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Analysis of a Compressible Korteweg Fluid Model with Slip Boundary Conditions in a Three-Dimensional Half-Space

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ABSTRACT

This paper analyzes the resolvent system for the Navier-Stokes-Korteweg model with slip boundary conditions in a 3D half-space. The system is reduced using a partial Fourier transform, and the homogeneous resolvent problem is solved, demonstrating the existence of a solution operator under certain coefficient conditions. This result provides a foundational approach for studying the non-linear Navier-Stokes-Korteweg system

which is a key system equation for describing the motion of viscous fluids.

Keywords: Compressible Fluid; Navier Stokes Korteweg; Resolvent Problem; Partial Fourier Transform

INTRODUCTION

The Navier-Stokes-Korteweg (NSK) model is an essential tool for describing the behavior of compressible fluids, particularly in situations where capillary effects play a significant role, such as in phase transitions between gas and liquid states. This model extends the classical Navier-Stokes equations by incorporating capillarity, making it highly relevant for understanding phenomena in multi-phase fluid flows. Including capillary forces allows for a more accurate depiction of fluid dynamics at phase boundaries, critical in both natural and industrial applications involving multi-phase flows.

Previous research on the NSK model has explored its various aspects, including the existence and uniqueness of solutions under different boundary conditions and in varying spatial dimensions. Notably, studies by Liu et al., Danchin & Desjardins, and Hattori & Li have examined phase transitions and the behavior of compressible fluids in different settings [1]. In the research by *Danchin & Desjardins*, the existence and uniqueness of smooth solutions were shown, which are suitable for the isothermal compressible capillary fluid model, thus used as a phase transition model [2]. In the study by H. Hattori & Li, the existence of a strong unique solution for initial data requiring higher regularity than Danchin and Desjardins was demonstrated this system is a

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simplified isothermal version [3]. In *H. Hattori & Li*, solutions for the high-dimensional Navier-Stokes-Korteweg system were shown for small initial data [4].

In N-dimensional space, Saito proves the existence of the R-bounded solution operator for the compressible Navier–Stokes–Korteweg model with boundary conditions $\mathbf{n} \cdot \nabla\rho = g$ and $\mathbf{u} = 0$ [5]. Regarding space and time variables, various results have been presented such as the existence of weak solutions, the local and global well-posedness for strong solutions, large time decay of solutions, time-periodic solutions, the vanishing capillarity limit, and maximal regularity; see, e.g., ref. [6] Subsequently, Inna et al. Studied the Navier–Stokes–Korteweg with slip boundary conditions in the half-space for certain coefficient conditions [7 – 8]. This research was further continued by Inna et al. to demonstrate the R-bounded solution operator [9] for arbitrary coefficient conditions. Furthermore, Inna & Saito demonstrate a local strong solution in the L_p framework concerning time and L_q concerning space, with $p \in (1, \infty)$ and $q \in (N, \infty)$ taking into account the additional regularity level of the initial density of air, which depends on the maximum regularity level of the linear system [10].

This study builds on prior work by addressing the NSK model with slip boundary conditions in a three-dimensional half-space, which is less explored but highly relevant for real-world applications where fluid slip occurs at boundaries, such as in microfluidic systems or porous media. Furthermore, this research demonstrates the existence of a solution operator for the resolvent system of the NSK model, offering a novel contribution to the mathematical understanding of compressible fluid models with slip boundary conditions as follows.

$$\left.\begin{aligned}
\lambda\rho + \left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}u_i\right) &= d \quad \text{di } \mathbf{R}_+^3\\
\lambda u_1 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)u_1 - \nu\frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}u_i\right) - \kappa\frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\rho &= f_1 \quad \text{di } \mathbf{R}_+^3\\
\lambda u_2 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)u_2 - \nu\frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}u_i\right) - \kappa\frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\rho &= f_2 \quad \text{di } \mathbf{R}_+^3\\
\lambda u_3 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)u_1 - \nu\frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}u_i\right) - \kappa\frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\rho &= f_3 \quad \text{di } \mathbf{R}_+^3\\
\begin{bmatrix}0\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial}{\partial y_1}\rho\\ \frac{\partial}{\partial y_2}\rho\\ \frac{\partial}{\partial y_3}\rho\end{bmatrix} &= g \quad \text{di } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_3}u_1 + \frac{\partial}{\partial y_1}u_3 &= h_1 \quad \text{di } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_3}u_2 + \frac{\partial}{\partial y_2}u_3 &= h_2 \quad \text{di } \mathbf{R}_0^3\\
u_3 &= h_3 \quad \text{di } \mathbf{R}_0^3
\end{aligned}\right\} \qquad (1)$$

with

$$\mathbf{R}_+^3 = \{y = (y_1, y_2, y_3) \in \mathbf{R}^3, y_3 > 0\}$$
$$\mathbf{R}_0^3 = \{y = (y_1, y_2, y_3) \in \mathbf{R}^3, y_3 = 0\}$$

where λ being the resolvent parameter belonging to $\mathbf{C}_+ = \{z \in \mathbf{C} | \mathfrak{R}_z > 0\}$, where $\mathbf{C}$ denotes the set of complex numbers, $\rho = \rho(y)$ and $\mathbf{u} = \mathbf{u}(y) = \left(u_1(y), u_2(y), u_3(y)\right)^{\top}$ are unknown functions, where each represents density (a scalar value) and fluid velocity (a vector value) depending on the variables $y = (y_1, y_2, y_3)$. The coefficients μ and ν are used to express viscosity constants, while κ denotes the capillarity constant. Furthermore, $(0,0,-1)^{\mathsf{T}} \cdot \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\right) = g, \frac{\partial}{\partial y_3} u_1 + \frac{\partial}{\partial y_1} u_3 = h_1, \frac{\partial}{\partial y_3} u_2 + \frac{\partial}{\partial y_2} u_3 = h_2$, and $u_3 = h_3$ represent slip boundary conditions, where $(0,0,-1)^{\mathsf{T}}$ is the unit vector pointing outward from $\mathbf{R}_0^3$.

In three-dimensional space, the Navier-Stokes-Korteweg (NSK) model with slip boundary conditions has been explored by various researchers. Salsabila et al. developed the model for the case where the coefficient μ, ν and κ satisfying $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) < 0$, J $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) > 0$ with $\kappa \neq \mu\nu$ [11], and Jihandika et al. extended it for the case where the coefficients μ, ν and κ satisfying $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) > 0$ with $\kappa = \mu\nu$ [12]. Alfiyyah further analyzed the model for the case where the coefficients μ, ν and κ satisfying $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) = 0$ with $\kappa \neq \mu\nu$ [13].

Additionally, the NSK model with Dirichlet boundary conditions was examined by Khasanah et al. for the case where the coefficients μ, ν and κ satisfying $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) > 0$ with $\kappa = \mu\nu$ [14444] followed by Manaqib et al., who investigated the case where the coefficients satisfy μ, ν and κ satisfying $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) = 0$ with $\kappa \neq \mu\nu$ [15]. Furthermore, this research establishes the existence of a solution operator for the resolvent system of the Navier-Stokes-Korteweg (NSK) model, for the case where the coefficients satisfy $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) = 0$ with $\kappa = \mu\nu$.

METHODS

In this study, the primary goal is to analyze the resolvent system for the Navier-Stokes-Korteweg (NSK) model with slip boundary conditions in a three-dimensional half-space. To achieve this, a systematic approach is used to reduce the complexity of the original problem and find a solution operator.

The first step involves simplifying the inhomogeneous system by applying a **partial Fourier transform**. This method is chosen because it effectively transforms the system of partial differential equations into a system of ordinary differential equations, which are easier to solve. The partial Fourier transform is beneficial here as it reduces the dimensional complexity of the problem by transforming the spatial variables into frequency variables. This allows us to focus on the system's behaviour in terms of more straightforward functions of one variable, making the analysis more tractable.

Before applying the partial Fourier transform, we first reduce the inhomogeneous resolvent system. This involves isolating the terms that will be transformed simplifying the equations into a form where the transform can be applied efficiently. By converting the system into the Fourier domain, we can obtain a simplified set of equations that describe the system's behaviour in a more manageable way.

After this transformation, we focus on solving the **homogeneous resolvent system**, which emerges from the reduced problem. The solution of this system demonstrates the existence of a solution operator, which is a key result of the study. This operator confirms that the system can be solved under the given conditions, and it provides a framework for understanding how slip boundary conditions influence the dynamics of compressible fluids in the context of the Navier-Stokes-Korteweg model.

This approach is chosen because it breaks down the complex system into smaller, more solvable components, ultimately allowing us to conclude the overall system's behaviour. Each step is carefully structured to ensure that the mathematical complexities are handled clearly and logically, with the Fourier transform playing a crucial role in simplifying the analysis.

Several notations are used in this article to denote key results. For instance, $\mathbf{N}$ represents the set of all natural numbers, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Additionally, $\mathbf{C}$ and $\mathbf{R}$ denote the set of complex numbers and the set of real numbers, respectively. For any domain $\Omega \subseteq \mathbf{R}^3$, the Lebesgue space and Sobolev space are denoted as $L_q(\Omega)$ and $W_q^m(\Omega)$, respectively, where $m \in \mathbf{N}$ and $q \in [1, \infty)$. Furthermore, the norm of the Sobolev space $W_q^n(\Omega)$ with $n \in \mathbf{N}_0$ is denoted as $\| \cdot \|_{W_q^n(\Omega)}$.

Let X and Y be Banach spaces X^s, where $s \in \mathbf{N}$, denotes the s -fold Cartesian product of X, defined as $X^s = \{x_i = (x_1, \dots, x_s) | x_i \in X\}$, and the length of a vector in the space $X^s$is denoted as $\| \cdot \|_{X^s}$. The set of linear operators mapping from X to Y is denoted by $\mathcal{L}(X, Y)$, and the notation $\mathcal{L}(X, Y)$ can be written as $\mathcal{L}(X)$. Furthermore, for a domain $U \subseteq \mathbf{C}$, $\mathbf{Hol}\big(U, \mathcal{L}(X, Y)\big)$ represents the set of holomorphic functions mapping from U to $\mathcal{L}(X, Y)$.

To find the solution operator (1) we can first solve it with an approach of finding solutions in whole-space 3-dimensional where the system of equations in 3-dimensional whole-space ($\mathbf{R}^3$) is represented by the following equations:

$$\left.\begin{aligned}
\lambda\rho + \left(\textstyle\sum_{i=1}^{3} \frac{\partial}{\partial y_i} u_i\right) &= d \quad \text{di } \mathbf{R}^3\\
\lambda u_1 - \mu\left(\textstyle\sum_{i=1}^{3} \frac{\partial^2}{\partial {y_i}^2}\right) u_1 - \nu \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3} \frac{\partial}{\partial y_i} u_i\right) - \kappa \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3} \frac{\partial^2}{\partial {y_i}^2}\right)\rho &= f_1 \quad \text{di } \mathbf{R}^3\\
\lambda u_2 - \mu\left(\textstyle\sum_{i=1}^{3} \frac{\partial^2}{\partial {y_i}^2}\right) u_1 - \nu \frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3} \frac{\partial}{\partial y_i} u_i\right) - \kappa \frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3} \frac{\partial^2}{\partial {y_i}^2}\right)\rho &= f_2 \quad \text{di } \mathbf{R}^3\\
\lambda u_3 - \mu\left(\textstyle\sum_{i=1}^{3} \frac{\partial^2}{\partial {y_i}^2}\right) u_1 - \nu \frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3} \frac{\partial}{\partial y_i} u_i\right) - \kappa \frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3} \frac{\partial^2}{\partial {y_i}^2}\right)\rho &= f_3 \quad \text{di } \mathbf{R}^3
\end{aligned}\right\} \tag{2}$$

with $\mathbf{R}^3 = \{y | y = (y_1, y_2, y_3)\}$.

Let the space for functions on the right-hand side of system of equations (2) in the whole-space 3-demensional, be denoted as $\mathbf{F}^0 = (d, f_1, f_2, f_3)$ and defined as follows:

$$X_q^1(\mathbf{R}^3) = W_q^1(\mathbf{R}^3) \times L_q(\mathbf{R}^3) \times L_q(\mathbf{R}^3) \times L_q(\mathbf{R}^3)$$

Then, define $\mathcal{F}_\lambda \mathbf{F}^0$and $\mathfrak{X}_q^1(\mathbf{R}^3)$ as follows:

$$\begin{aligned}
\mathcal{F}_\lambda \mathbf{F}^0 &= \left(\begin{bmatrix} \frac{\partial}{\partial y_1} d \\ \frac{\partial}{\partial y_2} d \\ \frac{\partial}{\partial y_2} d \end{bmatrix}, (\lambda)^{\frac{1}{2}} d, \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right) \in \mathfrak{X}_q^1(\mathbf{R}^3)\\
\mathfrak{X}_q^1(\mathbf{R}^3) &= L_q(\mathbf{R}^3)^{K_1}\\
K_1 &= 3 + 1 + 3 = 7
\end{aligned}$$

Based on the results obtained by Saito [5] for the case in ($\mathbf{R}^N$), the following theorem can be proven.

Theorem 1. [5] Let $q \in (1, \infty)$ assume that μ, ν dan κ are positive constants satisfying $\left(\frac{\mu + \nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) = 0$ and $\kappa = \mu\nu$. Then, for every $\lambda \in \mathbf{C}_+$, there exist operators $\mathcal{A}^1(\lambda)$ dan $\mathcal{B}^1(\lambda)$ such that:

$$\mathcal{A}^1(\lambda) \in \mathrm{Hol}\left(\mathbf{C}_+, \mathcal{L}\left(\mathfrak{X}_q^1(\mathbf{R}^3), W_q^3(\mathbf{R}^3)\right)\right),$$

$$\mathcal{B}^1(\lambda) \in \mathrm{Hol}\left(\mathbf{C}_+, \mathcal{L}\left(\mathfrak{X}_q^1(\mathbf{R}^3), W_q^2(\mathbf{R}^3)^3\right)\right)$$

so that for every $\mathbf{F}^0 = (d, f_1, f_2, f_3) \in X_q^1(\mathbf{R}^3)$ a unique solution operator$(\rho, \mathbf{u}) = (\mathcal{A}^1(\lambda)\mathcal{F}_\lambda\mathbf{F}^0,\ \mathcal{B}^1(\lambda)\mathcal{F}_\lambda\mathbf{F}^0)$ is obtained for Equation (2).

Definition 1. [5] Given a function z defined in $\mathbf{R}^3$ the partial Fourier transform and the inverse partial Fourier transform of $z = z\,(y_1, y_2, y_3)$ can be written as follows:

$$\hat{z} = \hat{z}\,(y_3) = \mathcal{F}[z](\xi_1, \xi_2, y_3) = \int_R \int_R e^{-(y_1,y_2).(\xi_1,\xi_2)}\ z(y_1, y_2, y_3)\ dy_1 dy_2$$

and the inverse of the partial Fourier transform is as follows:

$$\mathcal{F}^{-1}_{\xi_1,\xi_2}\ [\hat{z}(\xi_1, \xi_2, y_3)](y_1, y_2) = \frac{1}{(2\,\pi)^2} \int_R \int_R e^{-(y_1,y_2).(\xi_1,\xi_2)}\ z(y_1, y_2, y_3)\ dy_1 dy_2$$

Or, it can be simplified to:

$$\mathcal{F}^{-1}_{\xi_1,\xi_2}\ [\hat{z}(\xi_1, \xi_2, y_3)](y_1, y_2) = z(y_1, y_2, y_3)$$

RESULTS AND DISCUSSION

In this discussion, the process of proving the existence of solutions for the system of equations (1) will be demonstrated. Let the function space on the right-hand side of the system of equations (1) in 3-dimensional half-space, be denoted as $\mathbf{F}^1 = (d, f_1, f_2, f_3, g, h_1, h_2, h_3) \in X_q^2(\mathbf{R}^3_+)$ defined as follows,

$$X_q^2(\mathbf{R}^3) = W_q^1(\mathbf{R}^3_+) \times L_q(\mathbf{R}^3_+)^3 \times W_q^2(\mathbf{R}^3_+) \times W_q^1(\mathbf{R}^3_+)^2 \times W_q^2(\mathbf{R}^3_+)$$

Then, define $\mathcal{F}_\lambda\mathbf{F}^1$ dan $\mathfrak{X}_q^2(\mathbf{R}^3)$ as follows:

$$X_q^2(R^3_+) = L_q(R^3_+)^K,$$

$$\mathcal{F}_\lambda\mathbf{F}^1 = \left(\left(\begin{bmatrix} \frac{\partial}{\partial y_1} d \\ \frac{\partial}{\partial y_2} d \\ \frac{\partial}{\partial y_2} d \end{bmatrix}, (\lambda)^{\frac{1}{2}} d\right), \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \left(\begin{bmatrix} \frac{\partial^2}{\partial {y_1}^2} g & \frac{\partial}{\partial y_1 \partial y_2} g & \frac{\partial}{\partial y_1 \partial y_3} g \\ \frac{\partial}{\partial y_1 \partial y_2} g & \frac{\partial^2}{\partial {y_2}^2} g & \frac{\partial}{\partial y_2 \partial y_3} g \\ \frac{\partial}{\partial y_1 \partial y_3} g & \frac{\partial}{\partial y_2 \partial y_3} g & \frac{\partial^2}{\partial {y_3}^2} g \end{bmatrix}, \begin{bmatrix} (\lambda)^{\frac{1}{2}} \frac{\partial}{\partial y_1} g \\ (\lambda)^{\frac{1}{2}} \frac{\partial}{\partial y_2} g \\ (\lambda)^{\frac{1}{2}} \frac{\partial}{\partial y_3} g \end{bmatrix}, \lambda g\right),\right.$$

$$\left(\begin{bmatrix} \frac{\partial}{\partial y_1} h_1 & \frac{\partial}{\partial y_2} h_1 & \frac{\partial}{\partial y_3} h_1 \\ \frac{\partial}{\partial y_1} h_2 & \frac{\partial}{\partial y_2} h_2 & \frac{\partial}{\partial y_3} h_2 \end{bmatrix}, \begin{bmatrix} (\lambda)^{\frac{1}{2}} h_1 \\ (\lambda)^{\frac{1}{2}} h_2 \end{bmatrix}\right), \left(\begin{bmatrix} \frac{\partial^2}{\partial {y_1}^2} h_3 & \frac{\partial}{\partial y_1 \partial y_2} h_3 & \frac{\partial}{\partial y_1 \partial y_3} h_3 \\ \frac{\partial}{\partial y_1 \partial y_2} h_3 & \frac{\partial^2}{\partial {y_2}^2} h_3 & \frac{\partial}{\partial y_2 \partial y_3} h_3 \\ \frac{\partial}{\partial y_1 \partial y_3} h_3 & \frac{\partial}{\partial y_2 \partial y_3} h_3 & \frac{\partial^2}{\partial {y_3}^2} h_3 \end{bmatrix},\right.$$

$$\left.\left.\begin{bmatrix} (\lambda)^{\frac{1}{2}} \frac{\partial}{\partial y_1} h_3 \\ (\lambda)^{\frac{1}{2}} \frac{\partial}{\partial y_2} h_3 \\ (\lambda)^{\frac{1}{2}} \frac{\partial}{\partial y_3} h_3 \end{bmatrix}, \lambda h_3\right)\right) \in \mathfrak{X}_q^2(\mathbf{R}^3_+)$$

$$\mathfrak{X}_q^2(\mathbf{R}^3) = L_q(\mathbf{R}^3_+)^K$$

$$K = (3+1) + 3 + (3^2 + 3 + 1) + (3-1)(3+1) + (3^2 + 3 + 1)$$

$$K = 41$$

The aim of this research is to find the solution operators of System of Equations (1), or in other words, to prove the following theorem.

Theorem 2. Let $p \in (1,\infty)$ and assume that μ, ν dan κ are positive constants satisfying $\left(\frac{\mu+\nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) = 0$ and $\kappa = \mu\nu$. Then, for every $\lambda \in \mathbf{C}_+$ there exist operators $\mathcal{A}^2(\lambda)$ dan $\mathcal{B}^2(\lambda)$ such that,

$$\mathcal{A}^2(\lambda) \in Hol\left(\mathbf{C}_+, \mathcal{L}\left(X_q^2(\mathbf{R}_+^3), W_q^3(\mathbf{R}_+^3)\right)\right),$$
$$\mathcal{B}^2(\lambda) \in Hol\left(\mathbf{C}_+, \mathcal{L}\left(X_q^2(\mathbf{R}_+^3), W_q^2(\mathbf{R}_+^3)^3\right)\right)$$

thus, for every $\mathbf{F}^1 = (d, f_1, f_2, f_3, g, h_1, h_2, h_3) \in X_q^2(\mathbf{R}^3)$, $(\rho, \mathbf{u}) = (\mathcal{A}^2(\lambda)\mathcal{F}_\lambda \mathbf{F}^1,\ \mathcal{B}^2(\lambda)\mathcal{F}_\lambda \mathbf{F}^1)$ represents the solution operator of Equation (1).

The next part involves proving Theorem 2 by reducing the inhomogeneous system (1) to a homogeneous system in the three-dimensional half-space. The subsequent step is solving the homogeneous system in the three-dimensional half-space.

Reduced system

Assume that $u_j = w_j; (j = 1,2)$ and $u_3(y_1, y_2, y_3) = w_3 + h_3$, and $\boldsymbol{w} = (w_1, w_2, w_3)^\intercal$, Then the reduced equation from equation (1) is obtained as follows:

$$\left.\begin{aligned}
\lambda\rho + \left(\textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} w_i\right) &= \tilde{d} && \text{in } \mathbf{R}_+^3\\
\lambda w_1 - \mu\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) w_1 - \nu \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} w_i\right) - \kappa \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right)\rho &= \widetilde{f_1} && \text{in } \mathbf{R}_+^3\\
\lambda w_2 - \mu\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) w_2 - \nu \frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} w_i\right) - \kappa \frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right)\rho &= \widetilde{f_2} && \text{in } \mathbf{R}_+^3\\
\lambda w_3 - \mu\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) w_3 - \nu \frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} w_i\right) - \kappa \frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right)\rho &= \widetilde{f_3} && \text{in } \mathbf{R}_+^3\\
\begin{bmatrix} 0\\ 0\\ -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial y_1}\rho\\ \frac{\partial}{\partial y_2}\rho\\ \frac{\partial}{\partial y_3}\rho \end{bmatrix} &= g && \text{on } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_3} w_1 + \frac{\partial}{\partial y_1} w_3 &= \tilde{h}_1 && \text{on } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_2} w_2 + \frac{\partial}{\partial y_2} w_3 &= \tilde{h}_2 && \text{on } \mathbf{R}_0^3\\
w_3 &= 0 && \text{on } \mathbf{R}_0^3
\end{aligned}\right\} \quad (3)$$

with

$$\tilde{d} = d - \frac{\partial}{\partial y_3} h_3, \widetilde{f_1} = f_1 + \nu \frac{\partial}{\partial y_1}\left(\frac{\partial}{\partial y_3} h_3\right), \widetilde{f_2} = f_2 + \nu \frac{\partial}{\partial y_2}\left(\frac{\partial}{\partial y_3} h_3\right),$$
$$\widetilde{f_3} = f_3 + \mu h_3\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}\right) + \nu \frac{\partial}{\partial y_3}\left(\frac{\partial}{\partial y_3} h_3\right) - \lambda h_3, \tilde{h}_1 = h_1 - \frac{\partial}{\partial y_1} h_3, \tilde{h}_2 = h_2 - \frac{\partial}{\partial y_2} h_3$$

Next, reduce equation (3) using even extension and odd extension to form the homogeneous system of equations $(d, y_1, y_2, y_3) = (0,0,0,0)$. For a function $f = f\ (y_1, y_2, y_3)$ di half-space 3-demensional, define the even extension E^e and odd extension E^o as follows:

$$E^e(f) = E^e f(y_1, y_2, y_3) = \begin{cases} f(y_1, y_2, y_3), & (y_3 > 0), \\ f(y_1, y_2, -y_3), & (y_3 < 0) \end{cases}$$
$$E^o(f) = E^o f(y_1, y_2, y_3) = \begin{cases} f(y_1, y_2, y_3), & (y_3 > 0), \\ -f(y_1, y_2, -y_3), & (y_3 < 0) \end{cases}$$

Then, define the extension of a vector function $\mathbf{f} = (f_1, f_2, f_3)^{\top}$ as follows:

$$\mathbf{Ef} = (E^e f_1, E^e f_2, E^o f_3)^{\top} \text{in } \mathbf{R}^3_+ \tag{4}$$

Note that $\mathbf{E} \in \mathcal{L}\big(L_q(\mathbf{R}^3_+)^3, L_q(\mathbf{R}^3)^3\big)$ and $E^e \in \mathcal{L}\Big(W^1_p(\mathbf{R}^3_+), W^1_p(\mathbf{R}^3)\Big)$.

Let $\mathcal{A}^1(\lambda)\mathcal{F}_\lambda \mathbf{F}^0$, $\mathcal{B}^1(\lambda)\mathcal{F}_\lambda \mathbf{F}^0$ be the solution operator found in Theorem 1 in the whole space and $(\tilde{d}, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ be the righthand side function of equation (3) existing in the space $W^1_p(\mathbf{R}^3_+) \times L_q(\mathbf{R}^3)^3$. Define operators R dan $\mathbf{W}$ as follows,

$$\left.\begin{aligned} R &= \mathcal{A}^1(\lambda)\mathcal{F}_\lambda\big(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\big), \\ \mathbf{W} &= \mathcal{B}^1(\lambda)\mathcal{F}_\lambda\big(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\big) \end{aligned}\right\} \tag{5}$$

where

$$\mathcal{B}^1(\lambda)\mathcal{F}_\lambda\big(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\big) = \begin{bmatrix} \mathcal{B}^1_1(\lambda)\mathcal{F}_\lambda\big(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\big) \\ \mathcal{B}^1_2(\lambda)\mathcal{F}_\lambda\big(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\big) \\ \mathcal{B}^1_3(\lambda)\mathcal{F}_\lambda\big(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\big) \end{bmatrix}. \tag{6}$$

Let operators $S = S(y_1, y_2, y_3)$ and $\mathbf{T} = \mathbf{T}(y_1, y_2, y_3)$ are defined as follows:

$$\left.\begin{aligned} S &= R(y_1, y_2, -y_3), \\ \mathbf{T} &= \big(W_1(y_1, y_2, -y_3), W_2(y_1, y_2, -y_3), -W_3(y_1, y_2, -y_3)\big)^{\top} \end{aligned}\right\} \tag{7}$$

Then, W_j dan T_j represent the j-th components of the functions $\mathbf{W}$ dan $\mathbf{T}$ with $j = 1,2,3$. Consequently

$$T_3(y_1, y_2, y_3) = -W_3(y_1, y_2, -y_3) \tag{8}$$

The next step, substitute the operators R and $\mathbf{W}$ from Equation (5) into the first Equation (3) to obtain

$$\begin{aligned} \left(\lambda S + \textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} T_i\right)(y_1, y_2, y_3) &= \left(\lambda R + \textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} W_i\right)(y_1, y_2, -y_3) \\ &= (E^e\tilde{d}(y_1, y_2, -y_3) \\ &= E^e\tilde{d}(y_1, y_2, y_3). \end{aligned}$$

By similar step as previous, we obtain the rest Equation of (3) as follows:

$$\left(\lambda W_1 - \mu\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) W_1 - \nu \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} W_i\right) - \kappa \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) R\right)(y') = E^e\tilde{f}_1(y')$$
$$\left(\lambda W_2 - \mu\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) W_2 - \nu \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} W_i\right) - \kappa \frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) R\right)(y') = E^e\tilde{f}_2(y')$$
$$\left(\lambda W_3 - \mu\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) W_3 - \nu \frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^3 \frac{\partial}{\partial y_i} W_i\right) - \kappa \frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^3 \frac{\partial^2}{\partial {y_i}^2}\right) R\right)(y') = E^o\tilde{f}_3(y')$$

with $y' = (y_1, y_2, y_3)$. Thus, S dan $\mathbf{T}$ are also solution operators in the 3-dimensional whole-space. Therefore, based on the uniqueness of the solution of Equation (2), we have

$$\begin{aligned} T_1(y_1, y_2, y_3) &= W_1(y_1, y_2, y_3), \\ T_2(y_1, y_2, y_3) &= W_2(y_1, y_2, y_3), \\ T_3(y_1, y_2, y_3) &= W_3(y_1, y_2, y_3) \end{aligned} \tag{9}$$

As a result, based on Equations (8) and (9), we obtain $W_3(y_1, y_2, y_3) = -W_3(y_1, y_2, -y_3)$, and when $y_3 = 0$ we get $W_3(y_1, y_2, 0) = -W_3(y_1, y_2, 0)$ if and only if $W_3(y_1, y_2, 0) = 0$.

Let ρ and $\boldsymbol{w}$ be defined as follows:

$$\rho = R + \tilde{\rho}, \quad \text{and } \boldsymbol{w} = (W_1 + \tilde{w}_1, W_2 + \tilde{w}_2, W_3 + \tilde{w}_3) = \boldsymbol{W} + \tilde{\boldsymbol{w}}, \qquad (10)$$

then we obtain the reduced system of Equation (3) as follows :

$$\left.\begin{aligned}
\lambda\tilde{\rho} + \left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}\tilde{w}_i\right) &= 0 && \text{in } \mathbf{R}_+^3\\
\lambda\tilde{w}_1 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\tilde{w}_1 - \nu\frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}\tilde{w}_i\right) - \kappa\frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\tilde{\rho} &= 0 && \text{in } \mathbf{R}_+^3\\
\lambda\tilde{w}_2 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\tilde{w}_2 - \nu\frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}\tilde{w}_i\right) - \kappa\frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\tilde{\rho} &= 0 && \text{in } \mathbf{R}_+^3\\
\lambda\tilde{w}_3 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\tilde{w}_3 - \nu\frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}\tilde{w}_i\right) - \kappa\frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\tilde{\rho} &= 0 && \text{in } \mathbf{R}_+^3\\
\frac{\partial}{\partial y_3}\tilde{\rho} &= -\tilde{g} && \text{on } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_3}\tilde{w}_1 + \frac{\partial}{\partial y_1}\tilde{w}_3 &= \tilde{\tilde{h}}_1 && \text{on } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_2}\tilde{w}_2 + \frac{\partial}{\partial y_2}\tilde{w}_3 &= \tilde{\tilde{h}}_2 && \text{on } \mathbf{R}_0^3\\
\tilde{w}_3 &= 0 && \text{on } \mathbf{R}_0^3
\end{aligned}\right\} \qquad (11)$$

with

$\tilde{g} = g + \frac{\partial}{\partial y_3}\mathcal{A}^1(\lambda)\mathcal{F}_\lambda\left(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\right), \tilde{\tilde{h}}_1 = \frac{\partial}{\partial y_1}h_3 - \left(\frac{\partial}{\partial y_3}\mathcal{B}_1^1(\lambda)\mathcal{F}_\lambda\left(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\right) + \frac{\partial}{\partial y_1}\mathcal{B}_3^1(\lambda)\mathcal{F}_\lambda\left(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\right)\right), \tilde{\tilde{h}}_2 = h_2 - \frac{\partial}{\partial y_2}h_3 - \left(\frac{\partial}{\partial y_3}\mathcal{B}_2^1(\lambda)\mathcal{F}_\lambda\left(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\right) + \frac{\partial}{\partial y_2}\mathcal{B}_3^1(\lambda)\mathcal{F}_\lambda\left(E^e\tilde{d}, E^e\tilde{f}_1, E^e\tilde{f}_2, E^o\tilde{f}_3\right)\right)$

Next, to prove Theorem 2, it is sufficient to solve the Homogeneous System (11), which will be discussed in the following section.

Solution of the Homogeneous System in the Half-Space

The Homogeneous System of Equations (11) in the 3-dimensional half-space can be written simply as follows:

$$\left.\begin{aligned}
\lambda\rho + \left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}w_i\right) &= 0 && \text{in } \mathbf{R}_+^3\\
\lambda w_1 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)w_1 - \nu\frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}w_i\right) - \kappa\frac{\partial}{\partial y_1}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\rho &= 0 && \text{in } \mathbf{R}_+^3\\
\lambda w_2 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)w_2 - \nu\frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}w_i\right) - \kappa\frac{\partial}{\partial y_2}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\rho &= 0 && \text{in } \mathbf{R}_+^3\\
\lambda w_3 - \mu\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)w_3 - \nu\frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3}\frac{\partial}{\partial y_i}w_i\right) - \kappa\frac{\partial}{\partial y_3}\left(\textstyle\sum_{i=1}^{3}\frac{\partial^2}{\partial {y_i}^2}\right)\rho &= 0 && \text{in } \mathbf{R}_+^3\\
\frac{\partial}{\partial y_3}\rho &= -g && \text{on } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_3}w_1 + \frac{\partial}{\partial y_1}w_3 &= h_1 && \text{on } \mathbf{R}_0^3\\
\frac{\partial}{\partial y_2}w_2 + \frac{\partial}{\partial y_2}w_3 &= h_2 && \text{on } \mathbf{R}_0^3\\
w_3 &= 0 && \text{on } \mathbf{R}_0^3
\end{aligned}\right\} \qquad (12)$$

For the function on the right-hand side of (12), let $\mathbf{G} = (g, h_1, h_2)$, Define the space **G**

$$X_q^3(\mathbf{R}_+^3) = W_q^2(\mathbf{R}_+^3) \times W_q^1(\mathbf{R}_+^3) \times W_q^1(\mathbf{R}_+^3)$$

Then, let's define $\mathcal{G}_\lambda \mathbf{G}$ and $\mathfrak{X}_q^3(\mathbf{R}_+^3)$ as follows:

$$\mathcal{G}_\lambda \mathbf{G} = \left(\left(\begin{bmatrix} \frac{\partial^2}{\partial {y_1}^2} g & \frac{\partial}{\partial y_1 \partial y_2} g & \frac{\partial}{\partial y_1 \partial y_3} g \\ \frac{\partial}{\partial y_1 \partial y_2} g & \frac{\partial^2}{\partial {y_2}^2} g & \frac{\partial}{\partial y_2 \partial y_3} g \\ \frac{\partial}{\partial y_1 \partial y_3} g & \frac{\partial}{\partial y_2 \partial y_3} g & \frac{\partial^2}{\partial {y_3}^2} g \end{bmatrix}, \begin{bmatrix} \lambda^{\frac{1}{2}} \frac{\partial}{\partial y_1} g \\ \lambda^{\frac{1}{2}} \frac{\partial}{\partial y_2} g \\ \lambda^{\frac{1}{2}} \frac{\partial}{\partial y_3} g \end{bmatrix}, \lambda g \right), \left(\begin{bmatrix} \frac{\partial}{\partial y_1} h_1 & \frac{\partial}{\partial y_2} h_1 & \frac{\partial}{\partial y_3} h_1 \\ \frac{\partial}{\partial y_1} h_2 & \frac{\partial}{\partial y_2} h_2 & \frac{\partial}{\partial y_3} h_2 \end{bmatrix}, \begin{bmatrix} \lambda^{\frac{1}{2}} h_1 \\ \lambda^{\frac{1}{2}} h_2 \end{bmatrix} \right) \right) \in \mathfrak{X}_q^3(\mathbf{R}_+^3)$$

$\mathfrak{X}_q^3(\mathbf{R}_+^3) = L_q(\mathbf{R}_+^3)^{N_2}$

$N_2 = (3^2 + 3 + 1) + (3 - 1)(3 + 1) = 21$

The following theorem will be used to continue the proof of Theorem 2.

Theorem 3. Let $q \in (1, \infty)$ and assume that μ, ν and κ are positive constants satisfying $\left(\frac{\mu + \nu}{2\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) = 0$ and $\kappa = \mu\nu$. Then, for every $\lambda \in \mathbf{C}_+$ there exist operators $\mathcal{A}^3(\lambda)$ and $\mathcal{B}^3(\lambda)$ with

$$\mathcal{A}^3(\lambda) \in Hol\left(\mathbf{C}_+, \mathcal{L}\left(\mathfrak{X}_q^3\,(\mathbf{R}_+^3), W_q^3(R_+^3)\right)\right),$$

$$\mathcal{B}^3(\lambda) \in Hol\left(\mathbf{C}_+, \mathcal{L}\big(\mathfrak{X}_q^3\,(\mathbf{R}_+^3), W_q^2(R_+^3)^3\big)\right)$$

Thus, for every $\mathbf{G} = (g, h_1, h_2) \in X_q^3(\mathbf{R}_+^3)$, $(\rho, \mathbf{w}) = (\mathcal{A}^3(\lambda)\mathcal{G}_\lambda \boldsymbol{G}, \mathcal{B}^3(\lambda)\mathcal{G}_\lambda \boldsymbol{G})$ is the solution operator for Equation (12).

Proof

A partial Fourier transform is applied to the homogeneous system, resulting in a solvable ordinary differential equation. Assume that $\psi = \left(\sum_{i=1}^{3} \frac{\partial}{\partial y_i} w_i\right)$, then do the partial Fourier transform on the equation (12), then for $y_3 > 0$ we obtain the system of ordinary differential equations:

$$\lambda \hat{\rho} + \hat{\psi} = 0 \quad (13)$$

$$\lambda \widehat{w}_1 - \mu\big(\partial_{y_3}^2 - |(\xi_1)^2 + (\xi_2)^2|\big)\widehat{w}_1 - \nu i \xi_1 \hat{\psi} - \kappa i \xi_1 \left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\hat{\rho} = 0 \quad (14)$$

$$\lambda \widehat{w}_2 - \mu\big(\partial_{y_3}^2 - |(\xi_1)^2 + (\xi_2)^2|\big)\widehat{w}_2 - \nu i \xi_2 \hat{\psi} - \kappa i \xi_2 \left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\hat{\rho} = 0 \quad (15)$$

$$\lambda \widehat{w}_3 - \mu\left(\frac{\partial^2}{\partial {y_3}^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\widehat{w}_3 - \nu \frac{\partial}{\partial y_3}\hat{\psi} - \kappa \frac{\partial}{\partial y_3}\left(\frac{\partial^2}{\partial {y_3}^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\hat{\rho} = 0 \quad (16)$$

With the boundary conditions :

$$-\frac{\partial}{\partial y_3}\hat{\rho}(0) = \hat{g}(0) \qquad y_3 > 0, \quad (17)$$

$$\frac{\partial}{\partial y_3}\widehat{w}_1(0) + i\xi_1 \widehat{w}_3(0) = \hat{h}_1(0) \quad y_3 > 0, \quad (18)$$

$$\frac{\partial}{\partial y_3}\widehat{w}_2(0) + i\xi_2 \widehat{w}_3(0) = \hat{h}_2(0) \quad y_3 > 0, \quad (19)$$

$$\widehat{w}_3(0) = 0 \qquad y_3 > 0, \quad (20)$$

Where

$$\hat{\psi} = i\xi_1\hat{w}_1 + i\xi_2\hat{w}_2 + \frac{\partial}{\partial y_3}\hat{w}_3 \tag{21}$$

note the equation (13) we can written as

$$\hat{\rho} = -\frac{\hat{\psi}}{\lambda} \tag{22}$$

Replace the equation (14), (15), (16) with (22), then for $y_3 > 0$ we have the equations:

$$\lambda^2\hat{w}_1 - \lambda\mu\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\hat{w}_1 - i\xi_1\left(\lambda\nu - \kappa\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\right)\hat{\psi} = 0 \tag{23}$$

$$\lambda^2\hat{w}_2 - \lambda\mu\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\hat{w}_2 - i\xi_2\left(\lambda\nu - \kappa\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\right)\hat{\psi} = 0 \tag{24}$$

$$\lambda^2\hat{w}_3 - \lambda\mu\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\hat{w}_3 - \frac{\partial}{\partial y_3}\left(\lambda\nu - \kappa\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\right)\hat{\psi} = 0 \tag{25}$$

$$\frac{\partial}{\partial y_3}\hat{\psi}(0) = \hat{g}(0)\lambda \tag{26}$$

By multiplying (23) by $i\xi_1$, (24) by $i\xi_2$ and (25) by $\frac{\partial}{\partial y_3}$ and summing up the resultant equations, we can have from (21) :

$$\lambda^2\hat{\psi} - \lambda(\mu + \nu)\left(\frac{\partial^2}{\partial {y_3}^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\hat{\psi} + \kappa\left(\frac{\partial^2}{\partial {y_3}^2} - |(\xi_1)^2 + (\xi_2)^2|\right)^2\hat{\psi} = 0 \tag{27}$$

Let define a polynomial with $P_\lambda(t)$ as follows:

$$P_\lambda(t) = \lambda^2 - \lambda(\mu + \nu)(t^2 - |(\xi_1)^2 + (\xi_2)^2|) + \kappa(t^2 - |(\xi_1)^2 + (\xi_2)^2|)^2 \tag{28}$$

Thus, the form of the ordinary differential equation system in (27) can be written as follows:

$$P_\lambda\left(\frac{\partial}{\partial y_3}\right)\hat{\psi} = 0, \quad y_3 > 0. \tag{29}$$

Let

$$\omega_\lambda = \sqrt{|(\xi_1)^2 + (\xi_2)^2| + \frac{\lambda}{\mu}}.$$

Then by the definition ω_λ above, Equations (23), (24) and (25) can be written as

$$\lambda\mu\left(\frac{\partial^2}{\partial y_3^2} - (\omega_\lambda)^2\right)\hat{w}_1 + i\xi_1\left(\lambda\nu - \kappa\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\right)\hat{\psi} = 0 \qquad y_3 > 0, \tag{30}$$

$$\lambda\mu\left(\frac{\partial^2}{\partial y_3^2} - (\omega_\lambda)^2\right)\hat{w}_2 + i\xi_2\left(\lambda\nu - \kappa\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\right)\hat{\psi} = 0 \qquad y_3 > 0, \tag{31}$$

$$\lambda\mu\left(\frac{\partial^2}{\partial y_3^2} - (\omega_\lambda)^2\right)\hat{w}_3 + \frac{\partial}{\partial y_3}\left(\lambda\nu - \kappa\left(\frac{\partial^2}{\partial y_3^2} - |(\xi_1)^2 + (\xi_2)^2|\right)\right)\hat{\psi} = 0 \qquad y_3 > 0. \tag{32}$$

Apply $P_\lambda\left(\frac{\partial}{\partial y_3}\right)$ to equations (30), (31), and (32), and then substitute equation (29) into (30), (31), and (32) to obtain the following simpler ordinary differential equations

$$\left(\frac{\partial^2}{\partial y_3^2}-(\omega_\lambda)^2\right)P_\lambda\left(\frac{\partial}{\partial y_3}\right)\widehat{w}_i=0, y_3>0 \quad \text{for } i=1,2,3. \tag{33}$$

To obtain the characteristic roots 0f Equation (33), let us consider

$$P_\lambda(t)=\kappa\lambda^2\left(\frac{1}{\kappa}-\left(\frac{\mu+\nu}{\kappa}\right)\left(\frac{t^2-|(\xi_1)^2+(\xi_2)^2|}{\lambda}\right)+\left(\frac{t^2-|(\xi_1)^2+(\xi_2)^2|}{\lambda}\right)^2\right). \tag{34}$$

Suppose $c=\left(\frac{t^2-|(\xi_1)^2+(\xi_2)^2|}{\lambda}\right)$ then equation (34) becomes:

$$P_\lambda(t)=\kappa\lambda^2 p(c) \tag{35}$$

where $p(c)$ is given by

$$p(c)=\frac{1}{\kappa}-\frac{\mu+\nu}{\kappa}c+c^2.$$

Thus, the roots of the $p(c)$ are obtained as follows:

$$c_{1,2}=\frac{\mu+\nu}{2\kappa}\pm\sqrt{\left(\frac{\mu+\nu}{\kappa}\right)^2-\frac{1}{\kappa}}$$

In this study, the case of the coefficients satisfy $\kappa=\mu\nu$ and $\left(\frac{\mu+\nu}{\kappa}\right)^2-\frac{1}{\kappa}=0$, *t*hen the roots of $p(c)$ are

$$c_1=c_2=\frac{\mu+\nu}{2\kappa}=\frac{1}{\mu}.$$

Consider $c_i=\frac{t^2-|(\xi_1)^2+(\xi_2)^2|}{\lambda}$ with $i=1,2$. The solution for t is as follows:

$$t=\pm\sqrt{\lambda c_i+|(\xi_1)^2+(\xi_2)^2|}.$$

Then we have

$$t_{1i}=\sqrt{\lambda c_i+|(\xi_1)^2+(\xi_2)^2|}, \text{ for } i=1,2$$

are the roots of the equations of (32). Moreover, since $c_1=c_2=\frac{\mu+\nu}{2\kappa}=\frac{1}{\mu}$, the characteristic roots of (29) and (32) are given by :

$$t_1=t_2=\sqrt{\lambda\mu^{-1}+|(\xi_1)^2+(\xi_2)^2|}=\omega_\lambda.$$

Thus, the general solution of (29) and (32) is given by

$$\widehat{\psi}=\sigma e^{-\omega_\lambda y_3}+\tau y_3 e^{-\omega_\lambda y_3} \tag{36}$$

$$\widehat{w}_i=\alpha_j e^{-\omega_\lambda y_3}+\beta_j y_3 e^{-\omega_\lambda y_3}+\gamma_j y_3^2 e^{-\omega_\lambda y_3} \quad \text{for } j=1,2,3 \tag{37}$$

Next, the coefficient values of the general solution to Equations (36) and (37) will be found by substituting Equation (37) into Equation (21). Then, we obtain

$$\sigma \ = i\xi' \cdot \alpha' - \omega_\lambda \alpha_3 + \beta_3, \tag{38}$$
$$\tau \ = i\xi' \cdot \beta' - \omega_\lambda \beta_3 + 2\gamma_3, \tag{39}$$
$$0 \ = i\xi' \cdot \gamma' - \omega_\lambda \gamma_3 \tag{40}$$

By the assumption that $\mu = \nu$, then from Equations (23), (24), and (25) we obtain:

$$\lambda \left(\frac{\partial^2}{\partial {y_3}^2} - {\omega_\lambda}^2\right) \widehat{w}_1 - \mu i \xi_1 \left(\frac{\partial^2}{\partial {y_3}^2} - {\omega_\lambda}^2\right) \widehat{\psi} \ = 0, \tag{41}$$
$$\lambda \left(\frac{\partial^2}{\partial {y_3}^2} - {\omega_\lambda}^2\right) \widehat{w}_2 - \mu i \xi_2 \left(\frac{\partial^2}{\partial {y_3}^2} - {\omega_\lambda}^2\right) \widehat{\psi} \ = 0, \tag{42}$$
$$\lambda \left(\frac{\partial^2}{\partial {y_3}^2} - {\omega_\lambda}^2\right) \widehat{w}_3 - \mu \frac{\partial}{\partial y_3} \left(\frac{\partial^2}{\partial {y_3}^2} - {\omega_\lambda}^2\right) \widehat{\psi} \ = 0. \tag{43}$$

Substitute (36), and (37) into equations (30), (31) and (32) we obtain:

$$-4\lambda\gamma_1\omega_\lambda = 0, \lambda(-2\beta_1\omega_\lambda + 2\gamma_1) + 2\mu i \xi_1 \tau \omega_\lambda \ = 0, \tag{44}$$
$$-4\lambda\gamma_2\omega_\lambda = 0, \lambda(-2\beta_2\omega_\lambda + 2\gamma_2) + 2\mu i \xi_2 \tau \omega_\lambda \ = 0, \tag{45}$$
$$-4\lambda\gamma_3\omega_\lambda = 0, \lambda(-2\beta_3\omega_\lambda + 2\gamma_3) - 2\mu \tau {\omega_\lambda}^2 \ = 0 \tag{46}$$

Based on equations (44), (45), and (46), the following will be obtained

$$-\gamma_j \ = 0, j = 1,2,3 \tag{47}$$
$$-\lambda\beta_j + \mu i \xi_j \tau \ = 0, j = 1,2 \tag{48}$$
$$-\lambda\beta_3 - \mu\tau\omega_\lambda \ = 0 \tag{49}$$

From Equation (48) and (49) , we obtain

$$\beta_j \ = -\frac{i\xi_j}{\omega_\lambda}\beta_3 \ \text{ for } j = 1,2, \tag{50}$$

which imply

$$i\xi' \cdot \beta' \ = \left(\frac{|\xi'|^2}{\omega_\lambda}\right)\beta_3. \tag{51}$$

By substituting (47) and (51) into (39) then we obtain

$$\tau = \frac{1}{\omega_\lambda}(|\xi'|^2)\beta_3 - \omega_\lambda\beta_3 = \frac{1}{\omega_\lambda}(|\xi'|^2 - {\omega_\lambda}^2)\beta_3 = -\frac{\lambda}{\omega_\lambda \mu}\beta_3 \, . \tag{52}$$

Then, since $\gamma_j = 0$ based on equation (47), we obtain

$$\widehat{w}_J = \ \alpha_j e^{-\omega_\lambda y_3} + \beta_j y_3 e^{-\omega_\lambda y_3} \ \text{ for } j = 1,2,3 \tag{53}$$

By applying the first derivative with respect to y_3 to Equation (53), then we have

$$\frac{\partial}{\partial y_3}\widehat{w}_J = \ \left(-\alpha_j\omega_\lambda + \beta_j - \omega_\lambda\beta_j y_3\right)e^{-\omega_\lambda y_3} \ \text{ for } j = 1,2,3. \tag{54}$$

By boundary condition $\widehat{w}_3 = 0$ in Equation (20) and by Equation (53) combining with $y_3 = 0$, we derive

$$0 = \ \alpha_3. \tag{55}$$

From Equation (54) combined with (18), (19) and (20), we can express

$$\hat{h}_j(0) = -\alpha_j\omega_\lambda + \beta_j, \text{for } j = 1,2$$

which implies, by Equation (50)

$$\hat{h}_j(0) = -\alpha_j\omega_\lambda + -\frac{i\xi_j}{\omega_\lambda}\beta_3, \text{ for } j = 1,2.$$

Therefore, we obtain

$$\alpha_j = -\frac{\hat{h}_j(0)}{\omega_\lambda} - \frac{i\xi_j}{{\omega_\lambda}^2}\beta_3, \quad \text{for } j = 1,2 \tag{56}$$

Then multiply (56) by $i\xi_j$ with $j = 1,2$, we have

$$\alpha' \cdot \xi' = -\frac{\hat{\boldsymbol{h}}'(0)}{\omega_\lambda}\, i\xi' + \left(\frac{|\xi'|^2}{{\omega_\lambda}^2}\right)\beta_3. \tag{57}$$

By substituting equation (57) into equation (38), we obtain

$$\sigma = -\frac{\hat{\boldsymbol{h}}'(0)}{\omega_\lambda}\, i\xi' + \left(\frac{|\xi'|^2 + {\omega_\lambda}^2}{{\omega_\lambda}^2}\right)\beta_3. \tag{58}$$

Next, by the boundary condition (29) and Equation (36), we get:

$$\lambda\hat{g}(0) = -\omega_\lambda\sigma + \tau \tag{59}$$

which implies, by (58) and (52) and (59)

$$\hat{g}(0)\lambda = \hat{\boldsymbol{h}}'(0)\, i\xi' - 2\omega_\lambda\beta_3. \tag{60}$$

Then transform equation (60) into the form of β_3, resulting in

$$\beta_3 = \frac{\hat{\boldsymbol{h}}'(0)\, i\xi' - \hat{g}(0)\lambda}{2\omega_\lambda} \tag{61}$$

Next, substitute equation (61) into equations (58) and (52) as follows:

$$\sigma = -\left(\frac{\lambda}{2\mu{\omega_\lambda}^3}\right) i\xi' \cdot \hat{\boldsymbol{h}}'(0) - \frac{\left(|\xi'|^2 + {\omega_\lambda}^2\right)\lambda}{2{\omega_\lambda}^3}\hat{g}(0) \tag{62}$$

$$\tau = -\frac{\lambda}{2{\omega_\lambda}^2\mu}\left(\hat{\boldsymbol{h}}'(0)\, i\xi' - \hat{g}(0)\lambda\right) \tag{63}$$

And also substitute equation (61) into (58) and (52) with $j = 1,2$ we obtain

$$\alpha_j = -\frac{\hat{h}_j(0)}{\omega_\lambda} - \frac{i\xi_j}{2{\omega_\lambda}^3}\left(\hat{\boldsymbol{h}}'(0)\, i\xi' - \hat{g}(0)\lambda\right) \tag{64}$$

$$\beta_j = -\frac{i\xi_j}{2{\omega_\lambda}^2}\left(\hat{\boldsymbol{h}}'(0)\, i\xi' - \hat{g}(0)\lambda\right) \tag{65}$$

Now, we obtain solution operators for equations (29) and (33) with coefficients $\alpha_j, \beta_j, \beta_3$, σ, τ defined in equations (64), (65), (61), (62), and (63), as follows:

$$\hat{\psi} = \left(-\left(\frac{\lambda}{2\mu\omega_\lambda{}^3}\right) i\xi' \cdot \hat{\boldsymbol{h}}'(0) - \frac{\left(|\xi'|^2+\omega_\lambda{}^2\right)\lambda}{2\omega_\lambda{}^3}\hat{g}(0)\right) e^{-\omega_\lambda y_3} - \frac{\lambda}{2\omega_\lambda{}^2\mu_*}\left(\hat{\boldsymbol{h}}'(0)\, i\xi' - \hat{g}(0)\lambda\right) y_3 e^{-\omega_\lambda y_3} \tag{66}$$

$$\hat{w}_J = -\frac{\hat{h}_j(0)}{\omega_\lambda} - \frac{i\xi_j}{2\omega_\lambda{}^3}\left(\hat{\boldsymbol{h}}'(0)\, i\xi' - \hat{g}(0)\lambda\right)e^{-\omega_\lambda y_3} - \frac{i\xi_j}{2\omega_\lambda{}^2}\left(\hat{\boldsymbol{h}}'(0)\, i\xi' - \hat{g}(0)\lambda\right) y_3 e^{-\omega_\lambda y_3} \tag{67}$$

$$\hat{w}_3 = \left(\frac{\hat{\boldsymbol{h}}'(0)\, i\xi' - \lambda\hat{g}(0)}{2\omega_\lambda}\right) y_3 e^{-w_\lambda y_3} \tag{68}$$

To obtain ρ by substituting equation (66) into equation (29) as follows:

$$\rho = \frac{i\xi'}{2\omega_\lambda{}^3\mu}\hat{\boldsymbol{h}}'(0)e^{-\omega_\lambda y_3} + \frac{\left(|\xi'|^2+\omega_\lambda{}^2\right)}{2\omega_\lambda{}^3}\hat{g}(0)e^{-\omega_\lambda y_3} + \frac{i\xi'\hat{\boldsymbol{h}}'(0)}{2\omega_\lambda{}^2\mu} y_3 e^{-\omega_\lambda y_3} - \frac{\lambda\hat{g}(0)}{2\omega_\lambda{}^2\mu} y_3 e^{-w_\lambda y_3} \tag{69}$$

Next, by performing the inverse partial Fourier transform on ρ, $\hat{w}_J$, and $\hat{w}_3$ related to equations (67), (68), and (69) as follows:

$$\begin{aligned} \rho = & \sum_{n=1}^{2} \mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{i\xi_j}{2\mu w_\lambda{}^3}\hat{h}_n(0)e^{-w_\lambda y_3}\right](y_1,y_2) + \mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{(|\xi'|^2+w_\lambda{}^2)}{2w_\lambda{}^3}\hat{g}(0)e^{-w_\lambda y_3}\right](y_1,y_2) \\ & + \sum_{n=1}^{2} \mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{i\xi_j\hat{h}_n(0)}{2w_\lambda{}^2\mu} y_3 e^{-w_\lambda y_3}\right](y_1,y_2) - \mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{\lambda\hat{g}(0)}{2w_\lambda{}^2\mu} y_3 e^{-w_\lambda y_3}\right](y_1,y_2) \qquad (70) \\ & =: \mathcal{A}^3(\lambda)G_\lambda\boldsymbol{G} \end{aligned}$$

$$\begin{aligned} w_J = & -\mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{\hat{h}_j(0)}{\omega_\lambda}\right](y_1,y_2) - \sum_{n=1}^{2}\mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{i\xi_j}{2\omega_\lambda{}^3}\hat{h}_n(0)\, i\xi_n e^{-\omega_\lambda y_3}\right](y_1,y_2) \\ & + \mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{i\xi_j}{2\omega_\lambda{}^3}\lambda\hat{g}(0)e^{-\omega_\lambda y_3}\right](y_1,y_2) - \sum_{n=1}^{2}\mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{i\xi_j}{2\omega_\lambda{}^2}\hat{h}_n(0)\, i\xi_n y_3 e^{-\omega_\lambda y_3}\right](y_1,y_2) \\ & + \mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{i\xi_j}{2\omega_\lambda{}^2}\lambda y_3\hat{g}(0)e^{-\omega_\lambda y_3}\right](y_1,y_2) \qquad (71) \\ & =: \mathcal{B}_j^3(\lambda)G_\lambda\boldsymbol{G}, \text{for } j = 1,2 \end{aligned}$$

$$\begin{aligned} w_3 = & \sum_{n=1}^{2}\mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{\hat{h}_n(0)\, i\xi_n y_3 e^{-\omega_\lambda y_3}}{2\omega_\lambda}\right](y_1,y_2) - \mathcal{F}^{-1}_{\xi_1,\xi_2}\left[\frac{\lambda\hat{g}(0)y_3 e^{-\omega_\lambda y_3}}{2\omega_\lambda}\right](y_1,y_2) \qquad (72) \\ & =: \mathcal{B}_3^3(\lambda)\mathcal{G}_\lambda\boldsymbol{G}. \end{aligned}$$

Suppose $\mathcal{B}^3(\lambda) = \left(\mathcal{B}_1^3(\lambda), \mathcal{B}_2^3(\lambda), \mathcal{B}_3^3(\lambda)\right)^{\top}$, the operator solution **w** can be written as

$$\mathbf{w} =: \mathcal{B}^3(\lambda)\mathcal{G}_\lambda\boldsymbol{G}$$

Therefore, the solution operators $(\rho, \mathbf{w})$ can be expressed as follows:

$$(\rho, \mathbf{w}) = (\mathcal{A}^3(\lambda)\mathcal{G}_\lambda\boldsymbol{G}, \mathcal{B}^3(\lambda)\mathcal{G}_\lambda\boldsymbol{G})$$

Thus, it is proven in Theorem 3 that there exist the solution operators $(\rho, \mathbf{w}) = (\mathcal{A}^3(\lambda)\mathcal{G}_\lambda \boldsymbol{G}, \mathcal{B}^3(\lambda)\mathcal{G}_\lambda \boldsymbol{G})$ of the System (12) in 3-dimensional half-space.

Next, to prove Theorem 2, consider equation (10) from which the operator solution in the 3-dimensional half-space is obtained as follows:

$$\begin{aligned} \rho &= R + \tilde{\rho} \\ &= \mathcal{A}^1(\lambda)\mathcal{F}_\lambda \mathbf{F}^1 + \mathcal{A}^3(\lambda)\mathcal{G}_\lambda \boldsymbol{G} \\ &= \mathcal{A}^2(\lambda)\mathcal{F}_\lambda \mathbf{F}^2 \end{aligned}$$

suppose $\mathbf{u} = \boldsymbol{W} + \tilde{\boldsymbol{w}}$**,** then $\mathbf{u}$ can be written as follows

$$\begin{aligned} \mathbf{u} &= \boldsymbol{W} + \tilde{\boldsymbol{w}} \\ &= \mathcal{B}^1(\lambda)\mathcal{F}_\lambda \mathbf{F}^0 + \mathcal{B}^3(\lambda)\mathcal{G}_\lambda \boldsymbol{G} \\ &= \mathcal{B}^2(\lambda)\mathcal{F}_\lambda \mathbf{F}^2 \end{aligned}$$

Thus, equation (1) has an operator solution in the three-dimensional half-space for the case $\left(\frac{\mu + \nu}{2\,\kappa}\right)^2 - \left(\frac{1}{\kappa}\right) = 0$ with $\kappa = \mu\nu, \mu = \nu$, namely $(\rho, \mathbf{u}) = \mathcal{A}^2(\lambda)\mathcal{F}_\lambda \mathbf{F}^2, \mathcal{B}^2(\lambda)\mathcal{F}_\lambda \mathbf{F}^2$. Hence, **Theorem 2** is proved.

CONCLUSIONS

This study successfully demonstrates the existence of a solution operator for the Navier-Stokes-Korteweg (NSK) model with slip boundary conditions in a three-dimensional half-space. The originally complex system was simplified by applying a partial Fourier transform, allowing the resolvent system to be solved under specific coefficient conditions. The results confirm that the fluid dynamics of compressible two-phase fluids, influenced by capillarity effects, can be accurately modeled in confined geometries with slip boundary conditions. This contributes to a deeper understanding of how fluids behave near boundaries where slip conditions occur, such as in microfluidic devices or porous materials.

The method used here can be extended to other fluid models that include capillarity, viscosity, or other complex boundary conditions. Furthermore, the framework developed in this study could be applied to more intricate geometries, such as curved surfaces or domains with obstacles, which are common in practical applications. Future research could explore how the solution operator behaves in these more complex settings or investigate its applicability in non-isothermal conditions, expanding its relevance to a broader range of fluid dynamics problems.

In practical terms, the ability to solve the NSK model with slip boundary conditions offers insights for industries dealing with multi-phase fluid flows, such as in oil recovery, chemical engineering, and environmental sciences. By improving the accuracy of fluid models in these contexts, the results of this research could inform the development of more efficient simulation tools and optimization strategies for managing multi-phase flows in industrial processes.

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