for any $x, y \in D_m$ the point $\frac{x+y}{c_m} \in D_m$ with $c_m = 1 + \frac{1}{m}$.

Example 1 Set $A = \{x | 0 \leq x \leq b; x, y \in \mathbb{Q}\}$ is Jensen m-convex set for rational number $m \in (0,1]$.

Proof. Let $m \in (0,1]$ is rational number. Then for all $x, y \in (0, b]$ we have $\frac{x+y}{c_m}$ is rational number less than b. This shows the Jensen m-convexity of set A.

Definition 7 Let $m \in (0,1]$ and D_m be a Jensen m-convex set. Set-valued function $F: D_m \to n(Y)$ is said to be Jensen m-convex if satisfies the inclusion

$$\frac{F(x) + F(y)}{c_m} \subseteq F\left(\frac{x+y}{c_m}\right), \tag{8}$$

for all $x, y \in D_m$ with c_m is a constant value depend on m.

The distinction between Jensen m-convex and m-convex functions lies in the role of m within the inequality. In Definition 2 and Definition 5, the value of t is arbitrary within the interval [0,1], whereas for Jensen m-convex functions, the value of t_m is dependent on m. From Inclusion (8) by taking value $x = y$, we have

$$\frac{2F(x)}{c_m} \subseteq F\left(\frac{2x}{c_m}\right),$$

given that $\lambda_m = \frac{2}{c_m}$, we obtain

$$\lambda_m F(x) \subseteq F(\lambda_m x). \tag{9}$$

Proposition 1 Let $m \in (0,1]$. If $F: D_m \to n(Y)$ is a Jensen m-convex set-valued function and starshaped function, then F satisfies the inclusion

$$\frac{1}{c_m}[tF(c_m x) + (1-t)F(c_m y)] \subseteq F(tx + (1-t)y), \tag{10}$$

for any $x, y \in D_m$ and $t \in (0,1)$.

Proof.
Let $x, y \in D_m$ and $t \in (0,1)$, then we have

$$\begin{aligned}\frac{1}{c_m}[tF(c_m x) + (1-t)F(c_m y)] &\subseteq \frac{1}{c_m}[F(c_m tx) + F(c_m(1-t)y)] \\ &\subseteq F\left(\frac{c_m tx + c_m(1-t)y}{c_m}\right) \\ &= F(tx + (1-t)y)\end{aligned}$$

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Proposition 2 Let $m \in (0,1]$. If $F: D_m \to n(Y)$ is a Jensen m-convex set-valued function and starshaped function, then for all positive real number $x_1, x_2, y_1, y_2 \in D_m$ we have

$$\frac{1}{c_m}\left[y_1 F\left(\frac{c_m x_1}{y_1} + y_2 F\left(\frac{c_m x_2}{y_2}\right)\right)\right] \subseteq (y_1 + y_2) F\left(\frac{x_1 + x_2}{y_1 + y_2}\right). \quad (11)$$

Proof.
Let F Jensen m-convex set-valued function and starshaped, then for all real positive $x_1, x_2, y_1, y_2 \in D_m$ we have

$$\begin{aligned}&\frac{1}{c_m}\left[y_1 F\left(\frac{c_m x_1}{y_1}\right) + y_2 F\left(\frac{c_m x_2}{y_2}\right)\right] \\ &= \frac{y_1 + y_2}{c_m}\left[\frac{y_1}{y_1 + y_2} F\left(\frac{c_m x_1}{y_1}\right) + \frac{y_2}{y_1 + y_2} F\left(\frac{c_m x_2}{y_2}\right)\right] \\ &= \frac{y_1 + y_2}{c_m}\left[\frac{y_1}{y_1 + y_2} F\left(\frac{c_m x_1}{y_1}\right) + \left(1 - \frac{y_1}{y_1 + y_2}\right) F\left(\frac{c_m x_2}{y_2}\right)\right] \\ &\subseteq (y_1 + y_2) F\left(\frac{x_1 + x_2}{y_1 + y_2}\right)\end{aligned}$$

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Proposition 3 Let $m \in (0,1]$. If $F: D_m \to n(Y)$ Jensen m-convex, then image of F for any subset Jensen m-convex of D_m are Jensen m-convex set of linear space Y.

Proof.
Let A be a non-empty Jensen m-convex subset of D_m, and let a, b belong to $F(A)$, where $F(A)$ represents the union of F over $x \in A$. Then, by taking $a \in F(x)$ and $b \in F(y)$ for some $x, y \in A$, we have

$$\frac{a + b}{c_m} \in \frac{F(x) + F(y)}{c_m} \subseteq F\left(\frac{(x + y)}{c_m}\right).$$

Since A is Jensen m-convex, $\frac{x+y}{c_m} \in A$, implying that $\frac{a+b}{c_m} \in F(A)$. Therefore, $F(A)$ is a Jensen m-convex function on Y.

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Theorem 1 Let $F: D_m \subseteq X \to n(Y)$ be a set-valued function. F is said to be Jensen m-convex function if and only if its graph, denoted by $Gr_m(F) \coloneqq \{(x, y) \in X \times Y \mid x \in D_m, y \in F(x)\}$, is Jensen m-convex set in $X \times Y$.

Proof.
($\Longrightarrow$) Let $m \in (0,1]$. For any $(x_1, y_1), (x_2, y_2) \in Gr_m$ such that $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ and satisfies $\frac{y_1+y_2}{c_m} \in \frac{F(x_1)+F(x_2)}{c_m}$. Since, F is Jensen m-convex we have $\left(\frac{x_1+x_2}{c_m}, \frac{y_1+y_2}{c_m}\right) \in Gr_m$ that implies,

$$\frac{(x_1, y_1) + (x_2, y_2)}{c_m} \in Gr_m(F).$$

($\Longleftarrow$) Let $m \in (0,1]$. For any $x, y \in D_m$, select $y \in \frac{F(x_1)+F(x_2)}{c_m}$ such that y can be expressed as $\frac{y_1+y_2}{c_m}$ where $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$. Suppose Gr_m is a Jensen m-convex set over $X \times Y$ with respect to m. Then for $(x_1, y_1), (x_2, y_2) \in Gr_m$, we have $\frac{(x_1,y_1)+(x_2,y_2)}{c_m} \in Gr_m(F)$, that implies

$$\left(\frac{x_1 + x_2}{c_m}, \frac{y_1 + y_2}{c_m}\right) \in Gr_m.$$

Hence, we have $y = \frac{y_1+y_2}{c_m} \in F\left(\frac{x_1+x_2}{c_m}\right)$. Given that y is constructed from any $x_1, x_2 \in D_m$, it follows that F is Jensen m-convex.

Definition 8 Let $F_1, F_2: D_m \subseteq X \to n(Y)$ be a set-valued function, we have

1) The union function of F_1 and F_2 is a set-valued function $F_1 \cup F_2: D_m \to n(Y)$ which defined by $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$ for all $x \in D_m$.
2) The intersection function of F_1 and F_2 is a set-valued function $F_1 \cap F_2: D_m \to n(Y)$ which defined by $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$ for all $x \in D_m$.
3) The sum function of F_1 and F_2 is a set-valued function $(F_1 + F_2)(x) = F_1(x) + F_2(x)$ for all $x \in D_m$.

Definition 9 Let X, Y, Z be linear spaces and let D_m be a subset of X. Then we have

1) If $F_1: D_m \to n(Y)$ and $F_2: D_m \to n(Z)$ are set-valued functions. Then the Cartesian product of F_1 and F_2 is set-valued function $(F_1 \times F_2)(x): D_m \to n(Y) \times n(Z)$ which defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for all $x \in D_m$.
2) If $F_1: D_m \to n(Y)$ and $F_2: n(Y) \to n(Z)$ are set-valued function. Then the composition function of F_1 and F_2 is the set-valued function $F_2 \circ F_1: D_m \to n(Z)$ given by $(F_2 \circ F_1)(x) = F_2(F_1(x)) = \bigcup_{y \in F_1(x)} F_2(y)$ for all $x \in D_m$.

Proposition 4 Let $m \in (0,1]$. If $F_1, F_2: D_m \to n(Y)$ are Jensen m-convex set-valued functions which satisfies

$$F_1(x) \subseteq F_2(x)\big(or\ F_2(x) \subseteq F_1(x)\big),$$

for each $x \in D_m$. Then the union function of F_1 and F_2 is Jensen m-convex set-valued function.

Proof Let $m \in (0,1]$ and $x_1, x_2 \in D_m$. Without loss of generalization, let $F_1(x) \subseteq F_2(x)$. Then by properties of Jensen m-convex set-valued function of F_1, F_2 we have

$$\begin{aligned}
\frac{(F_1 \cup F_2)(x_1) + (F_1 \cup F_2\ )(x_2)}{c_m} &= \frac{F_1(x_1) \cup F_2(x_1) + F_1(x_2) \cup F_2(x_2)}{c_m} \\
&= \frac{F_2(x_1) + F_2(x_2)}{c_m} \\
&\subseteq F_2\left(\frac{x_1 + x_2}{c_m}\right) \\
&= F_1\left(\frac{x_1 + x_2}{c_m}\right) \cup F_2\left(\frac{x_1 + x_2}{c_m}\right) \\
&= (F_1 \cup F_2)\left(\frac{x_1 + x_2}{c_m}\right)
\end{aligned}$$

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Furthermore, $(x, y) \in Gr_m(F_1 \cup F_2)$ if and only if $y \in (F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$ which is equivalent with

$$Gr_m(F_1 \cup F_2) = Gr_m(F_1) \cup Gr_m(F_2). \tag{12}$$

Proposition 5 Let $m \in (0,1]$. If $F_1, F_2: D_m \to n(Y)$ are Jensen m-convex set-valued functions. Then the intersection function of F_1 and F_2 is Jensen m-convex set-valued function.

Proof Since F_1 and F_2 are Jensen m-convex set-valued function, we have $Gr_m(F_1)$ and $Gr_m(F_2)$ are Jensen m-convex and

$$\begin{aligned} Gr_m(F_1) \cap Gr_m(F_2) &= \{(x,y) | x \in D_m(F_1) \cap D_m(F_2), y \in F_1(x) \cap F_2(x)\} \\ &= \{(x,y) | x \in D_m(F_1 \cap F_2), y \in (F_1 \cap F_2)(x)\} \\ &= Gr_m(F_1 \cap F_2). \end{aligned}$$

We can conclude that $Gr_m(F_1 \cap F_2)$ is Jensen m-convex. Then by Theorem 1, $F_1 \cap F_2$ is Jensen m-convex set-valued function.

Proposition 6 Let $m \in (0,1]$. If $F_1, F_2: D_m \to n(Y)$ are Jensen m-convex set-valued functions. Then the sum function of F_1 and F_2 is Jensen m-convex set-valued function.

Proof Let $x, y \in D_m$, since F_1 and F_2 Jensen m-convex set-valued function we have

$$\begin{aligned} \frac{(F_1+F_2)(x) + (F_1+F_2)(y)}{c_m} &= \frac{F_1(x) + F_2(x) + F_1(y) + F_2(y)}{c_m} \\ &= \frac{F_1(x) + F_1(y)}{c_m} + \frac{F_2(x) + F_2(y)}{c_m} \\ &\subseteq F_1\left(\frac{x+y}{c_m}\right) + F_2\left(\frac{x+y}{c_m}\right) \\ &= (F_1+F_2)\left(\frac{x+y}{c_m}\right). \end{aligned}$$

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Proposition 7 Let $m \in (0,1]$. If $F_1, F_2: D_m \to n(Y)$ are Jensen m-convex set-valued functions. Then the Cartesian product of F_1 and F_2 is Jensen m-convex set-valued function.

Proof Let $x, y \in D_m$, since F_1 and F_2 Jensen m-convex set-valued function we have

$$\begin{aligned} \frac{(F_1 \times F_2)(x) + (F_1 \times F_2)(y)}{c_m} &= \frac{F_1(x) \times F_2(x) + F_1(y) \times F_2(y)}{c_m} \\ &= \frac{\big(F_1(x) + F_1(y)\big) \times \big(F_2(x) + F_2(y)\big)}{c_m} \\ &= \frac{F_1(x) + F_1(y)}{c_m} \times \frac{F_2(x) + F_2(y)}{c_m} \\ &\subseteq F_1\left(\frac{x+y}{c_m}\right) \times F_2\left(\frac{x+y}{c_m}\right) = (F_1 \times F_2)\left(\frac{x+y}{c_m}\right). \end{aligned}$$

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Proposition 8 Let $m \in (0,1]$. If $F_1: D_m \to n(Y)$ and $F_2: n(Y) \to n(Z)$ are Jensen m-convex set-valued functions. Then the composition function $(F_2 \circ F_1)$ is Jensen m-convex set-valued function.

Proof Let $x, y \in D_m$, since F_1 and F_2 Jensen m-convex set-valued function we have

$$\frac{(F_2 \circ F_1)(x) + (F_2 \circ F_1)(y)}{c_m} = \frac{F_2\big(F_1(x)\big) + F_2\big(F_1(y)\big)}{c_m}$$

$$\subseteq F_2\left(\frac{F_1(x)+F_1(y)}{c_m}\right)$$
$$\subseteq F_2\left(F_1\left(\frac{x+y}{c_m}\right)\right)$$
$$=(F_1\circ F_2)\left(\frac{x+y}{c_m}\right).$$

■

Theorem 2 Let $m \in (0,1]$ and a function $F: D_m \to n(Y)$ be a Jensen m-convex set-valued function and starshaped. Then for $n \geq 2$ and $A_1, A_2, \dots A_n$ are non-empty subset of D_m satisfies the inclusion,

$$\frac{1}{n}\left[\frac{1}{c_m^{n-1}}F(c_m^{n-1}A_1)+\sum_{k=1}^{n-1}\frac{F(c_m^{n-k}A_{k+1})}{c_m^{n-k}}\right]\subseteq F\left(\frac{1}{n}\sum_{k=1}^{n}A_k\right). \tag{13}$$

Proof Since F is starshaped, for all n we have

$$\frac{1}{n}F\left(\sum_{k=1}^{n}A_k\right)\subseteq F\left(\frac{1}{n}\sum_{k=1}^{n}A_k\right).$$

Therefore, to establish the argument, it suffices to demonstrate the following inclusions,

$$\frac{1}{c_m^{n-1}}F(c_m^{n-1}A_1)+\sum_{k=1}^{n-1}\frac{F(c_m^{n-k}A_{k+1})}{c_m^{n-k}}\subseteq F\left(\frac{1}{n}\sum_{k=1}^{n}A_k\right).$$

For $n=2$, by applying Proposition 1 and selecting $t=\frac{1}{2}$, we obtain

$$\frac{1}{2}\left[\frac{F(c_mA_1)}{c_m}+\frac{F(c_mA_2)}{c_m}\right]\subseteq F\left(\frac{1}{2}(A_1+A_2)\right).$$

Assume that Inclusion (13) holds for n, now for $n+1$

$$\frac{1}{c_m^{n}}F(c_m^{n}A_1)+\sum_{k=1}^{n}\frac{F(c_m^{n+1-k}A_{k+1})}{c_m^{n+1-k}}$$
$$=\frac{1}{c_m}\left[\frac{1}{c_m^{n-1}}F(c_m^{n-1}c_mA_1)+\sum_{k=1}^{n-1}\frac{F(c_m^{n-k}c_mA_{k+1})}{c_m^{n-k}}\right]+\frac{1}{c_m}F(c_mA_{n+1})$$
$$\subseteq\frac{1}{c_m}\left[F\left(\sum_{k=1}^{n}c_mA_k\right)+F(c_mA_{n+1})\right]$$
$$\subseteq F\left(\frac{\sum_{k=1}^{n}c_mA_k+c_mA_{k+1}}{c_m}\right)=F\left(\sum_{k=1}^{n+1}A_k\right).$$

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Corollary 1 Let $m \in (0,1]$ and a function $F: D_m \to n(Y)$ be a Jensen m-convex set-valued function and starshaped. Then for $n \geq 2$ and $x_1, x_2, \dots x_n \in D_m$ satisfies the inclusion,

$$\frac{1}{n}\left[\frac{1}{c_m^{n-1}}F(c_m^{n-1}x_1)+\sum_{k=1}^{n-1}\frac{F(c_m^{n-k}x_{k+1})}{c_m^{n-k}}\right]\subseteq F\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right). \tag{14}$$

CONCLUSIONS

Jensen m-convex set-valued function attain the same some properties from m-convexity set-valued function. The Jensen m-convex set-valued function F to meet the inclusion principle's properties for combinations of elements in D_m, it is deduced that F must additionally satisfy the condition of being starshaped. Upon examining the graph of a set-valued function, it is determined that a set-valued function F is Jensen m-convex set-valued function if and only if its graph exhibits Jensen m-convexity. Collection of Jensen m-convex set-valued functions are closed under set-operations. Specifically, we establish the discrete Jensen-type inclusion of Jensen m-convex set-valued function.

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